Concerning Menger regular curves.

By

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A continuum M will be called a Menger regular curve, or simply a regular curve[1]), provided it is true that for each point P of M and each positive number ϵ there exists a connected open subset of M containing P and of diameter less than ϵ whose boundary with respect[2]) to M is finite. The point P of a continuum M is said to be a regular point of M if for each $\epsilon > 0$ an open subset of M exists which contains P and is of diameter $< \epsilon$ and whose M-boundary is finite. Menger[3]) has shown that all regular curves are continuous curves (or, rather, that they are connected im kleinen continua).

A continuous curve M is said to be *cyclicly connected*[4]) provided that every two points of M lie together on some simple closed curve which is a subset of M. If the cyclicly connected continuous curve C is a subset of a continuous curve M, then C ist said to be a *maximal cyclic curve*[5]) of M provided that C is not a proper

[1]) Cf. K. Menger, *Grundzüge einer Theorie der Kurven*, Math. Ann. vol. 95 (1925), pp. 287—306.

[2]) If R is an open subset of M (i. e., no point of R is a limit point of $M - R$), then the boundary of R with respect to M is the set of all those points of $M - R$ which are limit points of R. Cf. R. L. Moore, *Concerning simple continuous curves*, Trans. Amer. Math. Soc., vol. 21 (1920), p. 345. In the present paper I shall use the term M-boundary to denote the boundary of an open subset of a continuum M with respect to M.

[3]) Loc. cit.

[4]) G. T. Whyburn, *Cyclicly connected continuous curves*, Proc. Nat. Acad. Sc., vol. 13 (1927), pp. 31—38.

[5]) Loc. cit.

subset of any cyclicly connected continuous curve which is also a subset of M.

In this paper a study will be made of the structure of Menger regular curves in general and also of some particular types of regular curves. Particular attention will be given to the structure of the *maximal cyclic curves* of these regular curves. An extensive theory of the structure of a continuous curve (whether it be a regular curve or not) with reference to its *cyclic elements*, i. e., its maximal cyclic curves, cut points, and end points, will be found in my paper *Concerning the structure of a continuous curve*[1]). Among other things it is there shown that *Every continuous curve is an acyclic continuous curve*[2]) *with respect to its cyclic elements.* Thus it is seen that the structure of any continuous curve can be fairly readily determined when once we know the nature of each of its maximal cyclic curves.

In this paper, unless otherwise specified, it is to be understood that the point sets considered lie in a Euclidean space of n dimensions. In many cases the theorems hold in even more general spaces. It is also to be understood, unless definitely stated to the contrary, that the supposition that „M is a continuous curve" implies that the point set M is bounded. Use will be made in n dimensions and for unbounded curves of a number of theorems of the author's on cyclicly connected continuous curves which were stated by him in the above mentioned papers only for the case of the plane. That these theorems are true in a space of n-dimensions and for unbounded curves has recently been established by W. L. Ayres[3]).

§ 1.

Regular Curves in General.

Menger[4]) has raised the question as to whether or not every regular curve is, for each $\epsilon > 0$, the sum of a finite number of

[1]) To appear soon in the American Journal of Mathematics.

[2]) That is, a continuous curve containing no simple closed curve, or a „baum" curve (Menger).

[3]) Cf. W. L. Ayres, *Cyclicly connected continuous curves in a space of n-dimensions*, presented to the Am. Math. Soc., Sept. 9, 1927.

[4]) K. Menger, *Zur allgemeinen Kurventheorie*, Fund. Math., vol. 10 (1927), pp. 96—115.

continua each of diameter $< \epsilon$ and no two of which have more than one point in common. In this section I shall give a partial solution to this problem. Although my solution is not a complete one, nevertheless it applies to a fairly general class of regular curves — much more general than the baum curve.

Definition. The two points (or point sets) A and B of a continuum M are said to be separated in M by the point (or point set) X of M provided that $M-X$ is the sum of two mutually separated point sets M_a and M_b containing A and B respectively.

In our paper „*On continuous curves in n-dimensions*“[1]) W. L. Ayres and I proved the following theorem (Theorem A) which will be used in this paper.

Theorem A. *In order that a continuum M should be a Menger regular curve it is necessary and sufficient that every two points of M should be separated in M by some finite subset of M.*

Theorem 1. *In order that a continuous curve M should be a Menger regular curve it is necessary and sufficient that every maximal curve of M should be a regular curve.*

Proof The condition is sufficient. For let A and B be any two points of a continuous curve M every maximal cyclic curve of which is a Menger regular curve. First suppose A and B lie together in a maximal cyclic curve C of M. Then since by hypothesis C is a Menger regular curve, it follows by Theorem A that there exists a finite subset T of C which separates A and B in C. Then T separates A and B in M. For suppose it does not. Then from a theorem of R. L. Moore's[2]) it readily follows that $M-T$ contains an arc t from A to B. But by a theorem of the author's[3]) it follows readily that C must contain every arc in M whose endpoints belong to C. Hence C contains t, and T does not separate A and B in C. Thus the supposition that T does not separate A and B in M leads to a contradiction. Second, suppose A and B

[1]) Offered to the Bulletin of the American Mathematical Society.

[2]) R. L. Moore, *Concerning continuous curves in the plane*, Math. Zeit., vol. 15 (1922) pp. 254—260, Theorem 1. Although stated by him for the case of plane continuous curves, it is easy to see that Moore's theorem holds in n-dimensions, since the axioms used by him are all satisfied in n-dimensions.

[3]) G. T. Whyburn, *Cyclicly connected continuous curves*, loc. cit., Theorem 2; see Ayres, loc. cit., for this theorem in n-dimensions and for unbounded curves.

do not lie together in the same maximal cyclic curve of M. Then A and B cannot lie together on any simple closed curve which belongs to M; and by a theorem of the author's[1]) it follows that there exists a point X of M which separates A and B in M.

Thus, in any case, there exists a finite subset of M which separates A and B in M. Therefore, by Theorem A, M is a Menger regular curve. Hence the condition of Theorem 1 is sufficient.

The condition is also necessary. For if a continuum M is a Menger regular curve, then it is an immediate consequence of the definition of a regular curve that *every subcontinuum* of M is also a regular curve. Hence all the maximal cyclic curves of M are regular curves.

Since every maximal cyclic curve of the boundary of a complementary domain of a plane continuous curve is a simple closed curve, we have the following

Corollary. *The boundary of every complementary domain of a plane continuous curve is a Menger regular curve.*

This corollary is not true in a space of 3 dimensions.

Theorem 2. *In order that a bounded continuous curve M should be, for each $\epsilon > 0$, the sum of a finite number of continua each of diameter $< \epsilon$ and no two of which have more than one point in common it is necessary and sufficient that each maximal cyclic curve of M should have this same property.*

Proof. The condition is sufficient. For let M be any continuous curve satisfying the condition, and let ϵ be any positive number. By a theorem of the author's[2]) there are not more than a finite number of maximal cyclic curves $C_{11}, C_{12}, C_{13}, \ldots, C_{1n_1}$ of M each of diameter $\geqslant \epsilon/4$. Let the components[3]) of $\sum_{i=1}^{n_1} C_{1i}$ be denoted by $C'_{11}, C'_{12}, \ldots, C'_{1n_2}$. For each $i \leqslant n_2$, C'_{1i} contains at most one limit point of each component of $M \sum_{i=1}^{n_1} C_{1i}$. Furthermore, there are at

[1]) G. T. Whyburn, *Some properties of continuous curves*, Bull. Am. Math. Soc. vol. 33 (1927) pp. 305—308, Theorem 3.

[2]) G. T. Whyburn, *Cyclicly connected continuous curves*, loc. cit. Theorem 6.

[3]) A component of a point set K is a maximal connected subset of K, i. e. a connected subset of K which is not a proper subset of any other connected subset of K; cf. F. Hausdorff, *Mengenlehre* 1927.

most a finite number of components X of $M-\sum_{i=1}^{n_1} C_{1i}$ such that there exist two distinct integers j and k such that both C'_j and C'_k contain a limit point of X. Let the components of $M-\sum_{i=1}^{n_1} C_{1i}$ for which this is true be denoted by $A_{11}, A_{12}, A_{13}, \dots, A_{1n_3}$. For each integer $i \leqslant n_3$, let X_{1i} and Y_{1i} be points belonging to different sets $\{C'_{1i}\}$ and such that each of these points is a limit point of A_{1i}. There exists an arc t_{1i} from X_{1i} to Y_{1i} which lies, except for the points X_{1i} and Y_{1i}, wholly in A_{1i}. Let T_{1i} denote the *simple cyclic chain*[1]) in M from X_{1i} to Y_{1i}, i. e., the set of points obtained by adding to t_{1i} all those maximal cyclic curves of M which contain a segment of t_{1i}. Then T_{1i} is a subset of $\bar{A}_{1i}$[2]). Now let $C'_{21}, C'_{22}, C'_{23}, \dots, C'_{2n_4}$ denote the components of the set of points $\sum_{i=1}^{n_1} C_{1i} + \sum_{i=1}^{n_3} T_{1i}$. Let $A_{21}, A_{22}, \dots, A_{2n_5}$ denote the components of $M - [\sum_{i=1}^{n_1} C_{1i} + \sum_{i=1}^{n_3} T_{1i}]$ which have the property that, for each integer $i \leqslant n_5$, there exist two distinct integers j and $k \leqslant n_4$ such that each of the sets C'_{2j} and C'_{2k} contains a limit point of A_{2i}. For each $i \leqslant n_5$, let X_{2i} and Y_{2i} be points which belong to different sets $\{C'_{2i}\}$ and which are limit points of A_{2i}. Let t_{2i} be an arc from X_{2i} to Y_{2i} which lies, except for the points X_{2i} and Y_{2i}, wholly in A_{2i}, and let T_{2i} be the simple cyclic chain in M from X_{2i} to Y_{2i}. Now let $C'_{31}, C'_{32}, \dots, C'_{3n_6}$ denote the components of $\sum_{i=1}^{n_1} C_{1i} + \sum_{i=1}^{n_3} T_{1i} + \sum_{i=1}^{n_5} T_{2i}$, and so on. Let this process be continued. Since M is a connected point set, it follows that this process must terminate after at most n_1 steps, i. e. there exists an integer k such that $\sum_{i=1}^{n_1} C_{1i} + \sum_{j=1}^{k} \sum_{i=1}^{n_{2j+1}} T_{ji}$ is a connected point set (a continuum), and hence such that no component of M minus this point set has more than one limit point in this set. Let H_1 denote this set of points.

[1]) See my paper *Concerning the structure of a continuous curve*, loc. cit. *Simple cyclic chain* is there defined in a different but equivalent way.

[2]) Wherever the symbol X is used to denote a point set, the symbol $\bar{X}$ will be used to denote the set X plus all of its limit points.

It follows from our hypothesis and from a theorem of Menger's[1]) that every maximal cyclic curve of M is a regular curve. Therefore by Theorem 1, M itself is a regular curve and therefore every subcontinuum of M is[2]) a continuous curve.

Since M and H_1 are bounded continuous curves, then there[3]) are not more than a finite number of components of $M-H_1$ of diameter $>\epsilon/4$. Let these be denoted by $R_{11}, R_{12}, R_{13}, \dots, R_{1j_1}$. For each positive integer $i \leqslant j_1$, H_1 contains exactly one limit point P_{1i} of R_{1i} and R_{1i} contains a point Q_{1i} whose distance from P_{1i} is $>\epsilon/8$. Let U_{1i} be an arc in M from P_{1i} to Q_{1i} and let W_{1i} be the simple cyclic chain im M from P_{1i} to Q_{1i} determined by the arc U_{1i}. Let H_2 denote the continuum $H_1 + \sum_{i=1}^{j_1} W_{1i}$. Then only a finite number of the components of $M-H_2$ are of diameter $>\epsilon/4$. Let these be ordered $R_{21}, R_{22}, R_{23}, \dots, R_{2j_2}$. For each positive integer $i \leqslant j_2$, H_2 contains exactly one limit point P_{2i}, and R_{2i} contains a point Q_{2i} whose distance from P_{2i} is $>\epsilon/8$. Let U_{2i} be an arc in M from P_{2i} to Q_{2i}, and let W_{2i} be the simple cyclic chain in M from P_{2i} to Q_{2i} determined by the arc U_{2i}. Let H_3 denote the continuum $H + \sum_{i=1}^{j_2} W_{2i}$. Denote the components of $M-H_3$ which are of diameter $>\epsilon/4$ by $R_{31}, R_{32}, R_{33}, \dots, R_{3j_3}$, and so on. Since every subcontinuum of M is a continuous curve, since each of the sets $\{W_{ij}\}$ is of a diameter $>\epsilon/8$, and since it is true that if a and b are integers such that $a \geqslant b+2$, then no one of the sets W_{aj} has a point in common with any one of the sets W_{bj}, it follows by a theorem of Gehman's[4]) that this process must terminate after a finite number of steps, i. e. there exists an integer g such that no component of $M-H_g$ is of diameter $>\epsilon/4$.

Now the continuum H_g is by definition, equal to the set of points $\sum_{i=1}^{n_1} C_{1i} + \sum_{i=1}^{k} \sum_{i=1}^{n_{2j+1}} T_{ji} + \sum_{n=1}^{g-1} \sum_{i=1}^{j_n} W_{ni}$. Since each set $\{C_{1i}\}$ is a ma-

[1]) K. Menger, *Grundzüge einer Theorie der Kurven,* loc. cit., Theorem 20.

[2]) K. Menger, loc. cit., Theorem 21.

[3]) Cf. W. L. Ayres, *Concerning continuous curves and correspondences*, Ann. of Math., vol. 28 (1927), pp. 396—418, Theorem 1. This also follows from the fact that every subcontinuum of M is a continuous curve and a theorem of H. M. Gehman's, cf. H. M. Gehman, *Concerning the subsets of a plane continuous curve*, Ann. of Math., vol. 27 (1925) pp. 29—46. Theorem 5.

[4]) H. M. Gehman, loc. cit.

ximal cyclic curve of M, then by hypothesis each such set can be expressed as the sum of a finite number of continua each of diameter $< \epsilon/4$ and no two of which have more than one point in common. And since no two sets $\{C_{1i}\}$ can have more than one point in common[1]), therefore $\sum_{i=1}^{n_1} C_{1i}$ is the sum of a finite number $K_1, K_2, K_3, \ldots, K_d$ of continua each of diameter $< \epsilon/4$ and no two of which have more than one point in common. Now for each pair of integers a and b such that T_{ab} is an element of $\sum_{j=1}^{k} \sum_{i=1}^{n_{2j+1}} T_{ji}$, the chain T_{ab} can be expressed as the sum of a finite number of continua each of diameter $< \epsilon/2$ and no two of which have more than one point in common as follows. Consider the arc t_{ab}. Every segment of this arc which is of diameter $> \epsilon/4$ must contain at least one point which separates X_{ab} and Y_{ab} in M, for otherwise by a theorem of the author's[2]), there would exist a maximal cyclic curve of M, distinct from every C_{1i}, which is of diameter $\geqslant \epsilon/4$, contrary to the fact that all such sets belong to $\{C_{1i}\}$. It follows[3]) that there exists a finite subset $F_1, F_2, F_3, \ldots, F_m$ of the set of all those points which separate X_{ab} and Y_{ab} in M such that each of the intervals $\widehat{X_{ab}F_1}, \widehat{F_1F_2}, \widehat{F_2F_3}, \ldots, \widehat{F_mY_{ab}}$ of t_{ab} is of diameter $\leqslant \epsilon/2$. Then the simple cyclic chain in M from X_{ab} to F_1, from F_1 to F_2, F_2 to F_3, and so on, are subsets of T_{ab}, no two of them have more than one point in common, and each of them is of diameter $\leqslant \epsilon$ (for each of the arcs $\widehat{X_{ab}F_1}, \widehat{F_1F_2}, \ldots$ is of diameter $\leqslant \epsilon/2$ and each maximal cyclic curve of M which belongs to one of these chains must be of diameter $< \epsilon/4$ and contain a point in common with that one of these arcs which determines the chain

[1]) See my paper *Cyclicly connected continuous curves*, loc. cit., Theorem 5.

[2]) G. T. Whyburn, *Some properties of continuous curves*, loc. cit., Theorem 2.

[3]) This may be proved as follows. In the order from X_{ab} to Y_{ab} on t_{ab} there exists a first point B_1 whose distance from X_{ab} is $= \epsilon/4$, on the interval B_1Y_{ab} a first point B_2 whose distance from B_1 is $= \epsilon/4$, on B_2Y_{ab} a first point B_3 whose distance from B_2 is $= \epsilon/4$, and so on. Since each interval $\widehat{B_iB_{i+1}}$ of $\widehat{X_{ab}Y_{ab}}$ is of diameter $\geqslant \epsilon/4$, this process must terminate, i. e., there exists an integer m such that every point of the interval $\widehat{B_mY_{ab}}$ is at a distance $< \epsilon/4$ from B_m. Each of the intervals $\widehat{X_{ab}B_1}, \widehat{B_1B_2}, \widehat{B_2B_3}, \ldots, \widehat{B_{m-1}B_m}$ must contain points $F_1, F_2, \ldots, F_m$ which separate X_{ab} and Y_{ab} in M.

to which it belongs). Then since no two of the chains $\{T_{ji}\}$ have more than one point in common, it follows that $\sum_{j=1}^{k} \sum_{i=1}^{n_{2j+1}} T_{ji}$ is the sum of a finite number $K_{d+1}, K_{d+2}, K_{d+3}, \ldots, K_{d+e}$ of continua each of diameter $< \epsilon$ and no two of which have more than one point in common. In an entirely similar way it is shown that $\sum_{n=1}^{g-1} \sum_{i=1}^{j_n} W_{ni}$ is the sum of finite number $K_{d+e+1}, K_{d+e+2}, \ldots, K_{d+e+f}$ of continua of diameter $< \epsilon$ and no two of which have more than one point in common.

It is readily seen, then, that no two of the continua $K_1, K_2, \ldots$ $\ldots K_{d+e+f}$ have more than one point in common. Now let N_1 denote the point set obtained by adding to K_1 all those components of $M - H_g$ which have a limit point in K_1 (each component of $M - H_g$ has just one limit point which belongs to H_g), let N_2 be the set obtained by adding to K_2 all the components of $M - H_g$ which have a limit point in K_2 but not in K_1, let N_3 be obtained by adding to H_3 all the components of $M - H_g$ which have a limit point in K_3 but not in $K_1 + K_2$, and so on; in general let N_i be obtained by adding to K_i all the components of $M - H_g$ which have a limit point in K_i but not in $K_1 + K_2 + \ldots + K_{i-1}$. Then since all the components of $M - H_g$ are of diameter $\leqslant \epsilon/4$, and since for any $\delta > 0$, not more than a finite number of them are of diameter $> \delta$, it follows that each of the sets $\{N_i\}$ is a continuum of diameter $< 2\epsilon$. It is easy to see that no two of these continua have more than one point in common. But $M = \sum_{i=1}^{d+e+f} N_i$; hence M is the sum of a finite number of continua each of diameter $< 2\epsilon$ and no two of which have more than one point in common.

The condition is also necessary. For suppose a bounded continuous curve satisfies the condition. Let C be any maximal cyclic curve of M and ϵ any positive number. By hypothesis M is the sum of a finite number of continua $K_1, K_2, K_3, \ldots, K_n$ each of diameter $< \epsilon$ and no two of which have more than one point in common. For each positive integer $i \leqslant n$, let N_i denote the set of points common to K_i and C. Then[1]), for each i, N_i is either va-

[1]) Cf. G. T. Whyburn, *Concerning the structure of a continuous curve*, Amer. Journ. Math., to appear, Theorem 30.

cuous, a point, or a continuum. Let $N_{n_1}, N_{n_2}, N_{n_3}, \ldots, N_{n_k}$ be the sets N_i which are continua. It is readily seen that $C = \sum_{i=1}^{k} N_{n_i}$. Since for each $i \leqslant k$, N_{n_i} is a subset of K_{n_i}, therefore each continuum N_{n_i} is of diameter $< \epsilon$ and no two of these continua have more than one point in common. This completes the proof of Theorem 2.

It is not difficult to prove the following theorem, which is more general than one part of Theorem 2: *If a continuum M is, for each $\epsilon > 0$, the sum of a finite number of continua of diameter $< \epsilon$ and no two of which have more than one point in common, then every subcontinuum of M also has this same property.* This theorem may readily be proved with the aid of a lemma on connected point sets proved by Knaster and Kuratowski and independently by the author (see § 6 below).

Corollary 1. *If every maximal cyclic curve of a continuous curve M is a simple closed curve, then for each positive number ϵ, M is the sum of a finite number of continua of diameter $< \epsilon$ and no two of which have more than one point in common.*

Corollary 2. *The boundary of every complementary domain of a plane bounded continuous curve is, for each $\epsilon > 0$, the sum of a finite number of continua of diameter $< \epsilon$ no two of which have more than one point in common.*

§ 2.

Accessibility of the Regular Points of a Continuum.

A boundary point P of a point set R is said to be *regularly accessible*[1]) from R provided that for every positive number ϵ there exists a positive number $\delta_{\epsilon P}$ such that every point X of R whose distance from P is $< \delta_{\epsilon P}$ can be joined to P by an arc XP of diameter $< \epsilon$ and such that $XP - P$ is a subset of R.

Fundamental Accessibility Theorem. Theorem 3. *If the limit point P of a point set R is not regularly accessible from R, then there exists a positive number ϵ and an infinite sequence of points $P_1, P_2, P_3, \ldots$ of R which has P as its sequential limit point and such that no two of these points can be joined by any arc of R which is of diameter $< \epsilon$.*

Proof. Suppose P is a limit point of a point set R such that no matter what positive number ϵ may be there exists no sequence $P_1, P_2, P_3, \ldots$ of distinct points of R having P as its sequential limit point and having the property that no two of these points can be joined by an arc in R of diameter $< \epsilon$. I shall show that under these conditions the point P is regularly accessible from R.

For each positive integer n and each point X of R let G_{nx} denote the set of all those points of R which can be joined in R to X by an arc of diameter $< 1/n$. Now for each positive integer n there must exist a neighborhood U_{rn} of P (i. e. the set of all points of the space whose distance from P is $< r > 0$) such that the subset H_{rn} of R which belongs to U_{rn} is a subset of the sum of the elements of some finite collection of the collection of sets $\{G_{nx}\}$. For suppose, on the contrary, that there exists an integer k such that every neighborhood of P contains a subset of R which is not contained in any finite collection of the sets $\{G_{kx}\}$. Let $r_1, r_2, r_3, \ldots$ be a sequence of positive numbers approaching zero. Let X_1 be a point belonging to H_{r_1k} (such a point exists, since P is a limit point of R). There exists a point X_2 in H_{r_2k} which does not belong to G_{kx_1}; there exists a point X_3 in H_{r_3k} which does not belong to $G_{kx_1} + G_{kx_2}$; there exists a point X_4 in H_{r_4k} not belonging to $G_{kx_1} + G_{kx_2} + G_{kx_3}$; and so on. Then P is the sequential limit point of the sequence of points $X_1, X_2, X_3, \ldots$, each of these points belongs to R, and no two of them can be joined in R by any arc which is of diameter $< 1/k$. This contradicts our original assumption. Thus it follows that for each integer $n > 0$ a neighborhood U_{rn} of P exists such that H_{rn} is a subset of the sum of the elements of some finite collection of the sets $\{G_{nx}\}$. For each integer $n > 0$, let us select some definite neighborhood U_{rn} of P having this property.

Let[1]) ϵ be any definite positive number. There exists an integer n_0 such that $1/n_0 < \epsilon/4$. The set of points $H_{r_{n_0}n_0}$ is contained

[1]) Cf. G. T. Whyburn, *Concerning the open subsets of a plane continuous curve*, Proc. Nat. Acad. Sc., vol. 13 (1927), pp. 650—657.

[1]) Compare the proof from this point on with that given in my paper *Concerning the open subsets of a plane continuous curve*, loc. cit., to prove Theorem 1.

in the sum of the elements of a finite collection G_0 of the sets $\{G_{n_0x}\}$. There exists an integer $n_0' > n_0$ such that $r_{n_0'} < r_{n_0}$ and such that each point of $H_{r_{n_0'} n_0'}$ belongs to at least one of the sets of the collection G_0 which has P for a limit point. I shall show that every point X of R whose distance from P is $< r_{n_0'}$ can be joined to P by an arc XP of diameter $< \epsilon$ and such that $XP - P$ belongs to R. Let X_0 denote any such point of R, i. e. any point of $H_{r_{n_0'} n_0'}$. The point X_0 belongs to some set g_0 of the collection G_0 which has P for a limit point. There exists an integer $n_1 > 2n_0'$ and such that $r_{n_1} < 1/2\, r_{n_0'}$ and also less than 1/2 the diameter of g_0. Then since $H_{r_{n_1} n_1}$ is contained in a finite number of the sets $\{G_{n_1x}\}$, it readily follows that there exists at least one of these sets g_1 which contains a point in g_0 and has P for a limit point. Let X_1 be a point belonging to $g_1 \cdot g_0$. There exists an integer $n_2 > 2n_1$ such that $r_{n_2} < 1/2\, r_{n_1}$ and also less than 1/2 the diameter of g_1. Then, just as above, there exists a set g_2 of the collection $\{G_{n_2x}\}$ which contains a point in g_1 and has P for a limit point. Let X_2 be a point of $g_2 \cdot g_1$. There exists an integer $n_3 > 2n_1$ such that $r_{n_3} < 1/2\, r_{n_2}$, and so on. This process may be continued indefinitely, giving an infinite sequence of point sets $g_0, g_1, g_2, \dots$ having the properties that for $n \geqslant 0$, g_n has P for a limit point, is arcwise connected, and contains the points X_n and X_{n+1}. For each integer $n \geqslant 0$, g_n contains an arc $X_n X_{n+1}$. From the properties of the sets $g_0, g_1, g_2, \dots$ it readily follows that the point set $P + X_1X_2 + X_2X_3 + X_3X_4 + \dots$ is closed and that it contains an arc X_1P from X_1 to P which lies, except for the point P, wholly in R. It is readily seen that this arc is of diameter $< \epsilon$. But now X_1 was any point of R whose distance from P is $< r_{n_0'}$. It follows that P is regularly accessible from R.

We have shown that if no positive number ϵ and no sequence of points $P_1, P_2, P_3, \dots$ exist satisfying the conditions of Theorem 3, then P is regularly accessible from R. Hence if P is not regularly accessible from R, a positive number ϵ and a sequence of points satisfying the conditions of Theorem 3 must exist. The trouth of Theorem 3 is therefore established.

Theorem 4. *If P is a regular point of a continuum M, then P is regularly accessible from every complementary domain of M to whose boundary it belongs.*

Lemma 1. *If P is a regular point of a plane continuum M then for each $\epsilon > 0$ there exists a simple closed curve J which encloses P and such that $J.\ M$ is finite (of power n, in case P is a point of order n of M).*

Lemma 1 is readily established with the use of a theorem of R. G. Lubben's[1]).

Proof of Theorem 4. First let us suppose that M is a plane continuum. Let P be any regular point of M and let R be any complementary domain of M whose boundary contains P, and suppose, contrary to Theorem 4, that P is not regularly accessible from R. Then by Theorem 3 it follows that there exists a positive number ϵ and an infinite sequence $P_1, P_2, \dots$ of points of R having P as its sequential limit point and no two of which can be joined by any arc of R of diameter $< \epsilon$. Now, by Lemma 1, there exists a simple closed curve J of diameter $< \epsilon/4$ which encloses P and such that $J.\ M$ is finite. There exists an integer k such that if i is $\geqslant k$ then P_i is within J. For each integer $i > k$, R contains an arc P_iP_k. Each arc P_iP_k contains an arc P_iQ_i, where Q_i is a point of J and such that $P_iQ_i - Q_i$ is within J. No two of the arcs $\{P_iQ_i\}$ can have a common point. Hence the set of points $\{Q_i\}$ is infinite. Then since $J.\ M$ is finite there exist points Q_a and Q_b of this set such that one of the arcs Q_aXQ_b of J from Q_a to Q_b contains no point of M. But then the set of points $P_aQ_a + Q_aXQ_b + P_bQ_b$ is an arc of diameter $< \epsilon$ which lies wholly in R and contains two points of the sequence $P_1, P_2, P_3, \dots$ This is contrary to the definition of this sequence. Thus the supposition that Theorem 4 is not true for the case of a plane continuum leads to contradiction.

That Theorem 4 is true in a space of $n (n \geqslant 3)$ dimensions is a corollary of the Theorem 5 below.

Zarankiewicz[2]) has shown that in a Euclidean space E_n of $n (n \geqslant 3)$ dimensions every one dimensional[3]) continuum M is

[1]) R. G. Lubben, *The separation of mutually separated subsets of a continuum by curves* (abstract), Bull. Amer. Math. Soc. vol. 32 (1926), p. 114.

[2]) C. Zarankiewicz, *Sur les points de division dans les ensembles connexes*, Fund. Math. vol. 9 (1927), p. 43.

[3]) In the Menger-Urysohn sense. Cf. P. Urysohn, *Mémoire sur les multiplicités cantoriennes*, Fund. Math. vol. 7, p. 65, and K. Menger, Monatshefte f. Math. u. Phys. 1 (1923), S. 148.

accessible from $E_n - M$. It is to be noted that Theorem 4 is not a corollary to the theorem of Zarankiewicz. For in Theorem 4 it is not assumed that the continuum M be regular or that it be one dimensional — it is merely assumed that it be regular, and hence one dimensional, at the point P in question. Furthermore Theorem 4 is true in two dimensions, whereas Zarankiewicz's theorem is not. The following extension of the theorem of Zarankiewicz is of interest.

Theorem 5. *If P is any one dimensional point of a continuum M in a Euclidean space of n dimensions ($n \geqslant 3$), then P is regularly accessible from every complementary domain*[1]) *of M whose boundary contains P.*

Proof. Let R be any complementary domain of M whose boundary contains P, and suppose, contrary to this theorem, that P is not regularly accessible from R. Then, by Theorem 3, there exists a positive number ϵ and an infinite sequence $P_1, P_2, P_3, \ldots$ of points of R which has P as its sequential limit point and such that no two of these points can be joined by any arc of R of diameter $< \epsilon$. Now since P is a 1 dimensional point of M, there exists a domain D of diameter $< \epsilon/2$ which contains P and whose boundary N has only a null-dimensional set in common with M. Let K denote the boundary of the unbounded complementary domain of N. Then since $K.M$ is a subset of $N.M$, therefore $K.M$, is null-dimensional; and as K is the common boundary of two $n(n \geqslant 3)$ dimensional domains, then by a theorem of Urysohn's[2]) $K - K.M$ is connected. Now as K is of diameter $< \epsilon/4$, and $K - K.M$ is a connected open subset of K which contains no point whatever of M it is easy to see that there exists a domain G of diameter $< \epsilon/2$ which contains $K - K.M$ but contains no point whatever of M.

Now there exists an integer k such that if $i \geqslant k$, then P_i lies in D. For each integer $i > k$, R contains an arc P_iP_k. Each such arc must contain at least one point of K (since it is of diameter $\geqslant \epsilon$), and hence for each $i > k$, the arc P_iP_k contains an arc P_iQ_i,

[1]) There can exist at most one complementary domain of M whose boundary contains P — a fact immediately obvious in view of Urysohn's Theorem (P. Urysohn, loc. cit. p. 94) that the common boundary of any two domains in $n(n \geqslant 3)$ dimensions is at least 2-dimensional.

[2]) P. Urysohn, loc. cit. p. 123.

where Q_i is on K and $P_iQ_i - Q_i$ contains no point of K. Now let a and b be two integers $> k$. Then as Q_a and Q_b belong to K they belong also to G. The domain G contains an arc Q_aQ_b which is necessarily of diameter $< \epsilon/2$. But then the arc $P_aQ_a + Q_aQ_b + P_bQ_b$ is of diameter $< \epsilon$ and lies wholly in R (since G contains no point of M), and it contains two distinct points of the sequence $P_1, P_2, P_3, \ldots$ But this is contrary to the definition of this sequence. Thus the supposition that theorem 5 is not true leads to a contradiction.

§ 3.

Some Particular Types of Regular Curves.

If P is a point of a continuous curve M, then M will be said to be a *baum im kleinen at the point P* provided that for each $\epsilon > 0$ there exists a connected open subset U of M which contains P, is of diameter $< \epsilon$ and is such that U is a baum curve (i. e. an acyclic continuous curve). A continuous curve which is a baum im kleinen at each of its points is called a baum im kleinen[1]).

Theorem 6. *If P is a point of a continuous curve M at which M is not a baum im kleinen, then for each positive ϵ M contains a simple closed curve every point of which is at a distance less than ϵ from P.*

Proof. Let K denote the set of all those points whose distance from P is $= \epsilon/2$, and let N be the component of $M - M.K$ which contains P. Then N is a connected open subset of M and $\overline{N}$ is a continuous curve. But $\overline{N}$ is not a baum curve, for M is not a baum im kleinen at P. Hence $\overline{N}$ contains a simple closed curve J. Every point of J is at a distance $< \epsilon$ from P. This completes the proof of Theorem 6

Theorem 7. *If the continuous curve M does not contain any infinite collection of mutually exclusive simple closed curves, then M is a regular curve which is a baum im kleinen at all save possibly a finite number of its points.*

[1]) Cf. K. Menger, *Über reguläre Baumkurven*, Math. Ann., vol. 95 (1926-1927) p. 574 (footnote).

Proof. Suppose, on the contrary, that M contains an infinite set of points Q at no one of which M is a baum im kleinen. The set Q contains an infinite sequence of points $P_1, P_2, P_3, \ldots$ having a sequential limit point P. By theorem 6 it follows that there exists in M a simple closed curve J_1 every point of which is at a distance from P_1 less than the least distance from P to the set of points $P_2 + P_3 + \ldots$ Likewise M contains a simple closed curve J_2 every point of which is at a distance from P_2 less than the least distance from P_2 to the set of points $J_1 + P_3 + P_4 + \ldots$ In general for each $i > 0$ M contains a simple closed curve J_i every point of which is at a distance from P_i less than the least distance from P_i to the set of points $\sum_{n=1}^{i-1} J_n + \sum_{n=i+1}^{\infty} P_n$. It is easy to see that no two of the simple closed curves $J_1, J_2, J_3, \ldots$ can have a common point. But by hypothesis M contains no infinite collection of mutually exclusive simple closed curves. Thus the supposition that M contains an infinite set of points at which it is not a baum im kleinen leads to a contradiction.

Now obviously every point of the continuous curve M at which M is a baum im kleinen is a regular point of M. Hence by the proof given above, the non regular points of M must be finite. But by a theorem of Menger's[1]), the set of all the non-regular points of a continuum is either null or else it contains a continuum. Hence the non-regular points of M are null and M is therefore a regular curve.

Corollary. (Cf. also Theorem 1). *If no maximal cyclic curve of a continuous curve M contains an infinite sequence of mutually exclusive simple closed curves, then M is a Menger regular curve which is baum im kleinen at all save possibly a countable number of points.*

Theorem 8. *If every point of a bounded continuum M in a plane S is regularly accessible from $S-M$, then M is a continuous curve.*

Proof. Let D denote any complementary domain of M and B its boundary. Then from our hypothesis it follows that every point of B is regularly accessible from D. Hence, by a theorem of the author's[2]), every point of B is accessible from D from all sides in the

[1]) K. Menger, *Grundzüge einer Theorie der Kurven*, loc. cit., Theorem 8.

[2]) G. T. Whyburn, *Concerning the open subsets of plane continuous curve*, Proc. Nat. Acad. Sc., vol. 13 (1927) pp. 650—657, Theorem 2.

sense of Schoenflies[1]). Now if ϵ is any positive number, not more than a finite number of the complementary domains of M are of diameter $> \epsilon$. For suppose, on the contrary, that there exists an infinite sequence $D_1, D_2, D_3, \ldots$ of the complementary domains of M each of which is of diameter $> \epsilon$. Then by a Theorem of R. G. Lubben's[2]) there exists an infinite sequence $R_1, R_2, R_3, \ldots$ of these domains having a sequential limiting set T of diameter $\geqslant \epsilon$ which is a continuum belonging to M. Let X, Y and Z be three distinct points of T. Then each of these points must belong to the boundary of — and therefore be accessible from — all save a finite number of the domains $R_1, R_2, R_3, \ldots$; for otherwise there would exist an infinite sequence of points $P_{n_1}, P_{n_2}, P_{n_3}, \ldots$ having one of these points, say X, as a limit point and that, for each i, P_{n_i} belongs to R_{n_i}, but such that no one of these points can be joined to X by any arc which lies, except for the point X, in $S-M$, contrary to the fact that X is regularly accessible from $S-M$. Hence there exist three of these domains R_a, R_b, R_c such that each of the points X, Y and Z is accessible from all three of these domains. But this is contradictory to a theorem proved by the author[3]). Thus the supposition that there exist infinitely many complementary domains of M each of diameter $> \epsilon$ leads to a contradiction. Then by a theorem due to Schoenflies[4]) it follows that M is a continuous curve.

Theorem 9. *In order that every point of a non-dense continuous curve M in a plane S should be regularly accessible from $S-M$ it is necessary and sufficient that M should contain no infinite collection of mutually exclusive simple closed curves.*

Proof. The condition is necessary. For suppose, on the contrary, that there exists a non-dense continuous curve M in a plane S

[1]) A. Schoenflies, *Die Entwickelung der Lehre von den Punktmannigfaltigkeiten*, zweiter Teil, Jahr. der Deutschen Math.-Ver., Ergänzungsbände, vol. 2 (1908), p. 215.

[2]) R. G. Lubben, *Concerning Limiting sets*, (abstract) Bull. Amer. Math. Sc., vol. 32 (1926), p. 14.

[3]) Cf. abstract of my paper, *Some theorems concerning domains and their boundaries*, Bull. Amer. Math. Soc., vol. 32 (1926), p. 200. This theorem has been generalized and will appear in my paper *On irreducible cuttings of a continuum*.

[4]) A. Schoenflies, loc. cit.

every point of which is regularly accessible from $S-M$ but which contains an infinite collection G of mutually exclusive simple closed curves. It is readily seen that there exists an infinite sequence $J_1, J_2, \dots$ of curves of G and a point P of M such that either (1) for each positive integer i, J_i encloses $J_{i+1}+P$ and P is a limit point of a sequence of points $P_1, P_2, P_3, \dots$ where P_i belongs to $S-M$ and lies in the domain between J_i and J_{i+1} or (2) P is without every one of these curves and is a limit point of a sequence of points $P_1, P_2, \dots$ where, for each i, P_i is a point of $S-M$ within J_i. In either case it is clear that no point P_i of the sequence $P_1, P_2, \dots$ can be joined to P by any arc which lies save for the point P in $S-M$. Then since P is a limit point of the set of points $P_1+P_2+P_3+\dots$ belonging to $S-M$, it follows that P is not regularly accessible from $S-M$, contrary to hypothesis. Thus it follows that the condition of theorem 9 is necessary.

The condition is also sufficient. For suppose a continuous curve M contains no infinite collection of simple closed curves, and suppose, contrary to this theorem, that M contains a point P which is not regularly accessible from $S-M$. Then there exists a positive number ϵ such that for every positive number δ points of $S-M$ exist whose distance from P is $<\delta$ which cannot be joined to P by any arc of diameter $<\epsilon$ which lies, save for P, in $S-M$. Now let G denote the set of all those points of $S-M$ which lie in some complementary domain of M whose boundary contains the point P. Then by the above quoted theorems of Schoenflies and the author it readily follows that P is regularly accessible from G. Hence there exists a positive number δ_ε such that every point of G whose distance from P is $<\delta_\varepsilon$ can be joined to P by an arc lying except for P in G (and hence in $S-M$). There exists a point P_1 of $S-M$ whose distance from P is $<\delta_\varepsilon$ which cannot belong to G, because, by supposition, P is not regularly accessible from $S-M$. The boundary of the complementary domain of M which contains P_1 contains[1]) a simple closed curve J_1. The curve J_1 cannot contain P, for P_1 does not belong to G. Now it has just been shown that P is a limit point of the collection of all those components of $S-M$ which are not subsets of G. And as P

[1]) Cf. R. L. Moore, ***Concerning continuous curves in the plane***, loc. cit., Theorem 4.

is not a limit point of any one such component it must be a limit point of their sum. Then since, by the above theorem of Schoenflies only a finite number of the components of M are of diameter $>$ any preassigned positive number, it follows that there exists a component D_2 of $S-M$ every point of which is at a distance from P less than the distance from P to J_1 and whose boundary does not contain P. The boundary of D_2 contains a simple closed curve J_2. Likewise there exists a component D_3 of $S-M$ every point of which is at a distance from P less than the distance from P to J_1+J_2 and whose boundary does not contain P. The boundary of D_3 contains a simple closed curve J_3. This process may be continued indefinitely, giving an infinite sequence of simple closed curves $J_1, J_2, J_3, \dots$ which belong to M. But it is readily seen that no two of these curves can have a common point, and by hypothesis M contains no infinite collection of mutually exclusive simple closed curves. Thus the supposition that the condition of this theorem is not necessary leads to a contradiction and the proof is complete.

Theorem 10. *If the bounded non-dense continuum M in a plane S is regularly accessible at each of its points from $S-M$ then M is a regular curve which is a baum im kleinen at all save possibly a finite number of its points.*

Theorem 10 is an immediate consequence of Theorems 7, 8, and 9.

Theorem 11. *If M is a plane non-dense continuous curve and G denotes the collection of all the complementary domains of M, then the limiting set*[1]) *T of the collection G is identical with the set of all those points of M at which M is not a baum im kleinen.*

Proof. It is clear that T is a subset of M. Let P be any point of T. Then M is not a baum im kleinen at P. For there exists an infinite sequence of domains $D_1, D_2, D_3, \dots$ of G such that if, for each i, P_i is a point of D_i, then P is the sequential limit point of the sequence of points $P_1, P_2, P_3, \dots$ Now, for each positive integer i, the boundary of D_i contains[2]) a simple closed curve J_i. Then since P belongs to the limiting set of the collection of curves

[1]) i. e. the set of all those points which are limit points of some set of points containing just one point in each element of G.

[2]) Cf. R. L. Moore, loc. cit.

$[J_i]$, and since at most a finite number of these curves are of diameter greater than any preassigned positive number, it follows that every open subset of M which contains P must contain at least one of them. Hence M is not a baum im kleinen at P.

We have just shown that M is a baum im kleinen at no point of T. Now let P be any point of M at which M is not a baum im kleinen. Then P must belong to T. For, by Theorem 6, there exists in M a simple closed curve J_1 every point of which is at a distance < 1 from P. Since M is non-dense, there exists a point P_1 of $S-M$ which is within J_1. By Theorem 6, there exists a simple closed curve J_2 in M every point of which is at a distance from P less than $\frac{1}{2}$ the distance from P_1 to P. There exists within J_2 a point P_2 of $S-M$. By Theorem 6, M contains a simple closed curve J_3 every point of which is at a distance from P less than $\frac{1}{2}$ the distance from P_2 to P. The curve J_3 encloses a point P_3 of $S-M$. This process may be continued indefinitely, giving an infinite sequence $P_1, P_2, P_3, \ldots$, of points of M which has P as its sequential limit point. Now if P_i and P_j are any two distinct points of this sequence, $i > j$, then the simple closed curve J_j encloses P_j but does not enclose P_i. Hence P_i and P_j belong to different complementary domains of M, and therefore P belongs to the limiting set of the collection G and hence belongs to T. Thus we have shown that T is identical with the set of all those points of M at which M is not a baum im kleinen.

Theorem 12. *If P is a point of order w*[1] *of a cyclicly connected continuous curve M, then M is not a baum im kleinen at P.*

Proof. Suppose, on the contrary, that M is a baum im kleinen at P. Then since P is a regular point of M, there exists a connected open subset K of M which contains P and is such that $\overline{K}$ is an acyclic continuous curve, and furthermore $\overline{K}-K$, i. e. the M boundary of K, is finite. Let the points of $\overline{K}-K$ be denoted by $P_1, P_2, \ldots, P_m$. Then, for each integer $n \leqslant m$, $\overline{K}$ contains an arc $P_n P$. Let N denote the sum of this finite set of arcs $[P_n P]$. Then N must be identical with $\overline{K}$. For suppose there exists a component R of $\overline{K}-N$. Since N contains the set of points $P_1 + P_2 + \ldots + P_m$, it follows that R is an open subset of M. Then since, by a Theorem of the author's[1]), the M-boundary of R must contain at least two points, it is readily seen that $\overline{K}$ must contain a simple closed curve, contrary to the fact that $\overline{K}$ is acyclic. Hence N is identical with $\overline{K}$. Now since $\overline{K}$ is the sum of m arcs having P as a common endpoint, it readily follows that, for each $\epsilon > 0$, there exists a connected open subset of M of diameter $< \epsilon$ which contains P and whose M-boundary contains at most m points. But this is contrary to the hypothesis that P is a point of order w of M. Thus the supposition that Theorem 12 is not true leads to a contradiction.

Corollary. *If the cyclicly connected continuous curve M is a baum im kleinen at one of its points P, then there exists an integer n such that P is a point of Menger order n*[2]) *of M.*

Theorem 13. *If a cyclicly connected continuous curve M contains infinitely many points of order w, then M contains an infinite collection of mutually exclusive simple closed curves.*

Theorem 14. *If a cyclicly connected continuous curve M contains a point of order w, then M contains infinitely many (not necessarily mutually exclusive) simple closed curves.*

Theorems 13 and 14 are immediate consequences of Theorems 6, 7 and 12.

Theorem 15. *In order that a cyclicly connected continuous curve M should be a baum im kleinen it is necessary and sufficient* (1) *that M contain no infinite collection of mutually exclusive simple closed curves and* (2) *that M contain no point of order w.*

Proof. That condition (1) is necessary follows immediately from the fact that M is bounded. Condition (2) is necessary by virtue of Theorem 12. I shall proceed to show that these conditions are sufficient. Let P be any point of a cyclicly connected continuous curve M satisfying the conditions, and let ϵ be any positive number. By Theorem 7, M is a Menger regular curve. Then since, by condition (2), P is not a point of order w of M, there exists an integer n such that P is a point of order n of M. By a theorem

[1]) If P is a regular point of a continuous curve M but there exists no integer n such that for each $\epsilon > 0$ a connected open subset of M exists which contains P and whose M-boundary contains not more than n points, then P is said to be a point of order w of M. Cf. K. Menger, *Grundzüge einer Theorie der Kurven*, loc. cit.

[1]) G. T. Whyburn, *Cyclicly connected continuous curves*, loc. cit., Theorem 9.

[2]) Cf. K. Menger, loc. cit.

of Menger's[1]) there exists a subset N of M which is the sum of n arcs having P as common endpoint and no two of which have any other point in common. I shall now show that P is not a limit point of $M-N$. Suppose the contrary is true. Now as P is a point of order n of M, obviously P is not a limit point of any single component of $M-N$. Hence there exists an infinite collection G of components of $M-N$ such that P belongs to the limiting set of the collection G. Now since N contains[2]) at least two limit points of each element of G, it follows that either (1) there exists one arc t of the finite set of arcs of which N is the sum such that t contains at least two limit points of each element of an infinite subcollection G_1 of the collection G, or (2) there exist two of these arcs t_1 and t_2 such that there exists an infinite collection G_2 of the elements of G each of which has one limit point in each of the arcs t_1 and t_2. In case (1), let g_1 be an element of G_1. Then since $t-P$ contains at least two limit points of g_1, it is readily seen that $g_1+(t-P)$ contains a simple closed curve J_1. Since J_1 cannot contain P, and since, for each $\epsilon>0$, not more than a finite number of elements of G are of diameter[3]) $>\epsilon$, there exists an element g_2 of G_1 which has all of its M-boundary on an arc s_1 of t which contains no point of J_1. Just as above, g_2+s_1 contains a simple closed curve J_2 which does not contain P. The curves J_1 and J_2 can have no point in common. There exists an element g_3 of G_1 which has all its M-boundary on an arc s_2 of t which contains no point of J_1+J_2, and so on. This process may be continued indefinitely, giving an infinite collection of mutually exclusive simple closed curves $J_1, J_2, J_3, \ldots$, belonging to M. But this contradicts hypothesis (1). Now in case (2), let d_1 and e_1 be elements of G_2. It is easy to see that $d_1+e_1+t_1+t_2$ contains a simple closed curve J_1 which does not contain P. There exist elements d_2 and e_2 of G_2 whose M-boundaries are subsets of subarcs a_1 and b_1 respectively of t_1 and t_2 which contain P but contain no point of J_1. Then the set of points $d_2+e_2+a_1+b_1$ contains a simple closed curve J_2 which does not contain P. This pro-

[1]) *Zur allgemeinen Kurventheorie*, loc. cit., Theorem p. 98.

[2]) This follows from the fact that M is cyclicly connected Cf G. T. Whyburn, loc. cit.

[3]) Cf. W. L. Ayres, *Concerning continuous curves and correspondences*, loc. cit., Theorem 1.

cess may be continued indefinitely, giving an infinite collection of mutually exclusive simple closed curves $J_1, J_2, J_3, \ldots$, belonging to M. This again is contradictory to hypothesis (1). Thus, in any case, the supposition that P is a limit point of $M-N$ leads to a contradiction. Now since P is not a limit point of $M-N$, there exists a connected open subset K of M of diameter $<\epsilon$ which contains P and is a subset of N. Since N is the sum of a finite number of arcs having only the point P common to any two of them, it is clear that $\overline{K}$ is an acyclic continuous curve. Therefore M is a baum im kleinen at each of its points P, and our theorem is proved.

Theorem 16. *If P is a regular point of order n of a plane continuum M, then P belongs to the boundaries of not more than n complementary domains of M.*

Theorem 16 is readily proved with the aid of Lemma 1 to Theorem 4. We have the following

Corollary. *If the regular point P of a plane continuum M belongs to the boundaries of infinitely many complementary domains of M, then P is a point of order w of M.*

§ 4.

The Ramification Points of Curves.

A point P of a continuous curve M is said to be a ramification point[1]) of M provided there exist three arcs in M having the point P in common but such that no two of them have any other point in common.

Theorem 17. *The set of points of ramification of each maximal cyclic curve of a continuous curve M which contains only a finite number of simple closed curves*[2]) *is finite.*

[1]) Cf. W. Sierpiński, Comptes Rendus, vol. 160 (1915), p. 302.

[2]) Continuous curves containing only a finite number of simple closed curves have been studied by Alexandroff, *Kombinatorische Eigenschaften von Kurven*, Math. Ann. vol. 96, (1926), pp. 512—554, W. L. Ayres, *Concerning continuous curves of certain types*, Fund. Math., vol. 11, Kuratowski and Zarankiewicz, *A theorem on connected point sets*, Bull. Amer. Math. Soc., vol. 33 (1927), p. 575. Cf. also G. T. Whyburn, *On a problem of W. L. Ayres* Fund. Math., vol. 11.

Proof. Let C be any maximal cyclic curve of M. In my paper *On a problem of W. L. Ayres* [1]) I showed that C is the sum of a finite number $S_1, S_2, S_3, \ldots, S_n$ of mutually exclusive arc segments plus their endpoints. Let K denote the set of all points $[X]$ of C such that X is an endpoint of some one of the segments $S_1, S_2, \ldots, S_n$. Then clearly K contains not more than $2n$ points. Furthermore every ramification point of C belongs to K. Hence the ramification points of C are finite.

Theorem 18. *For cyclicly connected continuous curves M, the following properties are equivalent: (α) M contains only a finite number of simple closed curves, (β) M is a baum im kleinen, (γ) M has only a finite number of points of ramification.*

The proof of Theorem 18 is not difficult with the aid of Theorem 17. Properties (α) and (β) are equivalent [2]) for all continuous curves, cyclicly connected or not.

Theorem 19. *In order that the ramification points of a continuous curve M should be countable it is necessary and sufficient that the ramification points of each maximal cyclic curve of M should be countable.*

Proof. The condition is obviously necessary. I shall show that it is sufficient. Let K denote the set of all the ramification points of M. Let K_1 be the set of all points $\{X\}$ such that X is a ramification point of some maximal cyclic curve of M. Then since [3]) M has only a countable number of maximal cyclic curves and since, by hypothesis, the ramification points of each of these are countable, it follows that K_1 is countable. Now let K_2 be the set of all those points $\{Y\}$ of $K - K_1$ such that Y belongs to some maximal cyclic curve of M. Then each point Y of K_2 is a cut point of M; for if not, then since Y is a ramification point of M, it follows readily with the aid of a theorem of R. L. Moore's [4]) that Y is a ramification point of some maximal cyclic curve of M, contrary to the fact that Y does not belong to K_1. Hence each point of K_2 is a cut point of M which belongs to some simple closed curve in M, and therefore, by a theorem of the author's [1]), K_2 must be countable. Let K_3 be the set of points $K - (K_1 + K_2)$. Then no point of K_3 can belong to any maximal cyclic curve of M. And since each point Z of K_3 is a ramification point of M it easily follows with the aid of the above quoted theorem of R. L. Moore's that, for each point Z of K_3, $M - Z$ is neither connected nor the sum of two connected point sets. Therefore by a theorem proved by Kuratowski and Zarankiewicz [2]) and independently by the author [3]) it follows that K_3 must be countable. Then since K is the sum of three countable sets K_1, K_2 and K_3 therefore K is countable.

[1]) Loc. cit.

[2]) Cf. K. Menger, *Über reguläre Baumkurven*, loc. cit. p. 574 (footnote). In this connection see also Kuratowski and Zarankiewicz, loc. cit.

[3]) G. T. Whyburn, *Cyclicly connected continuous curves*, loc. cit., Theorem 6.

[4]) R. L. Moore, *Concerning continuous curves in the plane*, loc. cit.

Theorem 20. *If for each positive number ϵ the continuous curve M contains not more than a finite number of simple closed curves of diameter $> \epsilon$, then the ramification points of each maximal cyclic curve of M are finite, and the ramification points of M are countable.*

Theorem 20 is an immediate consequence of Theorems 17 and 19 and of a result in my paper *On a problem of W. L. Ayres* [4]).

Theorem 21. *If no maximal cyclic curve of a continuous curve M contains an infinite sequence of mutually exclusive simple closed curves, then the ramification points of M are countable.*

Proof. Let C be any maximal cyclic curve of M. Then by hypothesis and Theorem 7 it follows that C is a baum im kleinen at all save a finite set of points K. Then if U is any open subset of M containing K, then since M is a baum im kleinen at each point of $C - U$ and since the ramification points of every baum is countable [5]), it readily follows that the ramification points of C belonging to $C - U$ are countable (finite in fact). Then with this fact established, by choosing a suitable sequence of open subsets U_1, $U_2, U_3, \ldots$ of M converging to the set K it easily follows that the

[1]) G. T. Whyburn, *Concerning continua in the plane*, Trans. Amer. Math. Soc., Vol. 29 (1927) pp. 369—400, Theorem 29.

[2]) Kuratowski and Zarankiewicz, *A theorem on connected point sets*, Bull. Amer. Math. Soc., vol. 33 (1927), pp. 571—575.

[3]) G. T. Whyburn, *Concerning the cut points of continua*, presented to the Amer. Math. Soc. Sept. 9, 1927. Offered to the Trans. Amer. Math. Soc.

[4]) Loc. cit. It is there shown that each maximal cyclic curve of a continuous curve satisfying the condition of Theorem 20 contains only a finite number of simple closed curves.

[5]) Cf. Ważewski, Ann. Soc. Pol. Math., vol. 2 (1923), p. 169; K. Menger, *Über reguläre Baumkurven*, loc. cit., p. 576.

ramification points of C are countable; and since C is any maximal cyclic curve of M, therefore by Theorem 19, the ramification points of M are countable.

The same method of proof yields the following theorem.

Theorem 22. *If each maximal cyclic curve of a continuous curve M is a baum im kleinen at all save possibly a countable number of its points, then the ramification points of M are countable.*

Theorems 20, 21 and 22 are decidedly more general theorems than that of Alexandroff[1]) to the effect that the ramification points of a continuous curve containing only a finite number of simple closed curves are countable.

Theorem 23. *Every non-regular point of a continuous curve M is a ramification point of M. Indeed, if A is a non-regular point of M, then there exists a point B of M such that if n is any integer then M contains at least n arcs from A to B no two of which have in common any points other than A and B.*

Proof. Let A be any non-regular point of a continuous curve M. Then there exists a point B of M which is not separated in M from A by any finite subset of M, for otherwise[2]) A would be a regular point of M. Then by a theorem of N. E. Rutt's[3]) it follows that if n is any integer there exist in M at least n arcs from A to B such that no two of these arcs have any other point in common. Then by taking $n=3$, it follows that A is a ramification point of M.

Theorem 24. *If the set K of all the ramification points of a continuous curve M is punctiform*[1]), *then M is a Menger regular curve.*

[1]) P. Alexandroff, *Über kombinatorische Eigenschaften allgemeinen Kurven*, Math. Ann. vol. 96 (1926), p. 252, corollary 4. See also W. L. Ayres, *Continuous curves and correspondences*, loc. cit., Theorem 12; and Kuratowski and Zarankiewicz, *A theorem on connected point sets*, loc. cit., p. 575.

[2]) This statement is proved precisely as in the proof of Theorem A. (above). See G. T. Whyburn and W. L. Ayres, *On continuous curves in n dimensions*, loc. cit. Theorem 5.

[3]) N. E. Rutt, *Concerning the cut points of continuous curves*, etc., (abstract), Bull. Amer. Math. Soc., vol. 33 (1927), p. 411. Rutt's theorem states that if there exist in a continuous curve M exactly n independent arcs between two points A and B of M, then there exist in M n points which separate A and B in M. This theorem has been proved only for the case of the plane. Hence Theorems 23 and 24 are established only for the plane. However they are true in n-dimensions provided Rutt's theorem is true in n-dimensions.

Proof. By hypothesis and Theorem 23, the set B of all the non-regular points of M must be punctiform. But by a theorem of Menger's[2]). B is either null or else it contains a continuum. Therefore B must be null, and hence M is a regular curve.

Corollary. *If the ramification points of a continuous curve M are countable, then M is a Menger regular curve.*

§ 5.

The Disconnection of Regular Curves.

Theorem 25. *If a continuum M is disconnected by the omission of every one of its countably infinite subsets, then M is a Menger regular curve which contains only a finite number of simple closed curves* (hence is a baum im kleinen).

Proof. I shall first show that M is a continuous curve. Suppose this is not so. Then by a theorem of R. L. Moore's[3]) there exists a positive number d and a countable infinity of mutually exclusive subcontinua of M: $W, M_1, M_2, \dots$ each of diameter $> d$ and such that W is the sequential limiting set of the sequence of continua $M_1, M_2, \dots$ Now there exists an uncountable collection G of point sets $\{X\}$ such that each element X of G is a countably infinite subset of W and such that no two different elements of G have a common point. Now[4]) by hypothesis it follows that for each element X of G, $M-X$ is the sum of two mutually separated sets S_1 and S_2. One of these sets must contain infinitely many of the continua $M_1, M_2, \dots$ Denote one which does by S_x and denote

[1]) A set is punctiform if and only if it contains no continuum.

[2]) K. Menger, *Grundzüge einer Theorie der Kurven*, loc. cit.

[3]) R. L. Moore, *A report on continuous curves from the viewpoint of Analysis situs*, Bull. Amer. Math. Soc., vol. 29 (1923), pp. 296—297.

[4]) Compare the proof from this point on with that given by R. L. Moore on page 338 of his paper *Concerning simple continuous curves*, Trans. Amer. Math. Soc., vol. 21 (1920). See also R. L. Moore, *Concerning the cut points of continuous curves and of other closed and connected point sets*, Proc. Nat. Acad. Sc., vol. 9 (1923), pp. 101—106, Theorem B*, and G. T. Whyburn, *Concerning the disconnection of continua by the omission of pairs of their points*, Fund. Math., vol. 10 (1927), pp. 180—186, Theorem 3.

the other by S'_x. Then clearly $W-X$ must be a subset of S_x, and therefore S'_x can contain not more than a finite number of the continua $M_1, M_2, \ldots$ Now if X and Y are two distinct sets of the collection G, then S'_x and S'_y can have no point in common. For suppose they do have a point in common. Let K denote the set of all points common to these two sets. Since K is a subset of S'_x and $W-X$ is a subset of S_x, no point of $W-X$ is a limit point of K; and since Y is a subset of $W-X$, therefore no point of Y is a limit point of K. Hence no point of S_y+Y is a limit point of K. Clearly no point of S'_y-K is a limit point of K. Then since $M=(S_y+Y+S'_y-K)+K$, it follows that K is closed. But since K is the common part of the two open subsets S'_x and S'_y of M, therefore K is an open subset of M. Hence K and $M-K$ are mutually separated sets, contrary to the fact that M is connected. Thus the supposition that S'_x and S'_y have a point in common leads to a contradiction.

Now by the Zermelo postulate, there exists a set of points Q such that (1) for each set X in G there exists in Q just one point which belongs to S'_x, and (2) for each point U in Q there exists in G, just one set X such that S'_x contains U. Since G is uncountable, Q is uncountable and therefore it contains a point Z which is a limit point of $Q-Z$. But there exists in G a set A such that Z belongs to S'_a. Since no point of $H'-Z$ belongs to S'_a, Z is not a limit point of $H'-Z$. Thus the supposition that M is not a continuous curve (or rather a connected im kleinen continuum) leads to a contradiction.

Now by a theorem of Zarankiewicz[1]), M can contain only a finite number of simple closed curves. Hence M is a regular curve — in fact a baum im kleinen.

Corollary. *If the continuum M is such that there exists an integer k such that M is disconnected by the omission of any k of its points, then M is a baum im kleinen (contains only a finite number of simple closed curves).*

This corollary is a generalization of a theorem of J. R. Kline's[2]).

[1]) C. Zarankiewicz, *Sur les points de division dans les ensembles connexes*, loc. cit.

[2]) J. R. Kline, *Closed connected sets which are disconnected by the omission of a finite number of points*, Proc. Nat. Acad. Sc., vol. 9 (1923), pp. 7—12, Theorem A. Kline's theorem contains a superfluous hypothesis, is stated only for two dimensional space, and has a weaker conclusion than that of our corollary.

Theorem 26. *If C is a cyclicly connected continuous curve, then in order that there should exist an integer k such that C is disconnected by the omission of any k of its points it is necessary and sufficient that M should be the sum of a finite number of mutually exclusive arc segments plus their endpoints.*

Proof. The condition is necessary. For if there exists an integer k such that C is disconnected by the omission of any k of its points, then by the corollary to Theorem 25, C contains only a finite number of simple closed curves. Therefore, since C is cyclicly connected, by a theorem of the author's[1]), C is the sum of a finite number of mutually exclusive arc segments plus their endpoints.

The condition is also sufficient. For suppose C is the sum of a finite number of mutually exclusive arc segments $S_1, S_2, \ldots S_k$ plus their endpoints. Then let $P_1, P_2, \ldots P_k$ be any k of the points of C. Let K denote the point set $P_1, P_2, \ldots P_k$. Now if any two points of K lie together on the same arc of the set $\overline{S}_1, \overline{S}_2, \ldots \overline{S}_k$, then clearly $M-K$ ist not connected. And if no two points of K lie on the same arc of this set, then each of these arcs must contain a point of K, and there exists an integer $i \leqslant k$ such that S_i contains a point P of K. Let A and B be points belonging to the two segments respectively of S_i-P. Then A and B can be joined by no arc which is a subset of $M-K$, for K contains a point of each of the segments $S_1, S_2, \ldots S_k$. Hence by a theorem of R. L. Moore's[2]) it follows that $M-K$ is not connected.

Theorem 27. *If a continuum M is disconnected by the omission of any of its countably infinite subsets (or of any k of its points, where k is some integer given in advance) then there exist two points of M whose omission disconnects M.*

Proof. By Theorem 25, M is a continuous curve which contains only a finite number of simple closed curves. Now if M has a cut point, then clearly M is disconnected by the omission of some two of its points. If M has no cut point then[3]) M is cyclicly connected.

[1]) *On a problem of W. L. Ayres*, loc. cit.

[2]) R. L. Moore, *Concerning continuous curves in the plane*, loc. cit., Theorem 1.

[3]) G. T. Whyburn, *Cyclicly connected continuous curves*, loc. cit. Theorem 1.

Then since M contains only a finite number of simple closed curves. M is the sum[1]) of a finite number of arc segments $S_1, S_2, \dots S_n$ plus their endpoints. Then clearly for each $i \leqslant n$, M is disconnected by the omission of any two points belonging to S_i.

As a corollary to Theorem 27 we have the following theorem of J. R. Kline's[2]): *There does not exist, in n-dimensional space, a continuum M and an integer $k > 2$ such that M is disconnected by the omission of any k of its points but by no $k-1$ of its points.*

Theorem 28. *In order that a continuous curve M (bounded or not) should be disconnected by the omission of any two of its points which lie together on some simple closed curve in M it is necessary and sufficient that every maximal cyclic curve of M should be a simple closed curve.*

Theorem 28 is an extension of Theorem 6 of my paper *Concerning certain types of continuous curves*[3]). It can be proved very easily with the aid of the properties of the maximal cyclic curves of a continuous curve[4]).

§ 6.

The Connected Subsets of Regular Curves.

Theorem 29. *Every connected subset of a Menger regular curve is connected im kleinen.*

Proof. Let H be any connected subset of a regular curve M, and let P be any point of M and ϵ any positive number. Since M is a regular curve, there exists a connected open subset U of M which contains P, is of diameter $< \epsilon$, and whose M-boundary B is finite. Now by a theorem proved by Knaster and Kuratowski[5]) and independently by the author[5]), it follows that the components of the point set $H.(U+B)$ are finite in number. Hence if K denotes that one of these components which contains P, then P is not a limit point of $H-K$. Therefore there exists a positive number $\delta_{\epsilon P}$ such that every point of H whose distance from P is $< \delta_{\epsilon P}$ belongs to K. And since K is of diameter $< \epsilon$, it follows that H is connected im kleinen.

[1]) G. T. Whyburn, *On a problem of W. L. Ayres*, loc. cit.

[2]) J. R. Kline, loc. cit. Theorem 5. Kline does not state whether or not this theorem is true in n-dimensions.

[3]) Proc. Nat. Acad. Sc., vol. 12 (1926), pp. 761—767.

[4]) Cf. G. T. Whyburn, *Cyclicly connected continuous curves*, loc. cit.

[5]) This theorem is as follows: *If the subset N of a connected point set M is the sum of a finite number, n, of connected point sets, and if $M-N$ is the sum of two mutually separated sets M_1 and M_2, then M_1+N is the sum of n connected point sets.* Cf. Knaster and Kuratowski, *Remark on a theorem of R. L. Moore*, Proc. Nat. Acad. Sc.. vol. 13 (1927); G. T. Whyburn, *On the separation of connected point sets*, (abstract), Bulletin of the American Mathematical Society, vol. 33 (1927), p. 388. This theorem is a generalization of a theorem of mine in my paper, *Concerning the disconnection of continua by the omission of pairs of their point*, loc. cit., Theorem 1

Theorem 30. *If each maximal cyclic curve of a continuous curve M is a baum im kleinen at all save possibly a finite number of its points, then every connected subset of M is arcwise connected.*

Proof. Let C be any maximal cyclic curve of M. I shall first show that every connected subset of C is arcwise connected. Let H be any connected subset of C, and let K denote the set of all those points of H at which C is not a baum im kleinen. Then by hypothesis K is finite. Since each point of C at which C is a baum im kleinen is a regular point of C, it follows from a theorem of Menger's[1]) that C is a Menger regular curve. Hence, by Theorem 29, H is connected im kleinen. Then since C is a baum im kleinen at every point of $H-K$, and since[2]) every connected subset of a baum is arcwise connected, it readily follows that H is arcwise connected im kleinen[3]) at every point of $H-K$, and that every component of $H-K$ is arcwise connected im kleinen at every one of its points. Then, by a theorem of the author's[4]), each component of $H-K$ is arcwise connected. Now let P be any point of K. Let G denote the sum of all those components of $H-K$ which have P as a limit point. Now if E is any component of G, then since E is arcwise connected, and every subcontinuum of C is a continuous curve, then by a theorem of the au-

[1]) *Grundzüge einer Theorie der Kurven*, loc. cit.

[2]) Cf. R. L. Wilder, *Concerning continuous curves*, Fund. Math., vol. 7 (1925), Theorem 20.

[3]) *A* set M is arcwise connected im kleinen at one of its points P if for each $\epsilon > 0$ a $\delta_{\epsilon P} > 0$ exists such that every point of M whose distance from P is $< \delta_{\epsilon P}$ can be joined in M to P by an arc of diameter $< \epsilon$.

[4]) G. T. Whyburn, *Concerning the complementary domains of continua* (to appear), Theorem 12.

thor's[1]), P is regularly accessible from E, and hence $E+P$ is arcwise connected. Hence it follows that $G+P$ is arcwise connected, and therefore, by another theorem of mine[2]) $G+P$ is arcwise connected im kleinen. But since H is connected im kleinen, it follows with the aid of a theorem of R. L. Wilder's[3]) that P is not a limit point of $H-(G+P)$. Hence H is arcwise connected im kleinen at every point of K. It was shown above that H is arcwise connected im kleinen at every point of $H-K$. Hence H is arcwise connected im kleinen at every one of its points. Then since H is connected, it follows by a theorem of the author's[4]) that H is arcwise connected.

I have just shown that every connected subset of each maximal cyclic curve of M is arcwise connected. Therefore, by a theorem of mine[5]), every connected subset of M is arcwise connected.

Theorem 31. *If no maximal cyclic curve of a continuous curve M contains an infinite collection of mutually exclusive simple closed curves, then every connected subset of M is arcwise connected.*

Theorem 31 follows immediately from Theorems 7 and 30.

Theorem 32. *If the ramification points of each maximal cyclic curve of a continuous curve M are finite in number, then every connected subset of M is arcwise connected.*

Theorem 32 is an immediate consequence of Theorems 18 and 32.

Problem. *If the ramification points of each maximal cyclic curve of a continuous curve M are countable in number, then is every connected subset of M arcwise connected?*

[1]) Loc. cit. Theorem 15.

[2]) G. T. Whyburn, *Concerning certain types of continuous curves*, loc. cit. Theorem 5.

[3]) *A Theorem on connected point sets which are connected im kleinen*, Bull. Amer. Math Soc., vol. 32 (1926), pp 338—340.

[4]) G. T. Whyburn, *Concerning the complementary domains of continua*, loc. cit., Theorem 12.

[5]) G. T. Whyburn, *Concerning the structure of a continuous curve*, loc. cit., Theorem 33. This theorem is as follows: *In order that every connected subset of a continuous curve M should be arcwise connected it is necessary and sufficient that every connected subset of each maximal cyclic curve of M should be arcwise connected.*

University of Texas, Nov. 14, 1927.

A separation theorem.

By

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In my paper *Concerning the separation of point sets by curves* [1]) it is stated that if T is a totally disconnected closed subset of the boundary of a simply connected domain D and there exists a continuum K containing T and such that $K-T$ is a subset of D then there exists a simple closed curve J containing T and enclosing $K-T$ and such that $J-T$ is a subset of D. That this proposition does not hold true, in the form in which it is stated, even for the case where T is a single point on the outer boundary of D, may be seen with the aid of the following example.

Example. Let T, A, B and C denote the points (0, 0), (2, 0), (2, 2) and (0, 2) respectively. For each positive integer n, let F_n denote the point $(1, 1/n)$ and let T_n denote the point whose abscissa is $1/n$ and whose ordinate is $(2n+1)/(n^2+n)$. Let M denote the continuum composed of the straight line intervals TA, AB, BC and CT together with all the straight line intervals of the sequence TF_1, TF_2, $TF_3, \dots$ Let K denote the sum of all the intervals of the sequence TT_1, TT_2, $TT_3, \dots$ Let D denote the bounded complementary domain of the continuum M. There exists no simple closed curve J containing T and enclosing $K-T$ and such that $J-T$ is a subset of D.

The following modification of the proposition in question holds true and suffices as a substitute in some of the applications in which the use of that proposition may seem to be indicated.

[1]) Proc. Nat. Acad. Sc., vol. 11 (1925), pp. 469—476.