On Polish spaces Lipschitz universal for separable metric spaces

by

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Abstract. It is proved that a separable complete metric space which is Lipschitz universal for all separable reflexive Banach spaces is Lipschitz universal for all separable metric spaces.

1. Introduction. I. Aharoni [1] proved that the Banach space $C(0, 1)$ can be Lipschitz embedded into the Banach space c_0 (the space of all real-valued sequences converging to zero). This, combined with the Kuratowski–Wojdysławski Theorem [3], implies that the Banach space c_0 is Lipschitz universal for all separable metric spaces [1]. In other words, a separable metric space is Lipschitz universal for all separable metric spaces if and only if it Lipschitz-contains c_0. The aim of this note is to prove that if a complete separable metric space (M, ϱ) is Lipschitz universal for all separable reflexive Banach spaces then (M, ϱ) Lipschitz contains c_0 and therefore is Lipschitz universal for all separable metric spaces. This means that the class of Lipschitz structures of separable reflexive Banach spaces is rich — "it spans" all possible separable Lipschitz structures. But, on the other hand, the Lipschitz structures of reflexive Banach spaces are "specific" — not every metric space can be Lipschitz embedded into a reflexive Banach space. More precisely, it is known [7] that if a Banach space X is Lipschitz embeddable into a reflexive Banach space then X is reflexive as well.

The proof presented in this note is based on H. P. Rosenthal's [9] version of the proof of the following result due to J. Bourgain [4]: A separable Banach space which is isomorphically universal for all separable reflexive Banach spaces is isomorphically universal for all separable Banach spaces.

2. Preliminaries. Let (M, ϱ) and (N, h) be metric spaces. A map $t\colon M \to N$ is said to be a *Lipschitz embedding of M into N* iff there are positive constants K and k such that

$$(1)\qquad k\varrho(x, y) \leqslant h(t(x), t(y)) \leqslant K\varrho(x, y) \quad \text{for all } x, y \in M.$$

We say that a metric space (M, ϱ) is *Lipschitz universal for a class of metric spaces* $\mathcal{N}$ iff every metric space from the class $\mathcal{N}$ is Lipschitz embeddable into (M, ϱ).

Let $\prec$ be an order relation on an arbitrary set T. The order relation $\prec$ is said to be *well founded* iff every subset A of T admits a maximal element in A with respect to the order relation $\prec$ restricted to A. The set of all such elements will be denoted by $\mathrm{Max}(A)$. Set $H_0(T) = T$ and assume that $H_\alpha(T)$ for all ordinals $\alpha < \beta$ have been defined. Then, if β is a limit ordinal, we define $H_\beta(T) = \bigcap_{\alpha<\beta} H_\alpha(T)$. Otherwise, i.e. if $\beta = \alpha_0 + 1$ for some ordinal α_0, we define $H_\beta(T) = H_{\alpha_0}(T) \setminus \mathrm{Max}(H_{\alpha_0}(T))$. Since $\prec$ is well founded we infer that $H_\alpha(T)$'s are strictly decreasing and therefore eventually are empty sets. The smallest α such that $H_\alpha(T)$ is an empty set is called the index of the well-founded ordered set $(T, \prec)$ and will be denoted by $\mathrm{Ind}(T, \prec)$. In the sequel we shall need the following well-known result, cf. [5].

THEOREM 1. (*Kunen–Martin Theorem*). *The index of a well-founded analytic order relation on a separable complete metric space is a countable ordinal.*

Using standard transfinite induction, one can easily prove

PROPOSITION 2. *Let $(T_1, \prec_1)$ and $(T_2, \prec_2)$ be well-founded ordered sets and let $\varphi\colon T_1 \to T_2$ be an order-preserving mapping. Then for each ordinal α we have*

$$\varphi(H_\alpha(T_1)) \subset H_\alpha(T_2)$$

and consequently $\mathrm{Ind}(T_1, \prec_1) \leqslant \mathrm{Ind}(T_2, \prec_2)$.

The smallest uncountable ordinal will be denoted by ω_1.

3. Lipschitz c_0-indices of separable complete metric spaces. By c_0 we shall mean the Banach space of all real-valued sequences converging to zero endowed with its standard supremum norm. For every $n \in N$ we set

$$A_n = \{(\lambda_i)_{i=1}^{\infty} \in c_0\colon |\lambda_i| \leqslant 2^n \text{ for } 1 \leqslant i \leqslant 2^n \text{ and } \lambda_i = 0 \text{ otherwise}\}.$$

For every separable complete metric space (M, ϱ) and every $n, m \in N$ we define $T_{n,m}(M, \varrho)$ to be the set of all Lipschitz mappings $t\colon A_n \to M$ satisfying

$$\|a-b\| \leqslant \varrho(t(a), t(b)) \leqslant m\|a-b\| \quad \text{for} \quad a, b \in A_n \tag{2}$$

endowed with the sup-metric. It is clear that for every $n, m \in N$, the space $T_{n,m}(M, \varrho)$ is a complete separable metric space. For every $m \in N$, let

$$T_m(M, \varrho) = \bigoplus_{n=1}^{\infty} T_{n,m}(M, \varrho),$$

be the discrete union of the corresponding $T_{n,m}$'s. Obviously, $T_m(M, \varrho)$ is a separable complete-metrizable space. Now, for every $m \in N$ define the order relation $\prec_m$ on $T_m(M, \varrho)$ in the following way: for $t_1, t_2 \in T_m(M, \varrho)$ we say that $t_1 \prec_m t_2$ iff $t_1 \in T_{n_1,m}(M, \varrho)$, $t_2 \in T_{n_2,m}(M, \varrho)$ with $n_1 \leqslant n_2$ and t_1 is equal to t_2 restricted to A_{n_1}. Trivially, $\prec_m$ is a closed relation. In the sequel we shall need

PROPOSITION 3. *A separable complete metric space (M, ϱ) Lipschitz contains c_0 if and only if for some $m \in N$ the relation $\prec_m$ on $T_m(M, \varrho)$ is not well founded.*

Proof. Let f be a Lipschitz embedding of c_0 into a separable complete metric space (M, ϱ) such that (1) holds for some $0 < k \leqslant K$. Then $t(x) = f(k^{-1}x)$ is a Lipschitz embedding of c_0 into (M, ϱ) satisfying (2) for every positive integer $m \geqslant k^{-1}K$. Fix such an m and set $t_n = t|A_n$ for $n \in N$. Since $t_n \in T_{n,m}(M, \varrho)$ for every $n \in N$ and $t_{n_1} \prec_m t_{n_2}$ for every $n_1 \leqslant n_2$, we infer that the set $\{t_n\colon n \in N\} \subset T_m(M, \varrho)$ has no maximal element. Thus $(T_m(M, \varrho), \prec_m)$ is not well founded.

Conversely, if for some $m \in N$ the ordered set $(T_m(M, \varrho), \prec_m)$ is not well founded, then there is a sequence $(t_n)_{n=1}^{\infty} \subset T_m(M, \varrho)$ such that $t_{n_1} \prec_m t_{n_2}$ for every $n_1 \leqslant n_2$. Setting $\tilde{t} = \lim_{n\to\infty} t_n$, we obtain a Lipschitz embedding of $\bigcup_{n=1}^{\infty} A_n \subset c_0$ into (M, ϱ) satisfying (2). Since $\bigcup_{n=1}^{\infty} A_n$ is dense in c_0, the map $\tilde{t}$ admits a unique extension to a Lipschitz embedding t of c_0 into (M, ϱ) satisfying (2), which completes the proof.

Note that for every separable complete metric space (M, ϱ), if $m_1 < m_2$ then $T_{m_1}(M, \varrho) \subset T_{m_2}(M, \varrho)$ and $\prec_{m_1}$ is equal to $\prec_{m_2}$ restricted to $T_{m_1}(M, \varrho)$. Thus, by Propositions 2 and 3, if (M, ϱ) does not contain c_0, then

$$\mathrm{Ind}(T_{m_1}(M, \varrho), \prec_{m_1}) \leqslant \mathrm{Ind}(T_{m_2}(M, \varrho), \prec_{m_2}),$$

and therefore $\lim_{n\to\infty} \mathrm{Ind}\, T_n(M, \varrho, \prec_n)$ exists.

For every separable complete metric space (M, ϱ) which does not Lipschitz contain c_0, we define

$$\mathrm{ind}(M, \varrho) = \lim_{n\to\infty} \mathrm{Ind}(T_n(M, \varrho), \prec_n).$$

If such a space Lipschitz contains c_0, we set

$$\mathrm{ind}(M, \varrho) = \omega_1.$$

PROPOSITION 4. *The function* ind *defined on the class of all separable complete metric spaces has the following properties*:

(i) $\mathrm{ind}(M, \varrho) \leqslant \omega_1$ *for every separable complete metric space* (M, ϱ),

(ii) $\mathrm{ind}(M, \varrho) = \omega_1$ *if and only if* (M, ϱ) *Lipschitz contains* c_0,

(iii) *if there is a Lipschitz embedding f of (M, ϱ) into (N, h) such that*

$$\varrho(x, y) \leqslant h(f(x), f(y)) \leqslant K_f \varrho(x, y) \quad \textit{for} \quad x, y \in M, \tag{3}$$

then

$$\operatorname{ind}(M, \varrho) \leqslant \operatorname{ind}(N, h).$$

Proof. (i), (ii) If (M, ϱ) Lipschitz contains c_0, then, by definition, $\operatorname{ind}(M, \varrho) = \omega_1$. If (M, ϱ) does not contain c_0, then for each $m \in N$, the order relation $\prec_m$ on $T_m(M, \varrho)$ is, by Proposition 3, well founded. Since $\prec_m$ is closed, by the Kunen–Martin Theorem we conclude that $\operatorname{Ind}(T_m(M, \varrho), \prec_m) < \omega_1$ for every $m \in N$, and therefore $\lim_{m\to\infty} \operatorname{Ind}(T_m(M, \varrho), \prec_m) < \omega_1$.

(iii) If (N, h) Lipschitz contains c_0, then (iii) follows from (i) and (ii). Assume that (N, h) does not Lipschitz contain c_0 and fix a positive integer $m_0 \geqslant K_f$. Observe that, for every $m \in N$, the map $\varphi_m\colon T_m(M, \varrho) \to T_{mm_0}(N, h)$ defined by $\varphi_m(t) = f \circ t$ is order preserving. Thus, by Proposition 2,

$$\operatorname{Ind}(T_m(M, \varrho), \prec_m) \leqslant \operatorname{Ind}(T_{mm_0}(N, h), \prec_{mm_0}).$$

Consequently,

$$\operatorname{ind}(M, \varrho) = \lim_{m\to\infty} \operatorname{Ind}(T_m(M, \varrho), \prec_m) \leqslant \lim_{m\to\infty} \operatorname{Ind}(T_{mm_0}(N, h), \prec_{mm_0}) = \operatorname{ind}(N, h).$$

4. Construction of separable reflexive Banach space with large Lipschitz c_0-indices. Since c_0 is not Lipschitz embeddable into a reflexive Banach space [7], by Proposition 4 (ii) we infer that $\operatorname{ind}(R, \|\ \|) < \omega_1$ for every separable reflexive Banach space $(R, \|\ \|)$. In the proof of the lemma below we shall give, for every countable ordinal α, a construction of a separable reflexive Banach space $(R_\alpha, \|\ \|_\alpha)$ with $\operatorname{ind}(R_\alpha, \|\ \|_\alpha) > \alpha$. To this end, let $e_1 = (1, 0, 0, \ldots) \in c_0$ and let, for a Banach space $(R, \|\ \|)$, let

(4) $$S_n(R) = \{t \in T_{n,1}(R, \|\ \|)\colon t(\lambda e_1) = \lambda t(e_1) \text{ for } -1 \leqslant \lambda \leqslant 1\}$$

(i.e. let $S_n(R)$ be the set of isometric embeddings of A_n into $(R, \|\ \|)$ which are "linear" on the interval $[-e_1, e_1] \subset c_0$) and $S(R) = \bigcup_{n=1}^{\infty} S_n(R)$. Finally, let $\prec$ be the restriction of $\prec_1$ to the set $S(R)$.

Lemma 5. *For every $\alpha < \omega_1$ there is a separable reflexive Banach space $(R_\alpha, \|\ \|_\alpha)$ with $\operatorname{Ind}(S(R_\alpha), \prec) > \alpha$.*

Proof (by transfinite induction). Set $(R_1, \|\ \|_1) = (\boldsymbol{R}, \|\ \|)$ and assume that for every $\gamma < \alpha$ a separable reflexive Banach space $(R_\gamma, \|\ \|_\gamma)$ with $\operatorname{Ind}(S(R_\gamma), \prec) > \gamma$ has been defined.

Case 1°. α has a predecesor; i.e. $\alpha = \beta + 1$ for some β. Define $R_\alpha = R_\beta \times R_\beta$ and $\|(x, y)\|_\alpha = \max\{\|x\|_\beta, \|y\|_\beta\}$ for $(x, y) \in R_\beta \times R_\beta$. It is clear that $(R_\alpha, \|.\|_\alpha)$ is a separable reflexive Banach space. Since $\operatorname{Ind}(S(R_\alpha), \prec) > \beta$, by definition, we have $H_\beta(S(R_\beta)) \neq \emptyset$. Let $t_0 \in H_\beta(S(R_\beta))$. It follows from the definition of $\prec$ that $t_1 = t_0|A_1 \in H_\beta(S(R_\beta))$. Now, for $n = 1, 2, \ldots$ define $\varphi_n\colon S_n(R_\beta) \to S_{n+1}(R_\alpha)$ by the formulae

$$\varphi_n(t)(a) = \left(2t(\tfrac{1}{2}a_1, \tfrac{1}{2}a_3, \tfrac{1}{2}a_5, \ldots), 2t(\tfrac{1}{2}a_2, \tfrac{1}{2}a_4, \tfrac{1}{2}a_6, \ldots)\right),$$

for $a = (a_1, a_2, \ldots) \in A_{n+1}$ and $t \in S_n(R_\beta)$. It can easily be checked that the mapping φ defined by $\varphi(t) = \varphi_n(t)$ for $t \in S_n(R_\beta)$ is an order-preserving mapping from $S(R_\beta)$ into $S(R_\alpha)$. Thus, by Proposition 2, $\varphi(H_\beta(S(R_\beta))) \subset H_\beta(S(R_\alpha))$ and therefore $\tilde{t} = \varphi(t_1) \in H_\beta(S(R_\alpha))$. But $\varphi(t_1) \in S_2(R_\alpha)$. Hence $\tilde{t}_1 = \tilde{t}|A_1 \prec \tilde{t}$ and this implies that $\tilde{t}_1 \in H_{\beta+1}(S(R_\alpha))$. Thus $H_\alpha(S(R_\alpha)) = H_{\beta+1}(S(R_\alpha)) \neq \emptyset$ and therefore $\operatorname{Ind}(S(R_\alpha), \prec) > \alpha$, which completes the proof in case 1°.

Case 2°. α is a limit ordinal. By the induction hypothesis, $H_\gamma(S(R_\gamma)) \neq \emptyset$ for every $\gamma < \alpha$. For every $\gamma < \alpha$ select $\tilde{t}_\gamma \in H_\gamma(S(R_\gamma))$ and set $t_\gamma = \tilde{t}_\gamma|A_1$. Obviously, $t_\gamma \in H_\gamma(S(R_\gamma))$ for every $\gamma < \alpha$. By (4), for every $\gamma < \alpha$ there is an $x_\gamma \in R_\gamma$ with $\|x_\gamma\|_\gamma = 1$ such that $t_\gamma(\lambda e_1) = \lambda x_\gamma$ for $-1 \leqslant \lambda \leqslant 1$. Let

$$R_\alpha = \{(\mu(x_\gamma)_{\gamma<\alpha} + (y_\gamma)_{\gamma<\alpha}) \in \prod_{\gamma<\alpha} R_\gamma\colon \mu \in \boldsymbol{R},\ y_\gamma \in R_\gamma \text{ and } \sum_{\gamma<\alpha} \|y\|_\gamma^2 < \infty\}$$

equipped with the norm

$$\|\mu(x_\gamma)_{\gamma<\alpha} + (y_\gamma)_{\gamma<\alpha}\|_\alpha = \max\{\sup_{\gamma<\alpha} \|\mu x_\gamma + y_\gamma\|_\gamma, \tfrac{1}{2}(\sum_{\gamma<\alpha} \|y_\gamma\|_\gamma^2)^{1/2}\}.$$

Since $(R_\alpha, \|\ \|_\alpha)$ is a one-dimensional enlargement of the l_2-sum of R_γ's it is obvious that $(R_\alpha, \|\ \|_\alpha)$ is a separable reflexive Banach space. Fix an arbitrary $\beta < \alpha$ and, for $n \in N$, define

(5) $$\varphi_n(t)(a) = (a_1(x_\gamma)_{\gamma<\alpha} + t(a) - a_1 x_\beta)$$

for $t \in S_n(R_\beta)$ and $a = (a_1, a_2, \ldots) \in A_n$. Note that, for $n \in N$, $t \in S_n(R_\beta)$ and $a, b \in A_n$ with $a = (a_1, a_2, \ldots)$, $b = (b_1, b_2, \ldots)$ we have

$$\|a_1 x_\gamma - b_1 x_\gamma\|_\gamma = |a_1 - b_1| \leqslant \|a - b\|$$

for $\gamma \neq \beta$, $\gamma < \alpha$, and

$$\|(a_1 x_\beta + t(a) - a_1 x_\beta) - (b_1 x_\beta + t(b) - b_1 x_\beta)\|_\beta = \|t(a) - t(b)\|_\beta = \|a - b\|,$$

and

$$\|(t(a) - a_1 x_\beta) - (t(b) - b_1 x_\beta)\|_\beta \leqslant \|t(a) - t(b)\|_\beta + |a_1 - b_1| \leqslant 2\|a - b\|.$$

Hence

(6) $$\|\varphi_n(t)(a) - \varphi_n(t)(b)\|_\alpha = \|a - b\|$$

for $a, b \in A_n$, $n \in N$ and $t \in S_n(R_\beta)$. On the other hand, it follows from (4) and (5) that $\varphi_n(t)$ restricted to the interval $[-e_1, e_1]$ is linear. This and (6) imply that φ_n maps $S_n(R_\beta)$ into $S_n(R_\alpha)$ for $n \in N$. Define $\varphi(t) = \varphi_n(t)$ for $t \in S_n(R_\beta)$ and $n \in N$. It easily follows from (5) that φ is an order-preserving mapping from $(S(R_\beta), \prec)$ into $(S(R_\alpha), \prec)$. By Proposition 2, we infer that $\varphi(t_\beta) \in H_\beta(R_\alpha)$. But

$$\varphi(t_\beta)(\lambda e_1) = \lambda\varphi(t_\beta)(e_1) = \lambda(x_\gamma)_{\gamma<\alpha} \quad \text{for} \quad -1 \leqslant \lambda \leqslant 1.$$

Since β was an arbitrary ordinal smaller than α, we conclude that for each $\beta < \alpha$ the map $\tilde{t}$ defined by

$$\tilde{t}(\lambda e_1) = \lambda(x_\gamma)_{\gamma<\alpha} \quad \text{for} \quad -1 \leqslant \lambda \leqslant 1,$$

belongs to the set $H_\beta(S(R_\alpha))$ and therefore $\tilde{t} \in \bigcap_{\beta<\alpha} H_\beta(S(R_\alpha)) = H_\alpha(S(R_\alpha))$. Thus $\operatorname{Ind}(S(R_\alpha), \prec) > \alpha$, which completes the proof.

Lemma 5 yields

PROPOSITION 6. *For each $\alpha < \omega_1$ there is a separable reflexive Banach space $(R_\alpha, \| \|_\alpha)$ with $\operatorname{ind}(R_\alpha, \| \|_\alpha) > \alpha$.*

Proof. For each $\alpha < \omega_1$, let $(R_\alpha, \| \|_\alpha)$ be the space constructed in Lemma 5. Since $S(R_\alpha) \subset T_1(R_\alpha, \| \|_\alpha)$ and $\prec$ is the restriction of $\prec_1$ to $S(R_\alpha)$, by Proposition 2, we have

$$\alpha < \operatorname{Ind}(S(R_\alpha), \prec) \leqslant \operatorname{Ind}(T_1(R_\alpha, \| \|_\alpha), \prec_1).$$

On the other hand, $\operatorname{Ind}(T_m(R_\alpha, \| \|_\alpha, \prec_m)$ is a nondecreasing function of m. Thus

$$\alpha < \operatorname{Ind}(T_1(R_\alpha, \| \|_\alpha, \prec_1) \leqslant \lim_{m\to\infty} \operatorname{Ind}(T_m(R_\alpha, \| \|_\alpha, \prec_m) = \operatorname{ind}(R_\alpha, \| \|_\alpha).$$

5. Main result and open problems. The main result of this note is the following

THEOREM 7. *For a separable complete metric space (M, ϱ) the following conditions are equivalent:*

(i) (M, ϱ) Lipschitz contains c_0,

(ii) (M, ϱ) is Lipschitz universal for all separable metric spaces,

(iii) (M, ϱ) is Lipschitz universal for all separable reflexive Banach spaces.

Proof. The implication (i)$\Rightarrow$(ii) is due to Aharoni [1] and the implication (ii)$\Rightarrow$(iii) is trivial. In order to prove that the implication (iii)$\Rightarrow$(i) holds, let (M, ϱ) be a separable complete metric space Lipschitz universal for all separable reflexive Banach spaces. Fix an arbitrary $\alpha < \omega_1$ and let t be the Lipschitz embedding of $(R_\alpha, \| \|_\alpha)$ from Proposition 6 into (M, ϱ) satisfying (1) for some $0 < k \leqslant K$. Set $f(x) = t(k^{-1}x)$ for $x \in R_\alpha$. Obviously, f satisfies (3) with $K_f = k^{-1}K$. By Proposition 4 (iii),

$$\alpha < \operatorname{ind}(R_\alpha, \| \|_\alpha) \leqslant \operatorname{ind}(M, \varrho).$$

Since α was an arbitrary countable ordinal, by Proposition 4 (i), $\operatorname{ind}(M, \varrho)$

$= \omega_1$. Finally, by Proposition 4 (ii), the space (M, ϱ) Lipschitz contains c_0, which completes the proof.

Remark. It follows from Aharoni's result that if (M, ϱ) Lipschitz contains c_0 then there is a universal constant K_M such that for every separable metric space (N, h) there is a Lipschitz embedding f of N into M with

$$h(x, y) \leqslant \varrho(f(x), f(y)) \leqslant K_M h(x, y) \quad \text{for all } x, y \in N.$$

It follows from the argument presented above and from the construction of H. P. Rosenthal [9] that the following version of the Kuratowski–Wojdysławski Theorem holds:

THEOREM 8. *For a separable complete metric space (M, ϱ) the following conditions are equivalent:*

(i) (M, ϱ) contains $C(0, 1)$ isometrically,

(ii) (M, ϱ) is isometrically universal for all separable metric spaces,

(iii) (M, ϱ) is isometrically universal for all separable reflexive Banach spaces.

It would be interesting to know whether the corresponding version of Theorems 7 and 8 holds for uniform embeddings. More precisely:

PROBLEM 1. Let a separable complete metric space (M, ϱ) be uniformly universal for all separable reflexive Banach spaces (i.e. for each separable reflexive Banach space $(R, \| \|)$ let there be an embedding f: $R \to M$ such that both f and f^{-1} are uniformly continuous). Is (M, ϱ) uniformly universal for all separable metric spaces?

On the other hand, from the point of view of the Banach space theory it would be nice to know the answer to the following

PROBLEM 2. Let a separable Banach space $(X, \| \|)$ be Lipschitz universal for all separable reflexive Banach spaces. Does $(X, \| \|)$ contain c_0 isomorphically?

In view of Theorem 7, Problem 2 reduces to the question whether every separable Banach space $(X, \| \|)$ Lipschitz containing c_0 contains c_0 isomorphically. It follows from Ribe's theorem [8] (see also [6], Th. 5.1) that in such a case X contains isomorphically l_∞^n's uniformly. On the other hand, it is known, [6], Th. 3.5, that if a conjugate Banach space $(Y^*, \| \|)$ Lipschitz contains c_0 then Y^* contains c_0 isomorphically and therefore, by [2], it contains l_∞.

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Recursion theoretic operators and morphims on numbered sets *

by

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Dedicated to Buffee Lys Nelson on her twelfth birthday

Abstract. An operator is a map $\Phi: P\omega \to P\omega$. By embedding $P\omega$ in two natural ways into the λ-calculus model $P\omega^2$ (and T^ω) the computable maps on this latter structure induce classes of recursion operators.

§ 0. Introduction. With the notion of (pre complete) numbered set Ershov [3] gave a general framework for certain results in classical recursion theory. In his theory the notion of morphism is central. In [6] there is a definition of enumeration operators and (implicitly) of Turing operators. Although enumeration operators (restricted to the r.e. sets as numbered set) are morphisms, Turing operators are not even partial morphisms.

There is a natural correspondence between these (and other) classes of recursion theoretic operators and morphisms on an appropriate numbered set, via the constructive part of the λ-calculus models $P\omega^2$ and T^ω. The different classes of operators on $P\omega$ are effective continuous maps obtained by embedding $P\omega$ into $P\omega^2$ or T^ω in two natural ways, giving $P\omega$ either the Cantor or the Scott topology.

In particular Turing operators work on $P\omega$ with the Cantor topology. This is implicit in Nerode's theorem, see [6], p. 154, relating tt-reducibility to total Turing operators. Also a different proof will be given of a theorem in [6], p. 151, relating enumeration and Turing reducibility. Finally an interpolation result, in the sense of algebra, will be proved for total Turing operators.

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