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Confluent local expansions

by

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Abstract. Every confluent local expansion of an arcwise connected continuum onto itself is open.

In paper [6] Professor I. Rosenholtz has proved that every open local expansion of a continuum onto itself has a fixed point. The following question is asked in [2], Problem 3.1:

(∗) Does there exist a confluent local expansion of a continuum onto itself which is fixed point free?

This paper gives a partial answer to this question: every confluent local expansion of an arcwise connected continuum onto itself is open, and so it has a fixed point. This will follow from two results proved for locally one-to-one mappings. An example shows that this method cannot be extended to continua which are not arcwise connected.

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A continuum means a compact connected metric space. Let X and Y be metric spaces with metrics d_X and d_Y respectively. A continuous surjection $f\colon X \to Y$ is said to be

– a *local expansion* if for each $x \in X$ there is a neighbourhood U of x and a number $M > 1$ such that

$$d_Y(f(y), f(z)) \geqslant M \cdot d_X(y, z) \quad \text{for} \quad y, z \in U$$

(cf. [6]),

– *open* if the image of any open set in X is open in Y,

– *confluent* if for every continuum $Q \subset Y$ and for every component C of $f^{-1}(Q)$ we have $f(C) = Q$,

– *locally one-to-one* if for each point $x \in X$ there is an open neighbourhood U of x such that the restriction $f|_U$ is one-to-one.

Let us recall (see [1], VI, p. 214) that

(i) any open mapping of a compact space is confluent.

We will need the following very useful characterization of open mappings (cf. [5], 1.2, p. 101):

(ii) Let X, Y be compact metric spaces and $f\colon X \to Y$ a mapping of X onto Y. Then f is open if and only if

$$\mathrm{Ls}\big(f^{-1}(y_n)\big) = f^{-1}(y) \quad \text{for each } y \in Y \text{ and } y_n \to y.$$

(Ls – denotes the upper topological limit; for a definition and properties see [3], § 29, III–V, p. 337–339.)

The remark below gives a negative answer to the question (*) for confluent local expansions of locally connected continua.

Recall that any confluent mapping of a locally connected continuum is quasi-monotone (see [1], IX, p. 215) and any quasi-monotone and light (i.e. 0-dimensional) mapping of a locally connected continuum is open (cf. [7], Theorem 8.2, p. 152). Since obviously any local expansion is light, we conclude that any confluent local expansion of a locally connected continuum onto itself is open, and so it has a fixed point (cf. [6]).

Now we try to extend this result to confluent local expansions of arcwise connected continua.

Given a mapping $f\colon X \to Y$, we define a mapping k from Y to cardinal numbers (called the degree of f) as follows:

$$k(y) = \mathrm{card}\, f^{-1}(y) \quad \text{for} \quad y \in Y$$

(cf. [7], p. 199).

(iii) Let f be a locally one-to-one mapping of a compact space X into Y. Then k is bounded by a natural number.

Indeed, take a covering $\{U_x;\ x \in X\}$ of X where U_x denotes an open neighbourhood of x such that the restriction $f|_{U_x}$ is one-to-one. By the compactness of X there is its finite subcovering $\{U_{x_1}, U_{x_2}, \ldots, U_{x_n}\}$ which leads to the inequality

$$k(y) \leqslant \mathrm{card}\,\{U_{x_1}, U_{x_2}, \ldots, U_{x_n}\} \quad \text{for each } y \in Y.$$

It is known that for any local homeomorphism (locally one-to-one open mapping) of an arcwise connected continuum the degree is constant (cf. [7], 6.1, p. 199). We extend this statement to locally one-to-one confluent mappings (by another method of proof).

THEOREM 1. *Let X, Y be compact metric spaces with metrics d_X, d_Y respectively and f a locally one-to-one mapping of X onto Y. If f is confluent on $f^{-1}(Z)$, where Z is any arcwise connected subset of Y, then k is constant on Z.*

Proof. Let δ_L denote the Lebesgue coefficient of the covering $\{U_x;\ x \in X,\ U_x\text{-open neighbourhood of } x, f|_{U_x}\text{-one-to-one}\}$ of X (cf. [4], p. 24). In other words, for every $A \subset X$, if $\mathrm{diam}_X A < \delta_L$, then $f|_A$ is one-to-one.

The set $D = \{(x_1, x_2) \in X \times X;\ \frac{1}{2}\delta_L \leqslant d_X(x_1, x_2) \leqslant \frac{2}{3}\delta_L\}$ is closed, and hence it is a compact subset of the product $X \times X$. Since the composition $d_Y \circ (f \times f)$ is continuous and positive on D, there exists such a number $\delta > 0$ that

$$d_Y\big(f(x_1), f(x_2)\big) > \delta \quad \text{for each } (x_1, x_2) \in D.$$

Now suppose on the contrary that the theorem does not hold, i.e. that there exist points $p, q \in Z$ such that $k(p) \neq k(q)$. Let pq be an arc joining p and q in Z.

There exists an arc $ab \subset pq$ such that

1° $k(a) \neq k(b)$ and

2° $\mathrm{diam}_Y\, ab < \delta$

(taking a sequence of successive points $p_0 = p, p_1, p_2, \ldots, p_n = q$ on the arc pq such that the diameter of $p_i p_{i+1} \subset pq$ is less than δ, one can find a pair of adjacent ones on which k is not constant).

Let C denote a component of $f^{-1}(ab)$. Obviously $f(C) = ab$ because f is confluent.

Observe that $\mathrm{diam}_X C < \frac{1}{2}\delta_L$. Really, otherwise there would be points $x, \hat{x} \in C$ such that $\frac{1}{2}\delta_L \leqslant d_X(x, \hat{x}) \leqslant \frac{2}{3}\delta_L$ (C is connected). Since $(x, \hat{x}) \in D$, we would have $\mathrm{diam}_Y\, ab \geqslant d_Y\big(f(x), f(\hat{x})\big) > \delta$, which contradicts 2°.

Then the restriction $f|_C\colon C \to ab$ is one-to-one and onto. Thus $k(a) = \mathrm{card}\,\{C\colon C\text{-component of } f^{-1}(ab)\} = k(b)$, which contradicts our assumption.

Observe that if we replace "locally one-to-one" by "light", the conclusion does not hold. As an example consider the projection of the harmonic fan onto its limit segment. Also the assumption that Z is arcwise connected is essential. It can be seen by the following

EXAMPLE. Let $X = Y = A \cup B$ be subspaces of the Euclidean plane, where

$$A = \{e^{it} \in R^2\colon t \in [0, 2\pi)\},$$
$$B = \left\{\frac{2+t}{1+t} e^{it} \in R^2\colon t \in [0, +\infty)\right\}.$$

Obviously X and Y are continua (not arcwise connected). Let $f\colon X \to Y$ be defined by

$$f(z) = \begin{cases} e^{2it} & \text{for} \quad z = e^{it} \in A, \\ \dfrac{2+2t}{1+2t} e^{2it} & \text{for} \quad z = \dfrac{2+t}{1+t} e^{it} \in B. \end{cases}$$

The mapping f is locally one-to-one and confluent on X but

$$k(y) = \begin{cases} 1 & \text{for} \quad y \in B, \\ 2 & \text{for} \quad y \in A. \end{cases}$$

Now to show the main result of the paper it is enough to prove the following

THEOREM 2. *Let X and Y be compact metric spaces and $f\colon X\to Y$ a locally one-to-one surjection such that the degree k of f is constant on Y. Then f is open (thus f is a local homeomorphism).*

Proof. To show the openness of f we use (ii). Let $y\in Y$ and $y_n\to y$ be arbitrary.

By the continuity of f we have $\operatorname{Ls} f^{-1}(y_n)\subset f^{-1}(y)$ (cf. [4], Theorem 1, p. 61). By (iii) there exists such a natural number s that $k(y)=s$ for each $y\in Y$.

To finish the proof it is enough to show that

$$\operatorname{card} f^{-1}(y)=\operatorname{card}\operatorname{Ls} f^{-1}(y_n).$$

If we denote $f^{-1}(y_n)=\{x_n^1, x_n^2,\ldots,x_n^s\}$, we have $x_n^i\neq x_n^j$ for $i\neq j$ and for $n=1, 2,\ldots$

Let us consider the sequence $\{x_n^1\}_{n=1}^{\infty}$. By the compactness of X there exist such a subsequence $\{x_{n_m}^1\}_{m=1}^{\infty}$ and a point x^1 that $x_{n_m}^1 \xrightarrow[m\to\infty]{} x^1$. Now, let us consider the sequence $\{x_{n_m}^2\}_{m=1}^{\infty}$; we may choose such a subsequence $\{x_{n_{m_l}}^2\}_{l=1}^{\infty}$ of it and such a point x^2 that $x_{n_{m_l}}^2 \xrightarrow[l\to\infty]{} x^2$. Further, by choosing consecutive subsequences of previously taken ones (and continuing this process up to the index s) we obtain at last such a subsequence $\{y_{n_k}\}_{k=1}^{\infty}$ of the sequence $\{y_n\}_{n=1}^{\infty}$ that

$$x_{n_k}^1 \xrightarrow[k\to\infty]{} x^1,\quad x_{n_k}^2 \xrightarrow[k\to\infty]{} x^2,\quad \ldots,\quad x_{n_k}^s \xrightarrow[k\to\infty]{} x^s.$$

By f being locally one-to-one we have $x^i\neq x^j$ for $i\neq j$ (otherwise there would be an open neighbourhood U of $x^i=x^j$ such that $f|_U$ is one-to-one; in U one could find two different points $x_{n_{k_0}}^i$, $x_{n_{k_0}}^j$ which are mapped onto $y_{n_{k_0}}$).

Then the following inequalities hold:

$$s=\operatorname{card}\{x^1, x^2,\ldots,x^s\}\leqslant \operatorname{card}\operatorname*{Ls}_p f^{-1}(y_{n_p}) \leqslant \operatorname{card}\operatorname*{Ls}_n f^{-1}(y_n)\leqslant \operatorname{card} f^{-1}(y)=s;$$

hence the theorem is proved.

COROLLARY 1. *Let $f\colon X\to Y$ be any locally one-to-one mapping of a compact metric space X onto an arcwise connected metric space Y. Then f is open if and only if f is confluent.*

The necessity holds by (ii). Theorems 1 and 2 give the sufficiency.

As an application to local expansions we have

COROLLARY 2. *Any confluent local expansion of an arcwise connected continuum onto itself is open, and so it has a fixed point.*

(The existence of a fixed point of an open local expansion has been proved by I. Rosenholtz in [6].)

To give a complete answer to question (*) one ought to consider local expansions on non arcwise connected continua.

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