On continuous functions without a derivative.

By

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Since Weierstrass [1]) many examples have been given of continuous functions which have no derivative at every point of their definiton. But Weierstrass' function has a one-sided infinite derivative at points of an everywhere dense set. The attention [2]) has been drawn more than once to the question of the existence of functions which have no one-sided derivative at every point. In a paper recently published [3]) A. S. Besicovitch has constructed such a function. However his proof that the function has no derivative is rather complicated.

In this paper I consider the same example, but give a new proof which seems to be considerably simpler.

§ 1. We first construct a figure (fig. 1), which we call a step-triangle (Stufendreieck). Take the points $A(0, 0)$, $D(a, 0)$, $B(2a, 0)$, $C(a, b)$, where $a > 0$, $b > 0$. Mark a segment

$$l_{1,1} = \frac{a}{4}$$

on the segment AD, so that the centre of $l_{1,1}$ coincides with the

[1]) Monatberichte der Akademie der Wissenschaften zu Berlin Aug. 1880.

[2]) Denjoy. Mémoire sur les nombres dérivés des fonctions continues. Journal de Math. (7-e série) tome I, 1915, p. 210.

Carathéodory. Lehrbuch der Funktionentheorie (1-st edition) 1918 p. 594.

[3]) Bulletin de l'Académie des Sciences de Russie 1925, p. 527. Diskussion der stetigen Funktionen im Zusammenhang mit der Frage über ihre Differentierbarkeit. This paper was published in Russian in 1922.

centre of AD (thus $l_{1,1}$ is the segment $(\frac{3}{8}a, \frac{5}{8}a)$ of the x-axis. Denote by

$$d_{1,1} \quad d_{1,2}$$

the two segments of AD outside of $l_{1,1}$. Obviously $d_{1,1} = d_{1,2}$. Mark the segments

$$l_{2,1} = l_{2,2} = \frac{a}{4^2}$$

on the segments $d_{1,1}$, $d_{1,2}$ so that the center of $l_{2,1}$ coincides with that of $d_{1,1}$ and the center of $l_{2,2}$ with that of $d_{1,2}$. Denote by

$$d_{2,1},\ d_{2,2},\ d_{2,3},\ d_{2,4}$$

the four segments of AD complementary to the segments $l_{1,1}$, $l_{2,1}$, $l_{2,2}$.

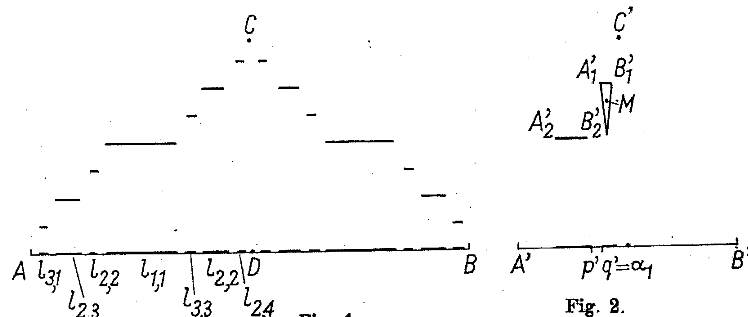

Fig. 1. Fig. 2.

Fig. 1: Step-triangle. For the sake of simplification of the fig. 2 the sides of $A'_1 B'_1 C'_1$ are given as straight lines instead of being „step-lines". The sides of $A' B' C'$ are not sewn at all except the segments $A'_1 B'_1$, $A'_2 B'_2$.

Obviously

$$d_{2,1} = d_{2,2} = d_{2,3} = d_{2,4}.$$

In the same way as before mark the segments

$$l_{3,1} = l_{3,2} = l_{3,3} = l_{3,4} = \frac{a}{4^3}$$

on the segments $d_{2,1}$, $d_{2,2}$, $d_{2,3}$, $d_{2,4}$ and so on.

Denote by L_k (k any positive integer) the set of segments

$$l_{k,1} = l_{k,2} = \ldots = l_{k,2^{k-1}} = \frac{a}{4^k}. \tag{1}$$

Each segment of L_k is of length $\frac{a}{4^k}$, consequently

$$mL_k = \frac{a}{2^{k+1}}.$$

Denote by L the aggregate of all L_k, then

(2) $$mL = \sum_{k=1}^{\infty} \frac{a}{2^{k+1}} = \frac{a}{2}.$$

Introduce further the notation

$$\Lambda_k = L_1 + L_2 + \dots + L_k$$

and denote the consecutive segments of Λ_k from left to right by

$$\lambda_{k,1},\ \lambda_{k,2}, \dots,\ \lambda_{k,2^k-1}.$$

Obviously the part of the segment AD complementary to Λ_k consists of all segments

$$d_{k,1},\ d_{k,2}, \dots,\ d_{k,2^k}.$$

Let

$$E = (0, a) - L.$$

We have

$$mE = \frac{a}{2}.$$

§ 2. Define now in the interval $(0, a)$ the curve

$$y = \varphi(x) = 2\frac{b}{a} m\,\{E \times (0,x)\}$$

where b is the ordinate of the point C.

We have

$$\varphi(0) = 0 \quad \text{and} \quad \varphi(a) = b.$$

This curve is monotone (increasing) and passes through the points A and C. Take it as the left side of the step-triangle. For the right side of the step-triangle we take the line BC simmetrical with respect to the line CD (the „height of the step-triangle"). The set E being complementary to L has no points on segments of L; consequently the part of E included in the interval $(0, x)$ remains unaltered when x takes different values on the same segment of L, thus $m\{E \times (0, x)\}$ is constant on any segment of L and to any segment of L corresponds a horizontal segment of the left side of the step-triangle.

Remark 1. Observe that all parts of the curve $AC(y = \varphi(x))$, corresponding to the segments

$$d_{k,1},\ d_{k,2}, \dots,\ d_{k,2^k}$$

are similar (congruent). Therefore if p, q are the ends of any segment $d_{k,i}$ we have

(3) $$\varphi(q) - \varphi(p) = \frac{\varphi(a) - \varphi(0)}{2^k} = \frac{b}{2^k}.$$

§ 3. Call segments L and similar segments of DB „L-segments of the first category", and the corresponding segments on the sides of the step-triangle „M-segments of the first category" On each of these M-segments, construct in a similar way, a step-triangle directed inside the triangle ABC. On equal segments construct equal triangles and the height of the triangle constructed on $l_{k,i}$ is to be equal to $\frac{b}{2^k}$, so that the vertex of the step-triangle corresponding to $l_{k,1}$ is (because of (3)) on the line AB. Call these triangles „triangles of the first category", and the operation of constructing these triangles „jagging of the step-triangle inside". Observe that the height of the largest triangle of the first category is equal to half the height of the original triangle AB, i. e. to $\frac{1}{2}b$. Applying the operation of „jagging inside" to each of the step triangles of the first category, we obtain step-triangles of the second category. Then „jagg inside" the triangles of the second category, and so on. To each of the triangles of the first category correspond segments similar to L and M-segments of the first category. Call them „L and M-segments of the second category". In the same way we obtain „L and M-segments of the third category" and so on. Observe that the height of the largest triangle of the second category is equal to $\frac{1}{4}b$, of the third category $\frac{1}{8}b$, and so on.

§ 4. Define a function $f(x)$ in the interval $(0, 2a)$ in the following way.

1. At points x outside L-segments of the first category, define $f(x)$ to be equal to the ordinates of the corresponding points on the side of the step-triangle ABC.

2. At points x which belong to L-segments of the first category

but not to those of the second cotegory, put $f(x)$ equal to the ordinates of the points on the sides of the triangles of the first category.

3. At points x which belong to L-segments of the second category but not to those of the third category, put $f(x)$ equal to the ordinates of the points on the sides of the triangles of the second category, and so on.

4. At points x which belong to L-segments of all categories, define $f(x)$ by the principle of continuity.

Evidently we have first to prove that the function $f(x)$ can be defined in this way. The possibility of the definition at the points 1., 2., 3. is obvious, and we have only to show that it is possible to define the function at points 4. by the principle of continuity. For this purpose, we introduce functions $\varphi_k(x)$, $k=1, 2, 3, \ldots$ Put

1). $\varphi_k(x)=f(x)$ at all points of AB outside the L-segments of the k-th category.

2) At all points of L-segments of the k-th category, define $\varphi_k(x)$ by the ordinates of the sides of the triangles of the k-th category.

The functions $\varphi_k(x)$ are obviously continuous. The maximum of $|\varphi_k(x)-\varphi_{k+1}(x)|$ is equal to the height of the largest triangle of the $k+1$ category, i. e. to $\frac{b}{2^{k+1}}$. Thus

$$|\varphi_k(x)-\varphi_{k+1}(x)| \leqq \frac{b}{2^{k+1}}$$

and consequently

$$|\varphi_k(x)-\varphi_{k+p}(x)| \leqq \frac{b}{2^{k+1}}+\frac{b}{2^{k+2}}+\ldots+\frac{b}{2^{k+p}}<\frac{b}{2^k}.$$

Hence the sequence of functions $\varphi_k(x)$, $k=1, 2, 3, \ldots$, is uniformly convergent and the limit function $\Phi(x)$ is continuous. Denote by G the set of all points of $(0, 2a)$ except the points which belong to L-segments of all categories. It is obvious that at every point of G, $f(x)=\Phi(x)$, and the definition of $f(x)$ outside G by the same equation $f(x)=\Phi(x)$ satisfies the condition of continuity.

Thus the function $f(x)$ is defined and is continuous. We shall now proceed to the proof that $f(x)$ has no derivative, *even no one-sided derivative* at every point of $(0, 2a)$.

§ 5. We shall first consider the question of the differentiability of $f(x)$ *at points of the interval* $(0, a)$ outside L-segments of the

first category, and at the ends of L-segments. For this purpose it will be sufficient to consider values of $f(x)$ only at these points and at the central points of L-segments of the first category. At points outside L-segments of the first category

$$f(x)=\varphi(x).$$

The point $x=a$ does not belong to L-segments and therefore the value of $f(x)$ for $x=a$ is defined by the ordinate of the vertex C of the triangle ABC. In the same way we see that at the central points of the L-segments of the first category the values of $f(x)$ are given by the ordinates of the centers of the triangles of the first category. Therefore if q and γ are an end and the center of a segment $l_{k,i}$, then $f(q)-f(\gamma)$ is equal to the height of the triangle constructed on $l_{k,i}$, i. e, (by § 3)

$$f(q)-f(\gamma)=\frac{b}{2^k}. \tag{4}$$

Consider separately the question of the existence of the right and, of the left derivative.

§ 6. Let x be a point outside the L-segments of the first category, or the right end of one of these segments. Since the set of L-segments of the first category is everywhere dense, there are some of these segments included in the interval $(x, x+h)$ for every positive h. Let $l_{k,i}$ be the largest of these segments. (In any interval, there is only one largest because between two equal consecutive intervals $l_{k,i}$, $l_{k,i+1}$ there is always a larger one). The segment $l_{k,i}$ is identical with the segment $\lambda_{k,2i-1}$. The point x lies in the interval d next left to $\lambda_{k,2i-1}$ i. e. in the interval $d_{k,2i-1}$. Denote by p, q the ends of $d_{k,2i-1}$ and by γ the center of $l_{k,i}$. We consider the ratio

$$\frac{f(x_1)-f(x)}{x_1-x}$$

for two values of x_1 in the interval $(x, x+h)$, viz. $x_1=q$ and $x_1=\gamma$. We have

$$\frac{f(q)-f(x)}{q-x}=\frac{\varphi(q)-\varphi(x)}{q-x}>0 \tag{5}$$

and by (4) and (3)

(6) $$\frac{f(\gamma)-f(x)}{\gamma-x}=\frac{f(\gamma)-f(q)+f(q)-f(x)}{\gamma-x}=$$

$$=\frac{\frac{-b}{2^k}+f(q)-f(x)}{\gamma-x}\leqq\frac{\frac{-b}{2^k}+\varphi(q)-\varphi(p)}{\gamma-x}=0$$

We have further

(7) $$\frac{f(q)-f(x)}{q-x}-\frac{f(\gamma)-f(x)}{\gamma-x}>\frac{\{f(q)-f(x)\}-\{f(\gamma)-f(x)\}}{\gamma-x}>$$

$$>\frac{\frac{b}{2^k}}{\frac{a}{2^k}}=\frac{b}{a}.$$

h being arbitrary, q and γ may be as near x as we please. From (5), (6) and (7) we conclude

Proposition 1. *At all points outside L-segments of the first category and at their right ends the right upper derivative is non-negative, the right lower derivative non-positive, and their difference is not less than $\frac{b}{a}$, i. e. at all such points no right derivative exists.*

§ 7. Let now x be a point outside L segments of the first category or the left end of one of these segments, and let h be any positive number. Denote by $l_{k,i}$ the largest L-segment of the first category completely included in $(x-h, x)$, by α and γ the left end and center of $l_{k,i}$. Consider the ratio $\frac{f(x)-f(x_1)}{x-x_1}$ for two values of x, in the interval $(x-h, x)$; for $x_1=\alpha$ and $x_1=\gamma$. We have

(8) $$\frac{f(x)-f(\alpha)}{x-\alpha}=\frac{\varphi(x)-\varphi(\alpha)}{x-\alpha}=\frac{2b}{a}\,\frac{m\{E\times(0,x)\}-m\{E\times(0,\alpha)\}}{x-\alpha}=$$

$$=\frac{2b}{a}\,\frac{m\{E\times(\alpha,x)\}}{x-\alpha}.$$

Thus

(9) $$0<\frac{f(x)-f(\alpha)}{x-\alpha}\leqq\frac{2b}{a}.$$

We have

(10) $$\frac{f(x)-f(\gamma)}{x-\gamma}-\frac{f(x)-f(\alpha)}{x-\alpha}>\frac{\{f(x)-f(\gamma)\}-\{f(x)-f(\alpha)\}}{x-\gamma}>$$

$$>\frac{f(\alpha)-f(\gamma)}{x-\gamma}>\frac{\frac{b}{2^k}}{\frac{a}{2^k}}=\frac{b}{a},$$

and thus

(11) $$\frac{f(x)-f(\gamma)}{x-\gamma}>\frac{b}{a}.$$

We conclude from (9), (10), and (11)

Proposition 2. *At all points outside L-segments of the first category and at their left ends, the left upper derivative is greater than or equal to $\frac{b}{a}$, the left lower derivative is less than or equal to $\frac{2b}{a}$ and their difference is greater than or equal to $\frac{b}{a}$, i. e. at all of these points no left derivative exists.*

We have to add yet that at the point $x=0$ no right derivative, and at the point $x=a$ no left derivative exists. Propositions 1. and 2. have been deduced for points of the interval $(0, a)$, but it is obvious that analogous propositions are also applicable to the interval $(a, 2a)$.

It is obvious further that the above argument is completely applicable to every triangle of the first category. Thus we have by combined application of proposition 1. and 2., *at all points inside segments of the first category, but outside segments of the second category no one-sided derivative exists, at left ends of segments of the first category and at right ends of segments of the second category no right derivative exists, and similarly at right ends of segments of the first category ant at left ends of segments of the second category no left derivative exists.* Proceeding in this way we finally conclude

Proposition 3. *At every point, except those which belongs to L-segments of all categories, no one-sided derivative exists.*

It remains to examine points which belong to L-segments of all categories. Let x_0 be such a point, and h any positive number. From some number onwards, the segments of all categories containing the point x_0 are included in the interval (x_0-h, x_0+h). Take among these intervals, any two consecutive ones (α,β), (α_1,β_1).

Let γ, γ_1 be their centers and let $A'B'C'$, $A_1'B_1'C_1'$ be the corresponding step-triangles (fig. 2). Corresponding to $A'B'C'$, we shall denote by a', b', $l'_{k,i}$, $d'_{k,i}, \ldots$, the elements similar to a, b, $l_{k,i}$, $d_{k,i}, \ldots$, of the original triangle ABC. Thus suppose that (α_1, β_1) is a segment $l'_{k,i}$ and that (p', q') is the interval $d'_{k,2i-1}$ (so that $q' = \alpha_1$): We consider the value of the ratio

$$\frac{f(x_1)-f(x_0)}{x_1-x_0}$$

for values of x_1 equal to α, β, α_1, β_1, p'. For these values, the function $f(x)$ is given by the ordinates of the points A', B', A_1', B_1', B_2'. The ordinate of C_1' being equal to that of B_2', we have

$$f(x) > f(p')$$

since the point M representing $f(x)$ is inside the triangle $A_1'B_1'C_1'$ We have

(12) $$\frac{f(x_0)-f(\alpha_1)}{x_0-\alpha_1} < 0$$

(13) $$\frac{f(x_0)-f(p')}{x_0-p'} > 0$$

$$\frac{f(x_0)-f(p')}{x_0-p'} - \frac{f(x_0)-f(\alpha_1)}{x_0-\alpha_1} > \frac{f(\alpha_1)-f(p')}{x_0-p'}$$

$$= \frac{\frac{b'}{2^k}}{x_0-p'} > \frac{\frac{b'}{2^k}}{\frac{2a'}{2^k}} = \frac{b'}{2a'} > \frac{b}{2a},$$

since it is easily seen that for the step-triangles of all categories $\frac{b'}{a'}$ is greater than the corresponding ratio for the original step-triangle ABC. Thus

(14) $$\frac{f(x_0)-f(p')}{x_0-p'} - \frac{f(x_0)-f(\alpha_1)}{x_0-\alpha_1} > \frac{b}{2a}.$$

We have further (for the disposition of the triangles given on the figure)

(15) $$\frac{f(\gamma)-f(x_0)}{\gamma-x_0} > 0$$

(16) $$\frac{f(\beta)-f(x_0)}{\beta-x_0} < 0$$

(17) $$\frac{f(\gamma)-f(x_0)}{\gamma-x_0} - \frac{f(\beta)-f(x_0)}{\beta-x_0} > \frac{f(\gamma)-f(\beta)}{\beta-x_0} > \frac{b'}{2a'} > \frac{b}{2a}.$$

It can easily be seen that formulae analogous to (12),..., (17) remain true for any other disposition of the triangles $A'B'C'$ and $A_1'B_1'C_1'$.

From (12),..., (17) we conclude that at the points x_0 the left upper derivative is non-negative, the left lower derivative non-positive, and their difference is greater than $\frac{b}{2a}$, with similar results for the right derivative. Thus at the points x_0 no one-sided derivative exists. Since the x_0 was an arbitrary point belonging to L-segments of all categories, we conclude.

Proposition 4. *At all points belonging to L-segments of all categories no one sided derivative exists.*

Combining propositions 3. and 4., we have the final result

At every point of its interval of definition, the function $f(x)$ has no one-sided derivative.