Undirected strict gammoids

by

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Abstract. In this short note we consider strict gammoids which arise from undirected graphs. We exhibit a minimal example of a strict gammoid which cannot arise in this way and we interpret Ingleton and Piff's characterisation of strict gammoids for the undirected case.

In a directed graph $\underset{\sim}{G} = (V, \underset{\sim}{E})$ we say that $X \subseteq V$ is *linked into* $Y \subseteq V$ if there exists a set of mutually disjoint paths in $\underset{\sim}{G}$ whose set of initial vertices is X and whose set of terminal vertices is a subset of Y. Given $A, B \subseteq V$, the collection of all subsets of A which can be linked into B is a special type of matroid known as a *gammoid*: in the case when $A = V$ the gammoid is said to be *strict*. This concept translates naturally to an undirected graph G: one can either replace paths by undirected paths in the definitions or one can regard G as a directed graph in which each of its edges $\{u, v\}$ is replaced by two directed edges uv and vu. This latter comment was made by Woodall in [3] and he, naturally enough, called (strict) gammoids arising from undirected graphs, *undirected* (*strict*) *gammoids*. In [3] Woodall gave an example of a strict gammoid which was not an undirected gammoid, and in this short note we exhibit a minimal such example and, in passing, we interpret Ingleton's and Piff's characterisation of strict gammoids for the undirected case.

PROPOSITION 1. *Any matroid of rank 2 or less is an undirected strict gammoid.*

Proof. Let $\mathcal{M}$ be the matroid in question, let V be its underlying set and let X be those points of V which form independent singletons in $\mathcal{M}$. Then the relation $\sim$ defined on X by

$$x \sim y \text{ if } x = y \text{ or if } \{x, y\} \text{ is a circuit of } \mathcal{M} \text{ with } x \neq y$$

is easily seen to be an equivalence relation on X. Let its distinct equivalence classes be $[x_1], \ldots, [x_n]$, and let G be the undirected graph with vertex set V and edge set given by

$$E = \{\{x_i, x_j\} \colon 1 \leqslant i < j \leqslant n\} \cup \bigcup_{1 \leqslant i \leqslant n} \{\{x_i, x\} \colon x \in [x_i]\}.$$

Then it is straightforward to check that $\mathcal{M}$ consists precisely of those subsets of V linked into B in G, where B is any subset of $\{x_1, \ldots, x_n\}$ of cardinality equal to the rank of $\mathcal{M}$. Hence $\mathcal{M}$ is an undirected strict gammoid. ∎

A *transversal* of a family of sets $\mathfrak{A} = (A_1, \ldots, A_n)$ is a set of n elements, $\{x_1, \ldots, x_n\}$ say, with $x_i \in A_i$ for each i. A *partial transversal* of $\mathfrak{A}$ is a transversal of some subfamily of $\mathfrak{A}$. It is well known that the set of partial transversals of $\mathfrak{A}$ form a matroid, and one arising in this way is called a *transversal matroid*. In that event $\mathfrak{A} = (A_1, \ldots, A_n)$ is a *presentation* of the matroid, and it is well known that a transversal matroid of rank n has a presentation of a family consisting of precisely n sets. Of the many presentations of a transversal matroid $\mathscr{M}$, naturally enough one is called a *minimal presentation* if it uses the smallest number of sets possible and if none of the sets used can be replaced by a proper subset to give another presentation of $\mathscr{M}$. Now a family $\mathfrak{A} = (A_1, \ldots, A_n)$ will be called *symmetric* if there exist distinct $x_1, \ldots, x_n$ with

(i) $x_i \in A_i$ for $1 \leqslant i \leqslant n$ and (ii) $x_i \in A_j$ implies $x_j \in A_i$ for $1 \leqslant i, j \leqslant n$;

and a transversal matroid will be called *symmetric* if it possesses such a presentation. So, for example, a transversal matroid of rank 2 or less is symmetric, a minimal presentation providing the required symmetric presentation. For if (A_1, A_2) is a minimal presentation of a matroid, then it is easy to check that neither A_i is a subset of the other; hence there exist $x_1 \in A_1 \setminus A_2$ and $x_2 \in A_2 \setminus A_1$, from which the symmetry is clear.

Proposition 2. *The duals of undirected strict gammoids are precisely the symmetric transversal matroids.*

Proof. In [1] Ingleton and Piff show that the duals of transversal matroids are precisely the strict gammoids. More particularly, it follows from a version of their result in [2, p. 217] that if $\mathscr{M}$ (on set V) has presentation $\mathfrak{A} = (A_1, \ldots, A_n)$ and a transversal $\{x_1, \ldots, x_n\}$ with $x_i \in A_i$ for each i, and if $\underset{\sim}{G} = (V, \underset{\sim}{E})$ is the directed graph given by

$$\underset{\sim}{E} = \bigcup_{1 \leqslant i \leqslant n} \{\{x_i, x\} \colon x \in A_i \setminus \{x_i\}\},$$

then $X \subseteq V$ is linked into $B = V \setminus \{x_1, \ldots, x_n\}$ if and only if $V \setminus X$ contains a transversal of $\mathfrak{A}$. It is therefore easy to check that in the special case when $\mathfrak{A}$ is symmetric (and the $x_1, \ldots, x_n$ are chosen accordingly) the same result holds for the corresponding undirected graph. Hence the dual of a symmetric transversal matroid is an undirected strict gammoid.

Conversely, if the dual of $\mathscr{M}$ is an undirected strict gammoid, and consists of sets linked into B in the undirected graph $G = (V, E)$ say, then from the same result referred to above it can be deduced that $V \setminus B$ has exactly n distinct elements, $x_1, \ldots, x_n$ say, and that $\mathscr{M}$ is the transversal matroid with presentation $\mathfrak{A} = (A_1, \ldots, A_n)$, where

$$A_i = \{x_i\} \cup \{x \colon \{x_i, x\} \in E\} \quad (1 \leqslant i \leqslant n).$$

It is clear that $\mathfrak{A}$ is symmetric, and the result follows. ∎

We remarked above that transversal matroids of rank 2 or less are symmetric, and we now see that sufficiently small transversal matroids of rank 3 are also symmetric.

Proposition 3. *A transversal matroid of rank 3 on a set of 6 or fewer points is symmetric.*

Proof. Let $\mathscr{M}$ be the matroid in question and let (A_1, A_2, A_3) be a minimal presentation of $\mathscr{M}$. Then, in particular, if $|A_1 \cup A_2 \cup A_3| = m$ ($\leqslant 6$), it follows that

(1) $|A_i| \leqslant m-2 \leqslant 4$ for each i and (2) $A_i \nsubseteq A_j$ if $i \neq j$.

We now, in cases, exhibit a symmetric presentation of $\mathscr{M}$.

Case I. $A_1 \nsubseteq A_2 \cup A_3$, $A_2 \nsubseteq A_1 \cup A_3$, and $A_3 \nsubseteq A_1 \cup A_2$.

In this case, of course, there exist $x_1 \in A_1 \setminus (A_2 \cup A_3)$, $x_2 \in A_2 \setminus (A_1 \cup A_3)$, and $x_3 \in A_3 \setminus (A_1 \cup A_2)$; and it is clear that (A_1, A_2, A_3) is symmetric.

Case II. $A_1 \subseteq A_2 \cup A_3$, $A_2 \nsubseteq A_1 \cup A_3$, and $A_3 \nsubseteq A_1 \cup A_2$ (say).

In this case there exist $x_2 \in A_1 \setminus A_3 = (A_1 \cap A_2) \setminus A_3$ and $x_3 \in A_1 \setminus A_2 = (A_1 \cap A_3) \setminus A_2$. If there exists $x_1 \in A_1 \cap A_2 \cap A_3$, then the symmetry of (A_1, A_2, A_3) is clear. So we may assume that $A_1 \cap A_2 \cap A_3 = \emptyset$ so that $|A_1| = |A_1 \cap A_2| + |A_1 \cap A_3|$. If $|A_1 \cap A_2| \geqslant 2$ then there exist distinct $x'_1, x'_2 \in A_1 \cap A_2 = (A_1 \cap A_2) \setminus A_3$ and $x'_3 \in A_3 \setminus (A_1 \cup A_2)$; and again the symmetry of (A_1, A_2, A_3) is clear. So finally we may suppose that $|A_1 \cap A_2| \leqslant 1$ and, similarly, that $|A_1 \cap A_3| \leqslant 1$. Then, using (2), it is easy to see that there exist four elements $x''_1, x''_2, x''_3, x''_4$ such that $A_1 = \{x''_1, x''_2\}$, $\{x''_1, x''_3\} \subseteq A_2 \setminus A_3$ and $\{x''_2, x''_4\} \subseteq A_3 \setminus A_2$. If we now replace the element x''_1 of A_2 by x''_2 we get a symmetric presentation of $\mathscr{M}$ with representatives x''_1, x''_3 and x''_4.

Case III. $A_1 \subseteq A_2 \cup A_3$, $A_2 \subseteq A_1 \cup A_3$, and $A_3 \nsubseteq A_1 \cup A_2$ (say).

In this case there exist distinct $x_1, x_2 \in (A_1 \cup A_2 \cup A_3) \setminus A_3 = A_1 \setminus A_3 = (A_1 \cap A_2) \setminus A_3$, and $x_3 \in A_3 \setminus (A_1 \cup A_2)$, and again the symmetry of (A_1, A_2, A_3) is clear.

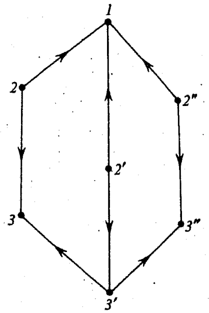
Case IV. $A_1 \subseteq A_2 \cup A_3$, $A_2 \subseteq A_1 \cup A_3$, and $A_3 \subseteq A_1 \cup A_2$.

It is not difficult to see that, in this case, any subset of $A_1 \cup A_2 \cup A_3$ which has cardinality at most three and is dependent must be contained in two of the sets A_1, A_2 and A_3, and be disjoint from the third. But then (1) and (2) lead to a contradiction in this particular case. Hence every subset of $A_1 \cup A_2 \cup A_3$ of cardinality at most three is in $\mathscr{M}$, and so $\mathscr{M}$ has symmetric presentation $(A_1 \cup A_2 \cup A_3, A_1 \cup A_2 \cup A_3, A_1 \cup A_2 \cup A_3)$. ∎

It is immediate from the above results that a strict gammoid which is not an undirected gammoid must be of rank at least 3 and on a set of at least 7 elements; below we present such a gammoid of rank precisely 3 and on a set of precisely 7 elements.

EXAMPLE. A minimal strict gammoid which is not an undirected gammoid.

Let $\mathscr{M}$ be the strict gammoid of sets linked into $1, 3, 3''$ in the directed graph illustrated in the figure:



Then the circuits of $\mathscr{M}$ of cardinality 3 are precisely $\{1, 2, 3\}$, $\{1, 2', 3'\}$, $\{1, 2'', 3''\}$ and $\{3, 3', 3''\}$: all other sets of cardinality 3 or less are independent. This example (and the verification below that $\mathscr{M}$ is not an undirected gammoid) is not dissimilar to Woodall's in [3].

Assume that $\mathscr{M}$ is an undirected gammoid consisting of the subsets of $\{1, 2, 3, 1', 2', 3', 1'', 2'', 3''\}$ $(\subseteq V)$ linked into a set B of cardinality 3 in the undirected graph $G = (V, E)$. Then since $\{3, 3', 3''\}$ is a circuit of $\mathscr{M}$ it follows from Menger's theorem that there exist $x, y \in V$ such that every path from $\{3, 3', 3''\}$ to B in G uses at least one of x and y. This means that, in addition, every path from $\{3, 3', 3''\}$ to $\{1, 2, 1', 2', 1'', 2''\}$ uses at least one of x and y (since, for example, the existence of a path from 3 to 1 avoiding x and y, together with the independence of $1, 3', 3''$, would imply the existence of a path from 3 to B avoiding x and y).

Now let us call a path from v to $\{x, y\}$ which meets $\{x, y\}$ only at its terminal vertex a *$v-x$ path* or a *$v-y$ path*, depending upon which member of $\{x, y\}$ it uses. Then, since $\{3, 1, 2'\} \in \mathscr{M}$ but $\{3', 1, 2'\} \notin \mathscr{M}$, it follows that either there exists a $3-x$ path but no $3'-x$ paths, or that there exists a $3-y$ path but no $3'-y$ paths: let us assume the former. A similar argument applied to $\{3', 1, 2\} \in \mathscr{M}$ and $\{3, 1, 2\} \notin \mathscr{M}$ then shows that there exists a $3'-y$ path but no $3-y$ paths. Similar arguments with respect to the pairs $3, 3''$ and $3', 3''$ show that there exists no $3''-x$ path and no $3''-y$ path (and hence no path from $3''$ to B). This contradiction shows that $\mathscr{M}$ is not an undirected gammoid.

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References

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