Confluent local expansions

by

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Abstract. Every confluent local expansion of an arcwise connected continuum onto itself is open.

In paper [6] Professor I. Rosenholtz has proved that every open local expansion of a continuum onto itself has a fixed point. The following question is asked in [2], Problem 3.1:

(*) Does there exist a confluent local expansion of a continuum onto itself which is fixed point free?

This paper gives a partial answer to this question: every confluent local expansion of an arcwise connected continuum onto itself is open, and so it has a fixed point. This will follow from two results proved for locally one-to-one mappings. An example shows that this method cannot be extended to continua which are not arcwise connected.

The author wishes to thank Professor J. J. Charatonik for his valuable suggestions and discussions on the subject of this paper.

A continuum means a compact connected metric space. Let $X$ and $Y$ be metric spaces with metrics $d_X$ and $d_Y$ respectively. A continuous surjection $f: X \to Y$ is said to be

- a local expansion if for each $x \in X$ there is a neighbourhood $U$ of $x$ and a number $M > 1$ such that

$$d_Y(f(y), f(z)) \geq M \cdot d_X(y, z) \quad \text{for} \quad y, z \in U$$

(cf. [6]),

- open if the image of any open set in $X$ is open in $Y$,

- confluent if for every continuum $Q \subset Y$ and for every component $C$ of $f^{-1}(Q)$ we have $f(C) = Q$,

- locally one-to-one if for each point $x \in X$ there is an open neighbourhood $U$ of $x$ such that the restriction $f|_U$ is one-to-one.

Let us recall (see [1], VI, p. 214) that

(i) any open mapping of a compact space is confluent.

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We will need the following very useful characterization of open mappings (cf. [5], I, 2, p. 101):

(ii) Let \( X, Y \) be compact metric spaces and \( f: X \to Y \) a mapping of \( X \) onto \( Y \). Then \( f \) is open if and only if

\[
Ls(f^{-1}(y)) = f^{-1}(y) \quad \text{for each } y \in Y \text{ and } y_n \to y.
\]

\((Ls = \text{denotes the upper topological limit}; \text{for a definition and properties see [3], § 29, III–V, p. 337–339.})\)

The remark below gives a negative answer to the question (*) for confluent local expansions of locally connected continua.

Recall that any confluent mapping of a locally connected continuum is quasi-monotone (see [7], IX, p. 215) and any quasi-monotone and light (i.e. \( 0 \)-dimensional) mapping of a locally connected continuum is open (cf. [7], Theorem 8.2, p. 152). Since obviously any local expansion is light, we conclude that any confluent local expansion of a locally connected continuum onto itself is open, and so it has a fixed point (cf. [6]).

Now we try to extend this result to confluent local expansions of arcwise connected continua.

Given a mapping \( f: X \to Y \), we define a mapping \( k \) from \( Y \) to cardinal numbers (called the degree of \( f \)) as follows:

\[
k(y) = \text{card } f^{-1}(y) \quad \text{for } y \in Y
\]

(cf. [7], p. 199).

(iii) Let \( f \) be a locally one-to-one mapping of a compact space \( X \) into \( Y \). Then \( k \) is bounded by a natural number.

Indeed, take a covering \( \{U_x: x \in X\}\) of \( X \) where \( U_x \) denotes an open neighbourhood of \( x \) such that the restriction \( f|_{U_x} \) is one-to-one. By the compactness of \( X \) there is its finite subcovering \( \{U_{x_1}, U_{x_2}, \ldots, U_{x_n}\} \) which leads to the inequality

\[
k(y) \leq \text{card } \{U_{x_1}, U_{x_2}, \ldots, U_{x_n}\} \quad \text{for each } y \in Y.
\]

It is known that for any local homeomorphism (locally one-to-one open mapping) of an arcwise connected continuum the degree is constant (cf. [7], 6.1, p. 199). We extend this statement to locally one-to-one confluent mappings (by another method of proof).

**Theorem 1.** Let \( X, Y \) be compact metric spaces with metrics \( d_x, d_Y \) respectively and \( f \) a locally one-to-one mapping of \( X \) onto \( Y \). If \( f \) is confluent on \( f^{-1}(Z) \), where \( Z \) is any arcwise connected subset of \( Y \), then \( k \) is constant on \( Z \).

**Proof.** Let \( \delta_z \) denote the Lebesgue coefficient of the covering \( \{U_x: x \in X, U_x \text{-open neighbourhood of } x, f|_{U_x} \text{-one-to-one}\} \) of \( X \) (cf. [4], p. 24). In other words, for every \( A \subset X \), if \( \text{diam}_X A < \delta_z \), then \( f(A) \) is one-to-one.

The set \( D = \{(x_1, x_2) \in X \times X; \delta_{x_1} \leq d_x(x_1, x_2) \leq \delta_{x_2}\} \) is closed, and hence it is a compact subset of the product \( X \times X \). Since the composition \( d_Y \circ (f \times f) \) is continuous and positive on \( D \), there exists such a number \( \delta > 0 \) that

\[
d_Y(f(x_1), f(x_2)) > \delta \quad \text{for each } (x_1, x_2) \in D.
\]

Now suppose on the contrary that the theorem does not hold, i.e. that there exist points \( p, q \in Z \) such that \( k(p) \neq k(q) \). Let \( pq \) be an arc joining \( p \) and \( q \) in \( Z \).

There exists an arc \( ab = pq \) such that \( 1^o \, k(a) \neq k(b) \) and

\[
2^o \, \text{diam}_A ab < \delta
\]

(taking a sequence of successive points \( p_0 = p, p_1, p_2, \ldots, p_k = q \) on the arc \( pq \) such that the diameter of \( p_i p_{i+1} = pq \) is less than \( \delta \), one can find a pair of adjacent ones on which \( k \) is not constant).

Let \( C \) denote a component of \( f^{-1}(ab) \). Obviously \( f(C) = ab \) because \( f \) is confluent.

Observe that \( \text{diam}_A C < \frac{1}{2} \delta_z \). Really, otherwise there would be points \( x, x \in C \) such that \( \frac{1}{2} \delta_z \leq d_y(x, x) \leq \frac{3}{2} \delta_z \) (\( C \) is connected). Since \( (x, x) \in D \), we would have \( \text{diam}_A ab \geq d_Y(f(x), f(x)) > \delta \), which contradicts \( 2^o \).

Then the restriction \( f|_C: C \to ab \) is one-to-one and onto. Thus \( k(a) = \text{card } (C) \); component of \( f^{-1}(ab) = k(b) \), which contradicts our assumption.

Observe that if we replace "locally one-to-one" by "light", the conclusion does not hold. As an example consider the projection of the harmonic fan onto its limit segment. Also the assumption that \( Z \) is arcwise connected is essential. It can be seen by the following

**Example.** Let \( X = Y = A \cup B \) be subspaces of the Euclidean plane, where

\[
A = \{e^{it} \in \mathbb{R}^2: t \in [0, 2\pi]\},
\]

\[
B = \left\{ \frac{2 + r e^{it}}{1 + r} \in \mathbb{R}^2: r \in [0, +\infty) \right\}.
\]

Obviously \( X \) and \( Y \) are continua (not arcwise connected). Let \( f: X \to Y \) be defined by

\[
f(z) = \begin{cases} e^{2it} & \text{for } z = e^{it} \in A, \\ \frac{2 + 2te^{it}}{1 + t} & \text{for } z = \frac{2 + r e^{it}}{1 + r} \in B. \end{cases}
\]

The mapping \( f \) is locally one-to-one and confluent on \( X \) but

\[
k(y) = \begin{cases} 1 & \text{for } y \in B, \\ 2 & \text{for } y \in A. \end{cases}
\]
Now to show the main result of the paper it is enough to prove the following

**Theorem 2.** Let $X$ and $Y$ be compact metric spaces and $f : X \to Y$ a locally one-to-one surjection such that the degree $k$ of $f$ is constant on $Y$. Then $f$ is open (thus $f$ is a local homeomorphism).

**Proof.** To show the openness of $f$ we use (ii). Let $y \in Y$ and $y_n \to y$ be arbitrary.

By the continuity of $f$ we have $Ls f^{-1}(y_n) \subseteq f^{-1}(y)$ (cf. [4], Theorem 1, p. 61). By (iii) there exists such a natural number $n$ that $k(y) = s$ for each $n$.

To finish the proof it is enough to show that

$$\text{card } f^{-1}(y) = \text{card } Ls f^{-1}(y_n).$$

If we denote $f^{-1}(y_n) = \{x^{(1)}_n, x^{(2)}_n, \ldots, x^{(s)}_n\}$, we have $x^{(i)}_n \neq x^{(j)}_n$ for $i \neq j$ and for $n = 1, 2, \ldots$.

Let us consider the sequence $\{x^{(1)}_n\}$. By the compactness of $X$ there exist such a subsequence $\{x^{(1)}_{n_k}\} \subseteq \{x^{(1)}_n\}$ and a point $x^1$ that $x^{(1)}_{n_k} \to x^1$. Now, let us consider the sequence $\{x^{(2)}_{n_k}\}$; we may choose such a subsequence $\{x^{(2)}_{n_k}\} \subseteq \{x^{(2)}_n\}$ of it and such a point $x^2$ that $x^{(2)}_{n_k} \to x^2$. Further, by choosing consecutive subsequences of previously taken ones (and continuing this process up to the index $s$) we obtain at last such a subsequence $\{y_n\} = \{y_{n_k}\}$ of the sequence $\{x_n\}$ that

$$x^{(1)}_{n_k} \to x^1, x^{(2)}_{n_k} \to x^2, \ldots, x^{(s)}_{n_k} \to x^s.$$

By $f$ being locally one-to-one we have $x^i \neq x^j$ for $i \neq j$ (otherwise there would be an open neighbourhood $U$ of $x^i$ in $X$ such that $f|U$ is one-to-one in $U$ one could find two different points $x^{(i)}_{n_k}, x^{(j)}_{n_k}$ which are mapped onto $y_{n_k}$).

Then the following inequalities hold:

$$s = \text{card } \{x^1, x^2, \ldots, x^s\} \leq \text{card } Ls f^{-1}(y_n)$$

$$\leq \text{card } Ls f^{-1}(y) \leq \text{card } f^{-1}(y) = s;$$

hence the theorem is proved.

**Corollary 1.** Let $f : X \to Y$ be any locally one-to-one mapping of a compact metric space $X$ onto an arcwise connected metric space $Y$. Then $f$ is open if and only if $f$ is confluent.

The necessity holds by (ii). Theorems 1 and 2 give the sufficiency.

As an application to local expansions we have

**Corollary 2.** Any confluent local expansion of an arcwise connected continuum onto itself is open, and so it has a fixed point.

(Comment: The existence of a fixed point of an open local expansion has been proved by I. Rosenholtz in [6].)

To give a complete answer to question (1) one ought to consider local expansions on non arcwise connected continua.

**References**


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Accepté par la Rédaction le 16.2.1981