countably many subfamilies each of which is a hereditarily closure-preserving family of subsets of X.

COROLLARY 1. *If a hereditarily normal space X can be represented as the union of a σ-hereditarily closure-preserving family $\{A_s\}_{s\in S}$ of F_σ-sets in X and A_s is A-w.i.d. for each $s\in S$, then the space X is A-w.i.d.*

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Some theorems on invariance of infinite dimension under open and closed mappings

by

L. Polkowski (Warszawa)

Abstract. We discuss here the invariance and the inverse invariance of three classes of spaces: A-weakly infinite-dimensional spaces, S-weakly infinite-dimensional spaces and spaces which have the transfinite dimension trInd under three classes of mappings: open mappings with finite fibres, closed mappings with finite fibres and open-and-closed mappings with no fibres dense-in-themselves; finite-dimensional spaces, countable-dimensional spaces and strongly countable-dimensional spaces also occasionally appear in our paper.

Our terminology and notation follow [7] and [8]. We shall quote from [7] and [8] all the necessary theorems of general topology and dimension theory. Similarly, we shall quote from [9] when the transfinite dimension trInd is concerned.

1. Definitions. We start with the definitions of the most important notions to be used in the sequel.

1.1. DEFINITION. Let X be a space and A, B a pair of disjoint closed subsets of X; a closed subset L of X is said to be a *partition between A and B* if there exist open subsets U, V of X which satisfy the conditions

$$A\subset U,\quad B\subset V,\quad U\cap V=\emptyset \quad\text{and}\quad X\setminus L=U\cup V.$$

1.2. DEFINITION. A normal space X is said to be *A-weakly infinite-dimensional* (abbrev. A-w.i.d.) if for every sequence $(A_1, B_1), (A_2, B_2),\ldots$ of pairs of disjoint closed subsets of X there exists a sequence $L_1, L_2,\ldots$, where L_i is a partition between A_i and B_i for $i=1, 2,\ldots$, with the property that $\bigcap_{i=1}^{\infty} L_i=\emptyset$.

1.3. DEFINITION. A Tychonoff space X is said to be *S-weakly infinite-dimensional* (abbrev. S-w.i.d.) if for every sequence $(A_1, B_1), (A_2, B_2),\ldots$ of pairs of disjoint functionally closed subsets of X there exists a sequence $L_1, L_2,\ldots$ where the functionally closed subset L_i of X is a partition between

A_i and B_i for $i = 1, 2, \ldots$, with the property that $\bigcap_{i=1}^{n} L_i = \emptyset$ for some integer n.

1.4. Remark. If the space X in Definition 1.3 is normal, then one can suppose that every sequence $L_1, L_2, \ldots$ of partitions, with the property that $\bigcap_{i=1}^{n} L_i = \emptyset$ for some integer n, consists of functionally closed sets (see [2], Ch. 10, § 4, Lemma 2). This, along with the fact that in a normal space any two disjoint closed sets are completely separated, implies that in the case where X is a normal space in Definition 1.3 one can replace the words "functionally closed" by the word "closed".

1.5. Definition. For a normal space X we let

(1) $\operatorname{tr\,Ind} X = -1$ *if the space* X *is empty*;

(2) $\operatorname{tr\,Ind} X \leqslant \alpha$ *if for every pair* A, B *of disjoint closed subsets of* X *there exists a partition* L *between* A *and* B *such that* $\operatorname{tr\,Ind} L < \alpha$;

(3) $\operatorname{tr\,Ind} X = \alpha$ *if* $\operatorname{tr\,Ind} X \leqslant \alpha$ *and there exists no ordinal number* $\beta < \alpha$ *such that* $\operatorname{tr\,Ind} X \leqslant b$.

A normal space X is said *to have* tr Ind if $\operatorname{tr\,Ind} X = \alpha$ for an ordinal number α.

1.6. Definition. A Tychonoff space X is said to be *countable-dimensional* (abbrev. c.d.) (*strongly countable-dimensional* (abbrev. s.c.d.)) if the space X can be represented as the union of a sequence $A_1, A_2, \ldots$ of subspaces (closed subspaces) each of which is finite-dimensional in the sense of the covering dimension dim.

2. Preliminaries. In this section we state some basic facts about infinite-dimensional spaces which will be frequently used in the sequel.

We begin with a countable sum theorem and a countable addition theorem; the former has been established by van Douwen [6] and by Levšenko [17] under the additional assumption that the space X is countably paracompact, the latter by Levšenko [17].

Theorem A. *If a normal space* X *can be represented as the union of a sequence* $F_1, F_2, \ldots$ *of closed* A*-w.i.d. subspaces, then the space* X *is* A*-w.i.d.*

Theorem B. *If a hereditarily normal space* X *can be represented as the union of a sequence* $A_1, A_2, \ldots$ *of* A*-w.i.d. subspaces, then the space* X *is* A*-w.i.d.*

2.1. Remark. It is obvious that every closed subspace of an A-w.i.d. space is A-w.i.d. Thus, Theorem A implies that every F_σ in a normal A-w.i.d. space is A-w.i.d.

2.2 Remark. It should be remembered that a subspace of an A-w.i.d. space need not be A-w.i.d.; indeed, Pol [25] defined an A-w.i.d. compact metric space which contains a dense subspace that is not A-w.i.d.

We now state in a slightly modified form a theorem on S-w.i.d. spaces due to Skljarenko [27].

Theorem C. *If a normal space* X *contains a closed* S*-w.i.d. subspace* C *with the property that every closed subspace* T *of* X *contained in* $X \setminus C$ *is* S*-w.i.d., then the space* X *is* S*-w.i.d. If a normal space* X *is* S*-w.i.d., then the closed subspace*

$$S_d(X) = X \setminus \bigcup \{U \subset X\colon\ U \text{ is open in } X \text{ and } \dim U < \infty\}$$

of X *is countably compact and every closed subspace* T *of* X *contained in* $X \setminus S_d(X)$ *satisfies the inclusion*

$$T \subset \bigcup \{U \subset X\colon\ U \text{ is open in } X \text{ and } \dim U \leqslant n\}$$

for some integer n.

2.3. Remark. Let us observe that if we assume in addition that the space X in Theorem C is weakly paracompact, then the subspace $S_d(X)$ of X is compact and every closed subspace T of X contained in $X \setminus S_d(X)$ satisfies the inequality $\dim T \leqslant n$ for some integer n (see [7], Theorem 5.3.2 and [8], Theorem 3.1.14 and Problem 3.1.D).

A similar theorem on spaces which have tr Ind also holds; the first part can be obtained by introducing obvious modifications in Lemmas 3 and 8 in [27] and applying Example 2.1 in [9], the second part is established by an easy induction (cf. the proof of Theorem 3.16 in [9]). Let us add that if a normal space X has tr Ind, then X is S-w.i.d. This was proved by Smirnov [30] for hereditarily normal spaces. A slightly different proof given in [2] (Ch. 10, § 6, Theorem 28) holds in normal spaces.

Theorem D. *If a hereditarily normal space* X *contains a closed subspace* C *with the property that* $\operatorname{tr\,Ind} C$ *is defined and every closed subspace* T *of* X *contained in* $X \setminus C$ *satisfies the inequality* $\operatorname{Ind} T < \infty$, *then the space* X *has* tr Ind. *If a normal space* X *has* tr Ind, *then the closed subspace*

$$S_I(X) = X \setminus \bigcup \{U \subset X\colon \text{ is open in } X \text{ and } \operatorname{Ind} \bar{U} < \infty\}$$

of X *is countably compact and every closed subspace* T *of* X *contained in* $X \setminus S_I(X)$ *satisfies the inclusion*

$$T \subset \bigcup \{U \subset X\colon\ U \text{ is open in } X \text{ and } \operatorname{Ind} \bar{U} \leqslant n\}$$

for some integer n.

2.4. Remark. Let us observe that if we assume in addition that the space X in Theorem D is strongly hereditarily normal and weakly paracompact, then the subspace $S_I(X)$ of X is compact and every closed subspace T of X contained in $X \setminus S_I(X)$ satisfies the inequality $\operatorname{Ind} T \leqslant n$ for some integer n (see [8], Definition 2.1.2 and Theorem 2.3.14).

2.5. Remark. Let us note that if X is a metrizable space, then $S_d(X) = S_I(X)$. Thus, we shall denote in the sequel both these sets by $S(X)$ whenever X is a metrizable space.

The following addition theorem for A-w.i.d. spaces is a strenghtened version of a theorem proved in [2] (Ch. 10, § 5, Proposition 1).

2.6. THEOREM. *If a normal and countably paracompact space X contains an A-w.i.d. subspace A, the set A is an F_σ-set in X and A has the property that every closed subspace T of X contained in $X \setminus A$ is A-w.i.d., then the space X is A-w.i.d.*

Proof. Let $(A_1, B_1), (A_2, B_2), \ldots$ be a sequence of pairs of disjoint closed subsets of the space X; we are going to show that there exists a sequence $L_1, L_2, \ldots$, where L_i is a partition between A_i and B_i for $i = 1, 2, \ldots$, such that $\bigcap_{i=1}^{\infty} L_i = \emptyset$. By the assumptions $A = \bigcup_{j=1}^{\infty} F_j$, where F_j is closed in X for $j = 1, 2, \ldots$ We decompose the set of positive integers into infinitely many disjoint infinite sets $N_0, N_1, \ldots$ (cf. Levšenko [17]). As F_j is A-w.i.d. for $j = 1, 2, \ldots$, it follows from the normality and the countable paracompactness of X that, for each positive integer j, there exists a sequence $\{L_k\}_{k \in N_j}$, where L_k is a partition between A_k and B_k in X for each $k \in N_j$, with the property that $F_j \cap \bigcap_{k \in N_j} L_k = \emptyset$ (see [2], Ch. 10, § 5, Lemma 1). We let $T = \bigcap_{j=1}^{\infty} \bigcap_{k \in N_j} L_k$. Since $T \subset X \setminus A$, it follows from the assumptions of the theorem that T is A-w.i.d., and thus there exists a sequence $\{L_k\}_{k \in N_0}$, where L_k is a partition between A_k and B_k in X for each $k \in N_0$, with the property that $T \cap \bigcap_{k \in N_0} L_k = \emptyset$. In this way the sets L_i are defined for $i = 1, 2, \ldots$ Clearly, $\bigcap_{i=1}^{\infty} L_i = \emptyset$. This completes the proof.

We now state one more addition theorem for A-w.i.d. spaces, due to Leĭbo [16].

THEOREM E. *If a hereditarily normal space X contains an A-w.i.d. subspace A and A has the property that every closed subspace T of X contained in $X \setminus A$ is A-w.i.d., then the space X is A-w.i.d.*

The following sum theorem for A-w.i.d. spaces is due to Leĭbo [16]. It should be observed, however, that the presence of weak paracompactness permits us to reduce the case of locally countable families to the case of countable families.

THEOREM F. *If a normal and weakly paracompact space X can be represented as the union of a locally countable family $\{F_s\}_{s \in S}$ of closed A-w.i.d. subspaces, then the space X is A-w.i.d.*

Let us mention another sum theorem for A-w.i.d. spaces (see [26]). Let us recall that a family $\{A_s\}_{s \in S}$ of subsets of a space X is said to be *hereditarily closure-preserving* if for every family $\{B_s\}_{s \in S}$ such that $B_s \subset A_s$ for every $s \in S$ we have $\overline{\bigcup_{s \in S} B_s} = \bigcup_{s \in S} \bar{B}_s$; a family $\{A_s\}_{s \in S}$ is *σ-hereditarily closure-preserving* if it can be represented as the countable union of hereditarily closure-preserving families. Let us note that every locally finite family is hereditarily closure-preserving.

THEOREM G. *If a hereditarily normal space X can be represented as the union of a σ-hereditarily closure-preserving family $\{A_s\}_{s \in S}$ of A-w.i.d. subspaces and A_s is an F_σ-set in X for every $s \in S$, then the space X is A-w.i.d.*

2.7. Remark. Let us add that if a normal and weakly paracompact space X is locally A-w.i.d., i.e. if for every $x \in X$ there exists an A-w.i.d. neighbourhood $U(x)$, then X is A-w.i.d. Indeed, take an open point-finite refinement $\mathscr{V} = \{V_s\}_{s \in S}$ of the cover $\mathscr{U} = \{U(x)\}_{x \in X}$ such that V_s is an F_σ-set in X and thus is A-w.i.d. for every $s \in S$ (see Remark 2.1 and [7], Theorem 1.5.18). Since X is countably paracompact (see [7], Theorem 5.2.6), one can apply Lemma 2.3.3 of [8] and Theorem 2.6 to verify that the sets $F_i = \{x \in X\colon x$ belongs to at most i members of $\mathscr{V}\}$ are all A-w.i.d. It follows from Theorem A that X is A-w.i.d.

We close this section by quoting a lemma which plays an important role in the dimension-theoretic study of open mappings.

LEMMA A ([8], Lemma 1.12.5). *Let $f\colon X \to Y$ be an open mapping of a metric space X onto a metric space Y. For every base $\mathscr{B} = \{U_s\}_{s \in S}$ for the space X there exists a family $\{A_s\}_{s \in S}$ of subsets of X such that $A_s \subset U_s$ for each $s \in S$ and*

(i) *A_s and $f(A_s)$ are F_σ-sets in X and Y, respectively,*
(ii) *$f|A_s\colon A_s \to f(A_s)$ is a homeomorphism,*
(iii) $X = \bigcup_{s \in S} A_s \cup \bigcup_{y \in Y} [f^{-1}(y)^d]$.

3. Open mappings with finite fibres. The invariance of dimension under open mappings was first studied by Alexandroff [1]; in that paper open mappings with countable fibres between compact metric spaces were discussed. Section 5 below will be devoted to a natural generalization of that class of mappings — open mappings with no fibres dense-in-themselves; here we discuss open mappings with finite fibres.

To begin with, we recall first a well-known lemma (see Nagami [20] and Arhangel'skiĭ [5]; cf. also Alexandroff [1]), which plays an important role in the dimension-theoretic study of open mappings with finite fibres, and then comment briefly upon the situation in the finite-dimensional, countable-dimensional and strongly countable-dimensional case.

LEMMA B. *Let $f\colon X \to Y$ be an open mapping of Hausdorff space X onto a space Y and let $Y_j = \{y \in Y\colon |f^{-1}(y)| = j\}$ for $j = 1, 2, \ldots$; if all fibres of the mapping f are finite, then*

(i) $\bigcup_{j=1}^{k} Y_j$ *is a closed subset of* Y *for* $k = 1, 2, \ldots$;

(ii) *the mapping* $f_{Y_j}\colon f^{-1}(Y_j) \to Y_j$ *is a local homeomorphism for* $j = 1, 2, \ldots$

The following theorem was proved by Nagami [20] under the assumption that X, Y are paracompact in the case of the dimension dim and X, Y are hereditarily paracompact in the case of the dimension Ind (announcement in [19]; cf. also [5]) and slightly generalized by Pears [23] to, respectively, X, Y normal weakly paracompact and X, Y totally normal weakly paracompact. One can easily see that weak paracompactness is an invariant of open mappings with finite fibres and thus in the first part of the theorem the assumption that Y is weakly paracompact is redundant; the second part of the theorem slightly generalizes the second part of Pears's theorem

3.1. THEOREM. *If* $f\colon X \to Y$ *is an open mapping of a normal space* X *onto a normal space* Y *and all fibres of the mapping* f *are finite, then*

(i) *if* Y *is weakly paracompact, then* $\dim X = \dim Y$;

(ii) *if* X *and* Y *are strongly hereditarily normal and either* X *or* Y *is hereditarily weakly paracompact, then* $\operatorname{Ind} X = \operatorname{Ind} Y$.

Proof. It suffices to show that $\operatorname{Ind} X_j = \operatorname{Ind} Y_j$, where $Y_j = \{y \in Y\colon |f^{-1}(y)| = j\}$ and $X_j = f^{-1}(Y_j)$, for $j = 1, 2, \ldots$, (see [8], Theorem 2.3.1 and (i) in Lemma B). Let us fix an integer j and consider the mapping $f_j = f|X_j\colon X_j \to Y_j$. By wirtue of (ii) in Lemma B the mapping f_j is a local homeomorphism and thus $\operatorname{loc Ind} X_j = \operatorname{loc Ind} Y_j$, where loc Ind stands for the local dimension Ind. By the assumptions either X_j or Y_j is weakly paracompact. The easily verified fact that the mapping f_j is open-and-closed implies that the spaces X_j and Y_j are both weakly paracompact. Thus $\operatorname{loc Ind} X_j = \operatorname{Ind} X_j$ and $\operatorname{loc Ind} Y_j = \operatorname{Ind} Y_j$ (see [7], Theorem 2.3.14), so that $\operatorname{Ind} X_j = \operatorname{Ind} Y_j$; thus the theorem is proved.

Arhangel'skiĭ [5] proved that if $f\colon X \to Y$ is an open mapping of a metrizable space X onto a normal space Y and all fibres of the mapping f are finite, then the space X is c.d. if and only if the space Y is c.d.; further information can be found in [10], Section 7.

The following theorem on s.c.d. spaces was proved by Arhangel'skiĭ [3]; as the proof in [3] is indirect, we include a short and direct one.

3.2. THEOREM. *If* $f\colon X \to Y$ *is an open mapping of a normal space* X *onto a normal space* Y, *all fibres of the mapping* f *are finite and the space* X *is s.c.d., then the space* Y *is s.c.d.*

Proof. By the assumptions $X = \bigcup_{j=1}^{\infty} F_j$, where F_j is closed in X and $\dim F_j = m_j < \infty$, for $j = 1, 2, \ldots$; we can assume that $F_j \subset F_{j+1}$ for $j = 1, 2, \ldots$ Since the mapping f is open, the subspace $Z_j = Y \setminus f(X \setminus F_j)$ is closed in Y for $j = 1, 2, \ldots$ As all fibres of the mapping f are finite, $Y = \bigcup_{j=1}^{\infty} Z_j$; thus it suffices to show that Z_j i.s. s.c.d. for $j = 1, 2, \ldots$ It follows from Lemma B that it suffices to prove that the subspace $Z_{j,k} = Z_j \cap \bigcup_{n=1}^{k} Y_n$ is finite-dimensional for $n = 1, 2, \ldots$ and $j = 1, 2, \ldots$ Let us consider a subspace $Z_{j,k}$; let $\mathcal{U} = \{U_1, U_2, \ldots, U_m\}$ be an open cover of $Z_{j,k}$; the open family $f^{-1}(\mathcal{U}) = \{f^{-1}(U_1), f^{-1}(U_2), \ldots, f^{-1}(U_m)\}$ is a cover of the subspace $f^{-1}(Z_{j,k}) \cap F_j$; as $\dim f^{-1}(Z_{j,k}) \leqslant m_j$, there exists an open refinement $\mathcal{V} = \{V_1, V_2, \ldots, V_s\}$ of the cover $f^{-1}(\mathcal{U})$ of order $\leqslant m_j$; one can easily check that the family $f(\mathcal{V}) = \{f(V_1), f(V_2), \ldots, f(V_s)\}$ is an open refinement of the cover $\mathcal{U}$ of order $\leqslant k(m_j+1)-1$; thus $\dim Z_{j,k} \leqslant k(m_j+1)-1$ and the theorem is proved.

Now, we are going to establish a theorem on A-w.i.d. spaces.

3.3 THEOREM. *If* $f\colon X \to Y$ *is an open mapping of a normal space* X *onto a normal space* Y *and all fibres of the mapping* f *are finite, then*:

(i) *if the space* Y *is weakly paracompact and the space* X *is* A-*w.i.d., then the space* Y *is* A-*w.i.d.*;

(ii) *if the space* X *is weakly paracompact and the space* Y *is* A-*w.i.d., then the space* X *is* A-*w.i.d.*;

(iii) *if the space* X *is either hereditarily normal or countably paracompact and the space* Y *is hereditarily normal, hereditarily weakly paracompact and* A-*w.i.d., then the space* X *is* A-*w.i.d.*

Proof. We first prove (i). It follows from the assumptions of (i) that Y is countably paracompact. By Theorem A and (i) in Lemma B it suffices to show that the subspace $Z_m = \bigcup_{j=1}^{m} Y_j$, where $Y_j = \{y \in Y\colon |f^{-1}(y)| = j\}$, is A-w.i.d. for $m = 1, 2, \ldots$; this will be done by induction with respect to the integer m. Let us consider the subspace $Z_1 = Y_1$. Since Z_1 is closed in Y, the subspace $X_1 = f^{-1}(Y_1)$ of X is closed and thus is A-w.i.d. The mapping $f_1 = f|X_1\colon X_1 \to Y_1$ being a homeomorphism, Z_1 is A-w.i.d. Let us assume that the subspace Z_m is A-w.i.d. for $m = 1, 2, \ldots, k$ and consider the subspace Z_{k+1}.

Now, Z_{k+1} is A-w.i.d.; indeed, if Z_{k+1} was not A-w.i.d., by Theorem 2.6 a closed subspace T of Y would exist such that $T \subset Y_{k+1}$ and T is not A-w.i.d. Consider the mapping $f_{k+1} = f|X_{k+1}\colon X_{k+1} \to Y_{k+1}$, where $X_{k+1} = f^{-1}(Y_{k+1})$, and the restriction $(f_{k+1})_T\colon f_{k+1}^{-1}(T) \to T$. By virtue of (ii) in Lemma B, there exists an open cover $\mathcal{U} = \{U_s\}_{s\in S}$ of X_{k+1} with the property that the mapping $f_{k+1}|U_s\colon U_s \to f_{k+1}(U_s)$ is a homeomorphism for every $s \in S$. By the normality of $f_{k+1}^{-1}(T)$ we can assume that $U_s \cap f_{k+1}^{-1}(T)$ is an F_σ-set in X and thus is A-w.i.d. for every $s \in S$. Thus, the open subspace $(f_{k+1})_T(U_s \cap f_{k+1}^{-1}(T))$ of T is A-w.i.d. for every $s \in S$. As the family

$(f_{k+1})_T(\mathscr{U}|f_{k+1}^{-1}(T)) = \{(f_{k+1})_T(U_s \cap f_{k+1}^{-1}(T))\}_{s\in S}$ is a cover of T, the weak paracompactness of T implies that T is A-w.i.d. (see Remark 2.7), a contradiction. Thus Z_{k+1} is A-w.i.d. and (i) is proved.

We now prove (ii). It follows from the assumptions of (ii) that X is countably paracompact. By Theorem A and (i) in Lemma B it suffices to show that the subspace $T_m = \bigcup_{j=1}^{m} X_j$, where $X_j = f^{-1}(Y_j)$, is A-w.i.d. for $m = 1, 2, \ldots$; this will be done by induction with respect to the integer m. Let us consider the subspace $T_1 = X_1$. Since Y_1 is closed in Y and thus is A-w.i.d. and the mapping f_1: $T_1 \to Y_1$ is a homeomorphism, T_1 is A-w.i.d. Let us assume that the subspace T_m is A-w.i.d. for $m = 1, 2, \ldots, k$ and consider the subspace T_{k+1}.

Now, if T_{k+1} was not A-w.i.d., by Theorem 2.6 a closed subspace K of X would exist such that $K \subset X_{k+1}$ and K would not be A-w.i.d. Consider the mapping f_{k+1}: $X_{k+1} \to Y_{k+1}$. By wirtue of (ii) in Lemma B there exists an open cover $\mathscr{U} = \{U_s\}_{s\in S}$ of X_{k+1} with the property that the mapping $f_{k+1}|U_s$: $U_s \to f_{k+1}(U_s)$ is a homeomorphism for every $s \in S$; moreover, we can assume, by the openness of X_{k+1} in T_{k+1}, that U_s is an F_σ-set in X for every $s \in S$. Similarly, by the openness of Y_{k+1} in $Z_{k+1} = f(T_{k+1})$, there exists an open refinement $\mathscr{V} = \{V_t\}_{t\in T}$ of the open cover $f_{k+1}(\mathscr{U}) = \{f_{k+1}(U_s)\}_{s\in S}$ of Y_{k+1} such that V_t is an F_σ-set in Y for every $t \in T$; for each $t \in T$, choose an index $s_t \in S$ such that $V_t \subset f_{k+1}(U_{s_t})$ and let $W_t = U_{s_t} \cap f_{k+1}^{-1}(V_t)$. Clearly, the mapping $f_{k+1}|W_t$: $W_t \to V_t$ is a homeomorphism and thus W_t is A-w.i.d. for every $t \in T$. The family $\mathscr{W} = \{W_t \cap K\}_{t\in T}$ is an open cover of K by A-w.i.d. subspaces and thus the weak paracompactness of K implies that K is A-w.i.d., a contradiction. Thus T_{k+1} is A-w.i.d. and (ii) is proved.

Finally, we prove (iii). It follows from the assumptions of (iii) that Y is strongly hereditarily normal (see [8], Theorem 2.1.5). Let us observe that every open subset of a strongly hereditarily normal A-w.i.d. space is A-w.i.d.; indeed, consider an open subset U of a strongly hereditarily normal A-w.i.d. space P and let A, B be a pair of disjoint closed subsets of U. Since the sets A and B are separated in P, there exist disjoint open sets U, $W \subset P$ such that $A \subset U$, $B \subset W$ and U, W can be represented as the union of a point-finite family of F_σ-sets in P; by the hereditary normality of U, W both these sets are A-w.i.d. (see Theorem B, Theorem E and Lemma 2.3.3 in [8]). Take an open subset H of P such that $A \subset H \subset \overline{H} \cap U \subset U \cap V$ and an open subset G of P such that $A \subset G \subset \overline{G} \cap V \subset H$ (cf. the proof of Lemma 2.3.5 in [8]). Clearly, the boundary $\mathrm{Fr}\, G$ of the set G in the space V is a partition between A and B in U; since $\mathrm{Fr}\, G$ is A-w.i.d., the hereditary normality of U implies that U is A-w.i.d. This implies in particular that by virtue of (i) in Lemma B the subspace Y_j of Y is A-w.i.d. for $j = 1, 2, \ldots$

To prove that X is A-w.i.d. it suffices to show, by virtue of (i) in Lemma B and Theorem A, that the subspace $T_m = \bigcup_{j=1}^{m} X_j$ is A-w.i.d. for $m = 1, 2, \ldots$; this will be done by induction with respect to the integer m. Let us consider the subspace $T_1 = X_1$ of X. Since Y_1 is closed in Y and thus is A-w.i.d. and the mapping f_1: $T_1 \to Y_1$ is a homeomorphism, T_1 is A-w.i.d. Let us assume that the subspace T_m of X is A-w.i.d. for $m = 1, 2, \ldots, k$ and consider the subspace T_{k+1}. Now, T_{k+1} is A-w.i.d.; indeed, if T_{k+1} was not A-w.i.d., then, either by virtue of Theorem E or by virtue of Theorem 2.6, a closed subset K of X would exist such that $K \subset X_{k+1}$ and K would not be A-w.i.d. Consider the mapping f_{k+1}: $X_{k+1} \to Y_{k+1}$. The easily verified fact that f_{k+1} is a closed mapping implies that the set $f_{k+1}(K)$ is closed in Y_{k+1} and thus is A-w.i.d.; moreover, $f_{k+1}(K)$ is weakly paracompact. Let us now consider the mapping $(f_{k+1})_{f_{k+1}(K)}$: $f_{k+1}^{-1}(f_{k+1}(K)) \to f_{k+1}(K)$. The mapping $(f_{k+1})_{f_{k+1}(K)}$ being open, it follows from the already proved part (ii), by virtue of the closedness of the mapping $(f_{k+1})_{f_{k+1}(K)}$ which implies the weak paracompactness of K (see [7], Problem 5.3.H(a)), that K is A-w.i.d., a contradiction. Thus T_{k+1} is A-w.i.d. and (iii) is proved. This completes the proof.

The next theorem describes the behaviour of S-w.i.d. spaces under open mappings with finite fibres.

3.4. THEOREM. *If f: $X \to Y$ is an open mapping of a weakly paracompact normal space X onto a normal space Y, all fibres of the mapping f are finite and the space X is S-w.i.d., then the space Y is S-w.i.d.*

Proof. It follows from the assumptions of the theorem that the subspace $S_d(X)$ of X is compact (see Remark 2.3). Consider the mapping $f|S_d(X)$: $S_d(X) \to f(S_d(X))$. The mapping $f|S_d(X)$ being closed, it follows from Theorem 4.5 below that the compact subspace $f(S_d(X))$ of Y is A-w.i.d. and thus is S-w.i.d. Consider now a closed subspace T of Y contained in $Y \setminus f(S_d(X))$. Since $f^{-1}(T) \subset X \setminus S_d(X)$, it follows from Theorem C that $\dim f^{-1}(T) < \infty$. The openness of the mapping f_T: $f^{-1}(T) \to T$ along with (i) in Theorem 3.1 implies that $\dim T < \infty$. Thus, by Theorem C, the space Y is S-w.i.d. and the theorem is proved.

We now pass to a theorem on spaces which have $\mathrm{tr\,Ind}$.

3.5. THEOREM. *If f: $X \to Y$ is an open mapping of a metrizable space X onto a hereditarily normal space Y, all fibres of the mapping f are finite and the space X has* $\mathrm{tr\,Ind}$, *then the space Y has* $\mathrm{tr\,Ind}$.

Proof. It follows from the assumptions of the theorem that the subspace $S_I(X)$ of X is compact (see Remark 2.4). Consider the mapping $f|S_I(X)$: $S_I(X) \to f(S_I(X))$. Since the subspace $S_I(X) \subset X$ is c.d. (see [9], Theorem 4.6), the subspace $f(S_I(X))$ of Y is, by virtue of Theorem I below, also c.d.; moreover, $f(S_I(X))$ is metrizable (see [7], Theorem 4.4.15). Thus

$f(S_I(X))$ has tr Ind (see [9], Theorem 4.2). Consider now a closed subspace T of Y contained in $Y \setminus f(S_I(X))$. Since $f^{-1}(T) \subset X \setminus S_I(X)$, it follows from Theorem D that $\operatorname{Ind}[f^{-1}(T)] < \infty$. Since $T = \bigcup_{j=1}^{\infty} (T \cap Y_j)$, where $Y_j = \{y \in Y: |f^{-1}(y)| = j\}$ for each j, it follows from (i) in Lemma B that to prove that $\operatorname{Ind} T < \infty$ it suffices to show that $\operatorname{Ind}(T \cap Y_j) \leqslant \operatorname{Ind}[f^{-1}(T)]$ for $j = 1, 2, \ldots$ (see [8], Theorem 2.3.1). Let us fix an integer j and consider the subspace $T \cap Y_j$ of T. The easily verified fact that the mapping $f_j = f|X_j$: $X_j \to Y_j$, where $X_j = f^{-1}(Y_j)$, is open-and-closed, implies that Y_j is metrizable (see [7], Theorem 4.4.18). Consider now the mapping $f_j|f^{-1}(T \cap Y_j)$: $f^{-1}(T \cap Y_j) \to T \cap Y_j$; since the mapping $f_j|f^{-1}(T \cap Y_j)$ is closed and $|[f_j|f^{-1}(T \cap Y_j)]^{-1}(y)| = j$ for every $y \in T \cap Y_j$, we have $\operatorname{Ind}(T \cap Y_j) = \operatorname{Ind}[f^{-1}(T \cap Y_j)] \leqslant \operatorname{Ind}[f^{-1}(T)]$ (see Remark 4.4, below). Thus $\operatorname{Ind} T < \infty$ and we infer from Theorem D that the space Y has tr Ind; the theorem is proved.

3.6. Remark. Let us observe that a space X which is the inverse image of an S-w.i.d. space Y (a space Y which has tr Ind) under an open mapping with finite fibres need not be S-w.i.d. (have tr Ind).

To give an example consider $Z = \bigoplus_{n=1}^{\infty} I^n$, where I^n is the n-cube for $n = 1, 2, \ldots$, $Y = \omega Z$, the one-point compactification of Z and $X = Z \oplus Y$. Let the mapping f: $X \to Y$ be defined as follows: $f(x) = x$ if $x \in Y$ and $f(x) = i(x)$ if $x \in Z$, where i: $Z \to Y$ is the embedding; f is open and has finite fibres, the space Y is S-w.i.d. and has tr Ind, and yet the space X neither is S-w.i.d. nor has tr Ind.

However, we have the following two theorems on open-and-closed mappings with finite fibres.

3.7. Theorem. *If f: $X \to Y$ is an open-and-closed mapping of a normal space X onto a weakly paracompact space Y, all fibres of the mapping f are finite and the space Y is S-w.i.d., then the space X is S-w.i.d.*

Proof. It follows from the assumptions of the theorem that the subspace $S_d(Y)$ of Y is compact (see Remark 2.3) and so is $f^{-1}(S_d(Y))$. By virtue of (ii) in Theorem 3.3, $f^{-1}(S_d(Y))$ is A-w.i.d. and thus is S-w.i.d. Consider a closed subspace T of X contained in $X \setminus f^{-1}(S_d(X))$. Since $f(T) \subset Y \setminus S_d(Y)$, it follows from Theorem C that $\dim[f(T)] < \infty$; by virtue of (i) in Theorem 3.1, $\dim[f^{-1}(f(T))] = \dim[f(T)]$ (see [7], Problem 5.3.H(a)), so that $\dim T \leqslant \dim[f^{-1}(f(T))] < \infty$ and, by Theorem C, the space X is S-w.i.d. Thus, the theorem is proved.

3.8. Theorem. *If f: $X \to Y$ is an open-and-closed mapping of a strongly hereditarily normal space X onto a metrizable space Y, all fibres of the mapping f are finite and the space Y has* tr Ind, *then the space X has* tr Ind.

Proof. It follows from the assumptions of the theorem that the subspace $S(Y)$ of Y is compact and so is $f^{-1}(S(Y))$. Since $S(Y)$ is c.d. (see [8], Theorem 4.6), $S(Y)$ can be represented as the union of a sequence $A_1, A_2, \ldots$ of 0-dimensional subspaces (see [8], Theorem 4.1.17). By virtue of (iii) in Theorem 3.1 we have $\operatorname{Ind} f^{-1}(A_i) = 0$ for $i = 1, 2, \ldots$ It follows from the equality $f^{-1}(S(Y)) = \bigcup_{i=1}^{\infty} f^{-1}(A_i)$ that $f^{-1}(S(Y))$ has tr Ind (to show this we can use the argument applied to prove Theorem 4.1 in [9], employing Theorem 2.2.4 of [8] instead of Theorem 4.1.13 of [8]). Consider now a closed subspace T of X contained in $X \setminus f^{-1}(S(Y))$. Since $f(T) \subset Y \setminus S(Y)$, it follows from Theorem D that $\operatorname{Ind}[f(T)] < \infty$. By virtue of (ii) in Theorem 3.1 $\operatorname{Ind}[f^{-1}(f(T))] = \operatorname{Ind}[f(T)]$ and thus $\operatorname{Ind} T \leqslant \operatorname{Ind}[f^{-1}(f(T))] < \infty$. By Theorem D the space X has tr Ind and the theorem is proved.

We conclude this section with some results having to do with more general classes of open mappings.

It is known that open mappings with countable fibres can arbitrarily raise dimension (see [8], Problems 1.12.E and 1.12.F). An example of an open mapping with countable fibres of a complete separable metric space X that is not c.d. onto the Cantor set is also known (see [10], Section 7). As shown by Pol [24], open mappings with discrete fibres can arbitrarily raise dimension.

Restricting ourselves to metrizable spaces, we obtain here some theorems on the invariance of the classes of infinite-dimensional spaces discussed in our paper under open mappings with separable fibres, no fibre of which is dense-in-itself.

The dimension-theoretic study of this class of open mappings was initiated by Pol [24]. That was the first paper in which the following lemma, due to Hansell (see [12], Proposition 3.11), was exploited in dimension theory.

Lemma C. *If f: $X \to Y$ is an open mapping of a collectionwise normal space X onto a metrizable space Y and all fibres of the mapping f are Lindelöf, then for every family $\{A_s\}_{s \in S}$, of subsets of X discrete in X, there exists a sequence $\{B_{s,1}\}_{s \in S}$, $\{B_{s,2}\}_{s \in S}, \ldots$ of discrete families, of subsets of Y, with the property that $f(A_s) = \bigcup_{n=1}^{\infty} B_{s,n}$ for each $s \in S$.*

The following theorem was proved in [24] by applying Lemma C.

Theorem H. *If f: $X \to Y$ is an open mapping of a metrizable space X onto a metrizable space Y, all fibres of the mapping f are separable and no fibre of the mapping f is dense-in-itself, then* $\dim Y \leqslant \dim X$.

We now pass to a theorem on A-w.i.d. spaces.

3.9. Theorem. *If f: $X \to Y$ is an open mapping of a metrizable space X onto a metrizable space Y, all fibres of the mapping f are separable, no fibre of* *the mapping f is dense-in-itself and the space X is A-w.i.d., then the space Y is A-w.i.d.*

Proof. Let us take a σ-discrete base $\mathscr{B} = \bigcup_{m=1}^{\infty} \mathscr{B}_m$ for the space X, where $\mathscr{B}_m = \{U_s\}_{s\in S_m}$ is discrete in X for $m = 1, 2, \ldots$; we let $S = \bigcup_{m=1}^{\infty} S_m$. Let the family $\{A_s\}_{s\in S}$ of subsets of X, where $A_s \subset U_s$ for each $s\in S$, satisfy (i)–(iii) in Lemma A. Let us fix an integer m and consider the family $\{A_s\}_{s\in S_m}$. By Lemma C there exists a sequence $\{B_{s,1}\}_{s\in S_m}, \{B_{s,2}\}_{s\in S_m}, \ldots$ of discrete families of subsets of Y such that $f(A_s) = \bigcup_{n=1}^{\infty} B_{s,n}$ for each $s\in S_m$. For each $s\in S_m$, enlarge the set $B_{s,n}$ to an set $G_{s,n}$ open in Y in such a way that the family $\{G_{s,n}\}_{s\in S_m}$ is discrete for $n = 1, 2$ (see [11], Lemma 4). By virtue of (i) in Lemma A the set $T_{s,n} = f(A_s)\cap G_{s,n}$ is an F_σ-set in Y and the set $A_s \cap [(f|A_s)^{-1}(T_{s,n})]$ is an F_σ-set in X and thus is A-w.i.d. for every $s\in S_m$ and $n = 1, 2, \ldots$ By virtue of (ii) in Lemma A the mapping $(f|A_s)_{T_{s,n}}\colon A_s\cap[(f|A_s)^{-1}(T_{s,n})] \to T_{s,n}$ is a homeomorphism and thus $T_{s,n}$ is A-w.i.d. for every $s\in S_m$ and $n = 1, 2, \ldots$ Since the family $\{T_{s,n}\}_{s\in S_m}$ is discrete, the subspace $T_{m,n} = \bigcup_{s\in S_m} T_{s,n}$ is A-w.i.d. for $n = 1, 2, \ldots$; it follows from Theorem B that the subspace $T_m = \bigcup_{n=1}^{\infty} T_{m,n}$ of Y is A-w.i.d. Now, it follows from (iii) in Lemma A that $Y = \bigcup_{m=1}^{\infty} T_m$ and thus Theorem B implies that the space Y is A-w.i.d. This completes the proof.

We now prove a theorem on S-w.i.d. spaces.

3.10. Theorem. *If $f\colon X\to Y$ is an open mapping of a metrizable space X onto a metrizable space Y, all fibres of the mapping f are separable, no fibre of the mapping f is dense-in-itself and the space X is S-w.i.d., then the space Y is S-w.i.d.*

Proof. It follows from the assumptions of the theorem that the subspace $S(X)$ of X is compact (see Remark 2.3) and so is the subspace $f(S(X))$ of Y. The subspace $f^{-1}[f(S(X))]$ of X is S-w.i.d., hence A-w.i.d. and thus Theorem 3.9 applied to the mapping $f_{f(S(X))}\colon f^{-1}[f(S(X))] \to f(S(X))$ implies that $f(S(X))$ is A-w.i.d., and so S-w.i.d. Consider now a closed subspace T of Y contained in $Y\setminus f(S(X))$. Since $f^{-1}(T)\subset X\setminus S(X)$, it follows from Theorem C that $\dim[f^{-1}(T)] < \infty$. Now, Theorem H applied to the mapping $f_T\colon f^{-1}(T)\to T$ implies that $\dim T < \infty$. Thus, by Theorem C, the space Y is S-w.i.d. and the theorem is proved.

Finally, we prove a theorem on spaces which have $\operatorname{tr}\operatorname{Ind}$.

3.11. Theorem. *If $f\colon X\to Y$ is an open mapping of a metrizable space X onto a metrizable space Y, all fibres of the mapping f are separable, no fibre of the mapping f is dense-in-itself and the space X has* $\operatorname{tr}\operatorname{Ind}$, *then the space Y has* $\operatorname{tr}\operatorname{Ind}$.

Proof. It follows from the assumptions of the theorem that the subspace $S(X)$ of X is compact (see Remark 2.4) and so is the subspace $f(S(X))$ of Y. The subspace $f^{-1}[f(S(X))]$ of X has $\operatorname{tr}\operatorname{Ind}$, and hence is c.d. (see [9], Theorem 4.6); thus $f(S(X))$ is c.d. (see Remark 3.12, below), and hence has $\operatorname{tr}\operatorname{Ind}$ (see [9], Theorem 4.2). Consider now a closed subset T of Y contained in $Y\setminus f(S(X))$. Since $f^{-1}(T)\subset X\setminus S(X)$, it follows from Theorem D that $\operatorname{Ind}[f^{-1}(T)] < \infty$. Now, Theorem H applied to the mapping $f_T\colon f^{-1}(T) \to T$ implies that $\operatorname{Ind} T < \infty$. Thus, by Theorem D, the space Y has $\operatorname{tr}\operatorname{Ind}$ and the theorem is proved.

3.12. Remark. Let us note that one can prove along the lines of Theorem 3.9 that if $f\colon X\to Y$ is an open mapping of a metrizable space X onto a metrizable space Y, all fibres of the mapping f are separable, no fibre of the mapping f is dense-in-itself and the space X is c.d. (s.c.d.), then the space Y is c.d. (s.c.d.).

Let us add that it follows immediately from Lemma A that if $f\colon X\to Y$ is an open mapping of a metrizable space X onto a metrizable space Y, all fibres of the mapping f are discrete and the space Y is A-w.i.d. (c.d., s.c.d.), then the space X is A-w.i.d. (c.d., s.c.d.).

4. Closed mappings with finite fibres. The theorems on the invariance of dimension under closed mappings go back to Hurewicz [14], who proved that if $f\colon X\to Y$ is a closed mapping of a separable metric space X onto a separable metric space Y and there exists an integer k such that $|f^{-1}(y)| = k$ for every $y\in Y$, then $\operatorname{ind} X = \operatorname{ind} Y$. Nagami [20] and Suzuki [31] proved along these lines a more general result (announced in [19] and, for $\operatorname{ind} X = 0$, in [18]), viz., if $f\colon X\to Y$ is a closed mapping of a metrizable space X onto a metrizable space Y and there exists an integer k such that $|f^{-1}(y)| = k$ for every $y\in Y$, then $\operatorname{Ind} X = \operatorname{Ind} Y$ (cf. Remark 4.4, below). The following theorem on c.d. spaces was proved by Nagami [21].

Theorem I. *If $f\colon X\to Y$ is a perfect mapping of a metrizable space X onto a metrizable space Y, no fibre of the mapping f is dense-in-itself and the space X is c.d., then the space Y is c.d.*

The first theorem on the behaviour of S-w.i.d. spaces under closed mappings was established by Skljarenko [28], who proved that if $f\colon X\to Y$ is a closed mapping of a separable completely metrizable space X onto a normal countably paracompact space Y, all fibres of the mapping f are of cardinality less that $\mathfrak{c}$ and the space X is S-w.i.d., then the space Y is A-w.i.d. Leĭbo [16] proved along these lines that if $f\colon X\to Y$ is a closed mapping of a weakly paracompact Čech-complete normal space X onto a normal space

Y, all fibres of the mapping f are countable and the space X is S-w.i.d., then the space Y is A-w.i.d.

We are now going to establish some theorems on the behaviour of A-w.i.d. spaces, S-w.i.d. spaces and spaces which have trInd under closed mappings with finite fibres.

We begin with the following two lemmas.

4.1. LEMMA. *If $f\colon X \to Y$ is a closed mapping of a normal space X onto a paracompact space Y, the diagonal Δ is a G_δ-set in $X \times X$ and there exists a positive integer k such that $|f^{-1}(y)| = k$ for every $y \in Y$, then there exists a closed cover $\mathcal{T} = \bigcup_{n=1}^{\infty} \mathcal{T}_n$ with the following properties:*

(a) *the family $\mathcal{T}_n$ is locally finite in Y for $n = 1, 2, \ldots$;*

(b) *for each $T \in \mathcal{T}$, the inverse image $f^{-1}(T)$ is the union of a sequence $T_1, T_2, \ldots, T_k$ of disjoint closed sets in X each of which is homeomorphic to T by the restriction of the mapping f.*

Proof. If follows from the assumptions of the lemma that the space X is paracompact (see [7], Theorem 5.1.35) and thus there exists a continuous one-to-one mapping $g\colon X \to Z$ of the space X onto a metrizable space Z (see [7], Problem 5.5.7(a)). We choose an admissible metric σ in Z and let $\varrho = \sigma \circ (g \times g)$. Let us fix a positive integer n.

To define the family $\mathcal{T}_n$ consider the set $A_n = \{y \in Y\colon \varrho(x_1, x_2) \geqslant 1/n$ whenever $x_1, x_2 \in f^{-1}(y)$ and $x_1 \neq x_2\}$. Clearly, A_n is closed in Y. For each $y \in A_n$, with $f^{-1}(y) = \{x_{1,y}, x_{2,y}, \ldots, x_{k,y}\}$, choose open subsets $U_{1,y}, U_{2,y}, \ldots, U_{k,y}$ of X with the properties:

(1) $$x_{i,y} \in U_{i,y} \quad \text{for} \quad i = 1, 2, \ldots, k;$$

(2) $$\delta(U_{i,y}) < 1/3n \quad \text{for} \quad i = 1, 2, \ldots, k;$$

(3) $$U_{i,y} \cap U_{j,y} = \emptyset \quad \text{whenever} \quad i \neq j \text{ for } i, j = 1, 2, \ldots, k.$$

Since the mapping f is closed, for each $y \in A_n$ there exists an open neighbourhood $W_{n,y}$ in A_n such that $f^{-1}(W_{n,y}) \subset \bigcup_{i=1}^{k} U_{i,y}$. Consider now a locally finite closed refinement $\mathcal{T}_n = \{T_{n,s}\}_{s \in S_n}$ of the open cover $\{W_{n,y}\}_{y \in A_n}$ of A_n. It is easy to see that, for each $T_{n,s} \in \mathcal{T}_n$, the sets $T_{n,s,1}, T_{n,s,2}, \ldots, T_{n,s,k}$, where $T_{n,s,i} = \overline{U_{i,y}} \cap f^{-1}(T_{n,s})$ for $i = 1, 2, \ldots, k$, satisfy condition (b) with $T = T_{n,s}$.

As $Y = \bigcup_{n=1}^{\infty} A_n$, the family $\mathcal{T} = \bigcup_{n=1}^{\infty} \mathcal{T}_n$ is a cover of Y and thus the proof is completed.

4.2. LEMMA. *If $f\colon X \to Y$ is a closed mapping of a normal space X onto a paracompact space Y, the diagonal Δ is a G_δ-set in $X \times X$ and all fibres of the mapping f are finite, then $\{y \in Y\colon |f^{-1}(y)| \geqslant k\}$ is an F_σ-set in Y for $k = 1, 2, \ldots$*

Proof. The proof will proceed along lines similar to those of Lemma 4.1. It is easy to see that $F_n = \overline{A_n \cap \{y \in Y\colon |f^{-1}(y)| \geqslant k\}} \subset \{y \in Y\colon |f^{-1}(y)| \geqslant k\}$ for $n = 1, 2, \ldots$, where the sets $A_1, A_2, \ldots$ were defined above. It follows from the equality $Y = \bigcup_{n=1}^{\infty} A_n$ that $\{y \in Y\colon |f^{-1}(y)| \geqslant k\} = \bigcup_{n=1}^{\infty} F_n$ and thus the lemma is proved.

We now prove a theorem on the inverse invariance of A-w.i. dimensionality.

4.3. THEOREM. *If $f\colon X \to Y$ is a closed mapping of a normal space X onto a paracompact space Y, the diagonal Δ is a G_δ-set in $X \times X$, there exists an integer k such that $|f^{-1}(y)| \leqslant k$ for every $y \in Y$ and the space Y is A-w.i.d., then the space X is A-w.i.d.*

Proof. We first consider the special case where $|f^{-1}(y)| = k$ for every $y \in Y$. It follows from the assumptions of the theorem that the space X is paracompact. By Lemma 4.1 there exists a closed cover $\mathcal{T} = \bigcup_{n=1}^{\infty} \mathcal{T}_n$ of Y such that the family $\mathcal{T}_n$ is locally finite for $n = 1, 2, \ldots$ and, for each $T \in \mathcal{T}$, the inverse image $f^{-1}(T)$ is the union of a sequence $T_1, T_2, \ldots, T_k$ of closed sets in X each of which is homeomorphic to T by the restriction of the mapping f.

Since Y is A-w.i.d., every $T \in \mathcal{T}$ is A-w.i.d. and thus the subspace T_i of X is A-w.i.d. for every $T \in \mathcal{T}$ and $i = 1, 2, \ldots, k$. The family $\{T_i\}_{T \in \mathcal{T}_n}$, where $i = 1, 2, \ldots, k$, being locally finite for $n = 1, 2, \ldots$, it follows from Theorems A and F that X is A-w.i.d.

We are now going to prove the theorem in the general case; this will be done by induction with respect to the integer k.

For $k = 1$ the theorem is true. Let us assume that if $|f^{-1}(y)| \leqslant k$ for every $y \in Y$, then the theorem is true and let us consider a mapping $f\colon X \to Y$ such that $|f^{-1}(y)| \leqslant k+1$ for every $y \in Y$. By Lemma 4.2 the set $Y_{k+1} = \{y \in Y\colon |f^{-1}(y)| = k+1\}$ is an F_σ-set in Y, and hence is A-w.i.d. By the already established special case of our theorem applied to the restriction $f_{Y_{k+1}}\colon X_{k+1} \to Y_{k+1}$, where $X_{k+1} = f^{-1}(Y_{k+1})$, the subspace X_{k+1} of X is A-w.i.d. Suppose that X is not A-w.i.d. Since X_{k+1} is an F_σ-set in X, by Theorem 2.6 there exists a closed subspace T of X contained in $X \setminus X_{k+1}$ which is not A-w.i.d. Consider the restriction $f|T\colon T \to f(T)$; since $|(f|T)^{-1}(y)| \leqslant k$ for every $y \in f(T)$, it follows from the inductive assumption that the closed subspace $f(T)$ of Y is not A-w.i.d., a contradiction. Thus X is A-w.i.d. and the theorem is proved.

4.4. Remark. Let us observe first that the argument used in the first part of the proof of Theorem 4.3 can be applied to show that if $f\colon X \to Y$ is a closed mapping of a normal space X onto a paracompact space Y, the diagonal Δ is a G_δ-set in $X \times X$ and there exists a positive integer k such that $|f^{-1}(y)| = k$ for every $y \in Y$, then $\dim X = \dim Y$ (as proved by Nagami [22], the inequality $\dim Y \leqslant \dim X$ holds in the case where $f\colon X \to Y$ is a closed mapping of a paracompact σ-space X onto a paracompact σ-space Y and $|f^{-1}(y)| = k$ for every $y \in Y$; let us note that if X is a paracompact σ-space and thus there exists in X a σ-discrete closed network, then the diagonal Δ is a G_δ-set in $X \times X$. Let us add that the same argument shows that if $f\colon X \to Y$ is a closed mapping of a strongly hereditarily normal space X onto a hereditarily paracompact space Y, the diagonal Δ is a G_δ-set in $X \times X$ and all fibres of the mapping f are finite, then the space X is c.d. if and only if the space Y is c.d. Indeed, represent Y as the union of the subspaces $Y_1, Y_2, \ldots$, where $Y_i = \{y \in Y\colon |f^{-1}(y)| = i\}$ and consider the mapping $f_{Y_i}\colon f^{-1}(Y_i) \to Y_i$ for $i = 1, 2, \ldots$; it follows from Lemma 4.1 and the locally finite sum theorem for c.d. spaces (see [10], Section 3) that, for each positive integer i, Y_i is c.d. if and only if $f^{-1}(Y_i)$ is c.d., and thus our result follows from the subspace theorem for c.d. spaces (see [10], Section 2) and the countable sum theorem for c.d. spaces (see [10], Section 3). Let us finally observe that a strongly hereditarily normal space X with the property that the diagonal Δ is a G_δ-set in $X \times X$ need not be perfectly normal. Indeed, the *Michael line* $\boldsymbol{R}_Q$ (see [7], Examples 5.1.22 and 5.1.32) is a hereditarily paracompact, and hence strongly hereditarily normal space, the diagonal Δ is a G_δ-set in $\boldsymbol{R}_Q \times \boldsymbol{R}_Q$ and yet $\boldsymbol{R}_Q$ is not perfectly normal.

We now pass to a theorem on the invariance of A-w.i. dimensionality under closed mappings. We begin with a lemma.

4.5. Lemma. *Let $f\colon X \to Y$ be a closed mapping of a normal countably paracompact A-w.i.d. space X onto a normal space Y. If the space Y is not A-w.i.d., then there exist disjoint closed subsets X_0, X_1 of X and a closed subset Y_0 of Y such that $f(X_0) = f(X_1) = Y_0$ and Y_0 is not A-w.i.d.*

Proof. Let $(E_1, F_1), (E_2, F_2), \ldots$ be a sequence of pairs of disjoint closed subsets of Y such that if L_i is a partition between E_i and F_i in Y for $i = 1, 2, \ldots$, then $\bigcap_{i=1}^{\infty} L_i \neq \emptyset$. For each positive integer i, consider the sets $A_i = f^{-1}(E_i)$ and $B_i = f^{-1}(F_i)$. As X is A-w.i.d., for $i = 1, 2, \ldots$ there exist closed subsets M_{2i}, N_{2i} of X such that

(1) $\quad X = M_{2i} \cup N_{2i}, \quad A_{2i} \subset M_{2i} \setminus N_{2i} \quad$ and $\quad B_{2i} \subset N_{2i} \setminus M_{2i};$

(2) $$\bigcap_{i=1}^{\infty} (M_{2i} \cap N_{2i}) = \emptyset;$$

Moreover, we can assume that $X \setminus M_{2i} = \bigcup_{j=1}^{\infty} P_{i,j}$ and $X \setminus N_{2i} = \bigcup_{j=1}^{\infty} R_{i,j}$, where the sets $P_{i,j}$, $R_{i,j}$ are closed in X for $j = 1, 2, \ldots$ (see [2], Ch. 10, § 4, Lemma 2).

By virtue of (1) the subset $K_{2i} = f(M_{2i}) \cap f(N_{2i})$ of Y is a partition between E_{2i} and F_{2i} in Y for $i = 1, 2, \ldots$ and thus the subset $K = \bigcap_{i=1}^{\infty} K_{2i}$ of Y is not A-w.i.d. It follows from (2) that for each $y \in K$ there exists a positive integer i such that $f^{-1}(y) \not\subset M_{2i} \cap N_{2i}$, and this implies that

$$K = \bigcup_{j=1}^{\infty} (K \cap f(P_{i,j})) \cup \bigcup_{j=1}^{\infty} (K \cap f(R_{i,j})).$$

By Theorem A there exists a set, say P_{i_0,j_0}, such that the subspace $K \cap f(P_{i_0,j_0})$ of Y is not A-w.i.d. It is easy to see that the sets $X_0 = f^{-1}(K) \cap P_{i_0,j_0}$, $X_1 = M_{2i_0} \cap f^{-1}(K \cap f(P_{i_0,j_0}))$ and $Y_0 = K \cap f(P_{i_0,j_0})$ satisfy our requirements. Thus the lemma is proved.

4.6. Theorem. *If $f\colon X \to Y$ is a closed mapping of a Čech-complete weakly paracompact normal space X onto a normal space Y, all fibres of the mappings f are scattered and the space X is A-w.i.d., then the space Y is A-w.i.d.*

Proof. It follows from the assumptions of the theorem that the space Y can be represented as the union of the sequence $Y_0, Y_1, Y_2, \ldots$ of subspaces such that Y_i is discrete in Y for $i = 1, 2, \ldots$ and the restriction $f_0 = f_{Y_0}\colon f^{-1}(Y_0) \to Y_0$ is perfect (see [4], Theorem 1.1). Since the space X is countably paracompact (see [7], Theorem 5.2.6), by Theorems 2.6 and A it suffices to show that each closed subset T of Y contained in Y_0 is A-w.i.d.

Thus consider a closed subset T_0 of Y contained in Y_0 and the restriction $(f_0)_{T_0}\colon f_0^{-1}(T_0) \to T_0$. As T_0 is Čech-complete (see [7], Theorem 3.9.10), there exists in T_0 a sequence $\mathscr{U}_1, \mathscr{U}_2, \ldots$ of open covers with the property that any family of closed sets in T_0 which has the finite intersection property and contains sets of diameter less than $\mathscr{U}_i$ for $i = 1, 2, \ldots$ has non-empty intersection (see [7], Theorem 3.9.2); we can assume that $\mathscr{U}_1 = \{T_0\}$.

Suppose now that T_0 is not A-w.i.d. For each positive integer k and for each sequence $(i_1, i_2, \ldots, i_k)$, where $i_j = 0$ or $i_j = 1$ for $j = 1, 2, \ldots, k$, we define a closed subset $S_{i_1 i_2 \ldots i_k}$ of $(f_0)_{T_0}^{-1}(T_0)$ and a closed subset T_k of T_0 such that

(1) $\quad S_{i_1 i_2 \ldots i_k} \cap S_{j_1 j_2 \ldots j_k} = \emptyset$ whenever $(i_1, i_2, \ldots, i_k) \neq (j_1, j_2, \ldots, j_k)$;

(2) $\quad S_{i_1 i_2 \ldots i_k 0} \cup S_{i_1 i_2 \ldots i_k 1} \subset S_{i_1 i_2 \ldots i_k}$;

(3) $\quad f(S_{i_1 i_2 \ldots i_k}) = T_k$;

(4) $\quad T_k$ is not A-w.i.d. and $\delta(T_k) < \mathscr{U}_k$;

(cf. the proofs of Theorems 1, 1′ in [28]).

The existence of the sets S_0, S_1 and T_1 follows from Lemma 4.5. Assume that the sets $S_{i_1 i_2 \ldots i_k}$ and $T_1, T_2, \ldots, T_k$ are defined. Since T_k is weakly paracompact (see [7], Theorem 5.3.7), we can assume that the open cover $\mathscr{U}_{k+1}|T_k = \{U \cap T_k\}_{U \in \mathscr{U}_{k+1}}$ of T_k consists of F_σ-sets in T_k. As T_k is not A-w.i.d., there exists a set $U \in \mathscr{U}_{k+1}|T_k$ that is not A-w.i.d. (see Remark 2.7) and thus, by Theorem A, there exists a closed subset Z of T_k contained in U which is not A-w.i.d. By applying Lemma 4.5 to the restriction $[(f_0)_{T_0}]_Z\colon (f_0)_{T_0}^{-1}(Z)$

$\to Z$ we can define the sets $S_{i_1 i_2 \ldots i_{k+1}}$ and T_{k+1} which satisfy conditions (1)–(4).

It follows from (2), (3), and (4) that $\bigcap_{k=1}^{\infty} T_k \neq \emptyset$; let $y \in \bigcap_{k=1}^{\infty} T_k$. By virtue of (2) the subspaces $S_{i_1 i_2 \ldots i_k} \cap f^{-1}(y)$ of $f^{-1}(y)$ form an inverse sequence whose limit is, by virtue of (1), a non-empty dense-in-itself subspace of $f^{-1}(y)$, a contradiction. Thus T_0 is A-w.i.d. and the theorem is proved.

4.7. Remark. Following Skljarenko's proof of his Theorem 1 in [28], one shows that if $f\colon X \to Y$ is a closed mapping of a normal space X onto a normal space Y, there exists an integer k such that $|f^{-1}(y)| \leqslant k$ for every $y \in Y$ and the space X is S-w.i.d., then the space Y is S-w.i.d. It is well known that the image of an S-w.i.d. space (a space which has tr Ind) under a closed mapping with finite fibres need not be S-w.i.d. (have tr Ind). Indeed, since the n-cube I^n is the image of the Cantor set C under a closed mapping with finite fibres, to give an example it suffices to let $X = \bigoplus_{n=1}^{\infty} C_n$, where C_n is homeomorphic to C for $n = 1, 2, \ldots$, $Y = \bigoplus_{n=1}^{\infty} I^n$ and $f = \bigoplus_{n=1}^{\infty} f_n$.

As with A-w.i. dimensionality, we only have the following theorem on the inverse invariance of S-w.i. dimensionality under closed mappings.

4.8. Theorem. *If $f\colon X \to Y$ is a closed mapping of a normal space X onto a paracompact space Y, the diagonal Δ is a G_δ-set in $X \times X$, there exists an integer k such that $|f^{-1}(y)| \leqslant k$ for every $y \in Y$ and the space Y is S-w.i.d., then the space X is S-w.i.d.*

Proof. It follows from the assumptions of the theorem that the subspace $S_d(Y)$ of Y is compact and so is the subspace $f^{-1}(S_d(Y))$ of X (see [7], Theorem 3.7.2). By Theorem 4.3, $f^{-1}(S_d(Y))$ is A-w.i.d. and thus is S-w.i.d. Consider now a closed subspace T of X contained in $X \setminus f^{-1}(S_d(Y))$. Since $f(T) \subset Y \setminus S_d(Y)$, it follows from Theorem C that $\dim[f(T)] < \infty$ (see Remark 2.3). By a theorem due to Skljarenko [29] applied to the mapping $f_{f(T)}\colon f^{-1}(f(T)) \to f(T)$ we have $\dim[f^{-1}(f(T))] < \infty$, so that $\dim T < \infty$. Thus, by Theorem C, the space X is S-w.i.d. and the theorem is proved.

We conclude this section with two theorems on spaces which have tr Ind.

4.9. Theorem. *If $f\colon X \to Y$ is a closed mapping of a metrizable space X onto a metrizable space Y, there exists an integer k such that $|f^{-1}(y)| \leqslant k$ for every $y \in Y$ and the space X has* tr Ind, *then the space Y has* tr Ind.

Proof. It follows from the assumptions of the theorem that the subspace $S(X)$ of X is compact and so is the subspace $f(S(X))$ of Y. The subspace $S(X)$ of X is c.d. (see [9], Theorem 4.6) and thus $f(S(X))$ is c.d. (see Remark 4.4.), so that $f(S(X))$ has tr Ind (see [9], Theorem 4.2). Consider now a closed subspace T of Y contained in $Y \setminus f(S(X))$. Since $f^{-1}(T) \subset X \setminus S(X)$, it follows from Theorem D that $\operatorname{Ind}[f^{-1}(T)] < \infty$ (see Remark 2.4). Thus $\operatorname{Ind} T < \infty$ (see [8], Theorem 4.3.3) and by Theorem D the space Y has tr Ind. Thus the theorem is proved.

The theorem on inverse invariance of the existence of tr Ind holds under much weaker assumptions than the finiteness of fibres.

4.10. Theorem. *If $f\colon X \to Y$ is a closed mapping of a metrizable space X onto a metrizable space Y, there exists an integer k such that* $\operatorname{Ind}[f^{-1}(y)] \leqslant k$ *for every $y \in Y$ and the space Y has* tr Ind, *then the space X has* tr Ind.

Proof. It follows from the assumptions of the theorem that the subspace $S(Y)$ of Y is compact and that $S(Y)$ is c.d. (see [9], Theorem 4.6). It follows from a theorem due to Arhangel'skiĭ [4] that the subspace $f^{-1}(S(Y))$ is c.d. Now, the Vaĭnšteĭn Lemma (see [7], 4.4.16) implies that $f^{-1}(S(Y))$ can be represented as the union of a compact subspace C and the union of the family $\{\operatorname{Int} f^{-1}(y)\}_{y \in S(Y)}$; as C is c.d., C has tr Ind (see [9], Theorem 4.2). Let K be a closed subset of $f^{-1}(S(Y))$ contained in $f^{-1}(S(Y)) \setminus C$. It follows from the inclusion $K \subset \bigcup_{y \in S(Y)} \operatorname{Int} f^{-1}(y)$ and the fact that $\operatorname{Ind}[\operatorname{Int} f^{-1}(y)] \leqslant k$ for every $y \in S(Y)$ that $\operatorname{Ind} K \leqslant k$. Thus, by Theorem D, the subspace $f^{-1}(S(Y))$ of X has tr Ind. Consider now a closed subset T of X contained in $X \setminus f^{-1}(S(Y))$. Since $f(T) \subset Y \setminus S(Y)$, it follows from Theorem D that $\operatorname{Ind}[f(T)] < \infty$ and thus $\operatorname{Ind}[f^{-1}(f(T))] < \infty$ (see [7], Theorem 4.3.6); this implies that $\operatorname{Ind} T < \infty$. It follows from Theorem D that the space X has tr Ind and thus our theorem is proved.

4.11. Remark. Let us note that in Theorem 4.10 the assumption that $\operatorname{Ind}[f^{-1}(y)] \leqslant k$ for every $y \in Y$ cannot be weakened to the assumption that $\operatorname{Ind}[f^{-1}(y)] < \infty$ for every $y \in Y$; indeed, it suffices to let $X = \bigoplus_{n=1}^{\infty} I^n$ and shrink each cube I^n to a point.

5. Open-and-closed mappings with no fibres dense-in-themselves. This class of spaces can be considered as a generalization of open mappings with countable fibres, defined on compact metric spaces (cf. Alexandroff [1], Hodel [13] and Vaĭnšteĭn [32]; cf. also Arhangel'skiĭ [5]). Hodel [13] proved that if $f\colon X \to Y$ is an open-and-closed mapping of a metrizable space X onto a metrizable space Y and no fibre of the mapping f is dense-in-itself, then $\dim Y \leqslant \dim X$. More general is the following theorem due to Keesling [15]. Let us recall that a mapping $f\colon X \to Y$ is said to be *σ-closed* if the space X can be represented as the union of a sequence $F_1, F_2, \ldots$ of closed subspaces such that the restriction $f|F_i\colon F_i \to f(F_i)$ is closed and $f(F_i)$ is closed in Y for $i = 1, 2, \ldots$; clearly, every closed mapping is σ-closed. We let $X_* = \{x \in X\colon x \text{ is open in } f^{-1}(f(x))\}$.

Theorem J. *If $f\colon X \to Y$ is an open-and-σ-closed mapping of a metrizable* *space X onto a metrizable space Y and the subspace K of X_* is closed in X_*, then* $\dim K = \dim f(K)$.

We are now going to establish some theorems on the behaviour of the classes of infinite-dimensional spaces discussed here under open-and-closed mappings with no fibres dense-in-themselves. We begin with a theorem on A-w.i.d. spaces.

5.1. THEOREM. *If $f\colon X \to Y$ is an open-and-closed mapping of a metrizable space X onto a metrizable space Y and no fibre of the mapping f is dense-in-itself, then*

(i) *if the space X is A-w.i.d., then the space Y is A-w.i.*

(ii) *the space Y is A-w.i.d. if and only if the space X_* is A-w.i.d.*

Proof. Let us take a σ-discrete base $\mathscr{B} = \bigcup_{m=1}^{\infty} \mathscr{B}_m$, where the family $\mathscr{B}_m = \{U_s\}_{s\in S_m}$ is discrete for $m = 1, 2, \ldots$, for the space X; we let $S = \bigcup_{m=1}^{\infty} S_m$. Let the family $\{A_s\}_{s\in S}$, where $A_s \subset U_s$ for each $s\in S$, satisfy (i)–(iii) in Lemma A. It follows from (iii) in Lemma A that $X_* = \bigcup_{s\in S} A_s$ and thus (i) in Lemma A implies that X_* is an F_σ-set in X. This implies that it suffices to prove (ii).

We suppose first that X_* is A-w.i.d. By the Vaĭnšteĭn Lemma (see [7], 4.4.16) the boundary $\operatorname{Fr} f^{-1}(y)$ is compact for every $y\in Y$. The openness of the mapping f implies that each one-point subset of the set $Y_0 = \{y\in Y\colon \operatorname{Int} f^{-1}(y) \neq \emptyset\}$ is open in Y; thus, by Theorem B, it suffices to prove that $Y\setminus Y_0$ is A-w.i.d. Consider the perfect mapping $\tilde{f} = f_{Y\setminus Y_0}\colon f^{-1}(Y\setminus Y_0) \to Y\setminus Y_0$. It follows from (ii) in Lemma A that the mapping $\tilde{f}|A_s \cap f^{-1}(Y\setminus Y_0)$ is a homeomorphism and thus $\tilde{f}[A_s \cap f^{-1}(Y\setminus Y_0)]$ is A-w.i.d. for every $s\in S$. Thus the family $\{\tilde{f}[A_s \cap f^{-1}(Y\setminus Y_0)]\}_{s\in S}$ is a σ-locally finite cover of $Y\setminus Y_0$ by closed A-w.i.d. subspaces. It follows from Theorem F that $Y\setminus Y_0$ is A-w.i.d.

We suppose now that Y is A-w.i.d. It follows from (i) in Lemma A that $f(A_s)$ is A-w.i.d. for every $s\in S$ and thus we infer from (ii) in Lemma A that A_s is A-w.i.d. for every $s\in S$. Since the family $\{A_s\}_{s\in S}$ is a σ-discrete cover of X_* by closed A-w.i.d. subspaces, it follows from Theorem F that X_* is A-w.i.d. Thus our theorem is proved.

5.2. Remark. The same argument, employing Theorem G instead of Theorem F and the Vaĭnšteĭn Lemma, proves a more general result, viz., if $f\colon X \to Y$ is an open-and-σ-closed mapping of a metrizable space X onto a metrizable space Y and the subspace K of X_* is closed in X_*, then K is A-w.i.d. if and only if $f(K)$ is A-w.i.d.

The next theorem is a partial counterpart for S-w.i.d. spaces.

5.3. THEOREM. *If $f\colon X \to Y$ is an open-and-closed mapping of a metrizable space X onto a metrizable space Y and no fibre of the mapping f is dense-in-itself, then:*

(i) *if the space X is S-w.i.d., then the space Y is S-w.i.d.;*

(ii) *if the space X_* is S-w.i.d., then the space Y is S-w.i.d.*

Proof. We prove first (i). It follows from the openness of the mapping f that the set $Y_0 = \{y\in Y\colon \operatorname{Int} f^{-1}(y) \neq \emptyset\}$ is open discrete and the Vaĭnšteĭn Lemma (see [7], 4.4.16) implies that the mapping $\tilde{f} = f_{Y\setminus Y_0}\colon f^{-1}(Y\setminus Y_0) \to Y\setminus Y_0$ is open-and-perfect. We let $X_0 = f^{-1}(Y\setminus Y_0)$. It follows from the assumptions of the theorem that the subspace $S(X_0)$ of X_0 is compact (see Remark 2.3) and so is the subspace $\tilde{f}(S(X_0))$ of $Y\setminus Y_0$. It follows from Theorem 5.1 applied to the restriction $\tilde{f}_{\tilde{f}(S(X_0))}\colon \tilde{f}^{-1}[\tilde{f}(S(X_0))] \to \tilde{f}(S(X_0))$ that $\tilde{f}(S(X_0))$ is A-w.i.d. and thus is S-w.i.d. Consider now a closed subspace T of $Y\setminus Y_0$ contained in $(Y\setminus Y_0)\setminus \tilde{f}(S(X_0))$. Since $\tilde{f}^{-1}(T) \subset X_0\setminus S(X_0)$, it follows from Theorem C that $\dim[\tilde{f}^{-1}(T)] < \infty$. It follows from Theorem J applied to the restriction $\tilde{f}_T\colon \tilde{f}^{-1}(T) \to T$ that $\dim T < \infty$. By Theorem C the subspace $Y\setminus Y_0$ of Y is S-w.i.d. The set Y_0 being discrete, one easily checks that Y is S-w.i.d. and thus (i) is proved.

We now prove (ii). Consider the mapping $f_* = f|X_*\colon X_* \to Y_*$. It follows from the assumption of the theorem that the subspace $S(X_*)$ of X_* is compact and so is the subspace $f_*(S(X_*))$ of Y. Since every fibre of the mapping f_* is discrete, it is finite. It follows from Theorem 4.5 applied to the restriction $f_*|S(X_*)\colon S(X_*) \to f_*(S(X_*))$ that $f_*(S(X_*))$ is A-w.i.d., and hence S-w.i.d. Consider now a closed subspace T of Y contained in $Y\setminus f_*(S(X_*))$. Since $f_*^{-1}(T) \subset X_*\setminus S(X_*)$, it follows from Theorem C that $\dim[f_*^{-1}(T)] < \infty$. It follows from Theorem J that $\dim T < \infty$. From Theorem C we infer that Y is S-w.i.d.; thus our theorem is proved.

5.4. Remark. As with Theorem 5.1, Theorem 5.3 can be generalized as follows: if $f\colon X \to Y$ is an open-and-σ-closed mapping of a metrizable space X onto a metrizable space Y, the subspace K of X_* is closed in X_* and K is S-w.i.d., then $f(K)$ is S-w.i.d.

A counterpart for spaces which have $\operatorname{tr}\operatorname{Ind}$ also holds.

5.5. THEOREM. *If $f\colon X \to Y$ is an open-and-closed mapping of a metrizable space X onto a metrizable space Y and no fibre of the mapping f is dense-in-itself, then:*

(i) *if the space X has* $\operatorname{tr}\operatorname{Ind}$, *then the space Y has* $\operatorname{tr}\operatorname{Ind}$;

(ii) *if the space X_* has* $\operatorname{tr}\operatorname{Ind}$, *then the space Y has* $\operatorname{tr}\operatorname{Ind}$.

Proof. We follow the proof of Theorem 5.3. To prove (i) we decompose Y into two disjoint sets Y_0 and Y_1 such that Y_0 is open discrete and the mapping $\tilde{f} = f_{Y_1}\colon f^{-1}(Y_1) \to Y_1$ is open-and-perfect. We let $X_1 = f^{-1}(Y_1)$. It follows from the assumptions of the theorem that the subspace $S(X_1)$ of X_1 is compact and so is the subspace $\tilde{f}(S(X_1))$ of Y_1. The space $S(X_1)$ is c.d. (see [9], Theorem 4.6). It follows from Theorem I applied to the restriction $\tilde{f}_{\tilde{f}(S(X))}\colon \tilde{f}^{-1}[\tilde{f}(S(X_1))] \to \tilde{f}(S(X_1))$ that $\tilde{f}(S(X_1))$ is c.d. and thus

$\operatorname{tr}\operatorname{Ind}\tilde{f}(S(X_1))$ is defined (see [9], Theorem 4.2). Consider now a closed subspace T of Y_1 contained in $Y_1 \setminus \tilde{f}(S(X_1))$. Since $\tilde{f}^{-1}(T) \subset X_1 \setminus S(X_1)$, it follows from Theorem D that $\operatorname{Ind}[\tilde{f}^{-1}(T)] < \infty$. It follows from Theorem J applied to the restriction $\tilde{f}_T$: $\tilde{f}^{-1}(T) \to T$ that $\operatorname{Ind} T < \infty$. From Theorem D we infer that Y_1 has tr Ind. The set Y_0 being discrete, one easily checks that Y has tr Ind; thus (i) is proved.

We now prove (ii). Consider the mapping $f_* = f|X_*$: $X_* \to Y$. It follows from the assumptions of the theorem that the subspace $S(X_*)$ of X_* is compact and so is the subspace $f_*(S(X_*))$ of Y. The space $S(X_*)$ is c.d. (see [9], Theorem 4.6). The finiteness of the fibres of the restriction $f_*|S(X_*)$: $S(X_*) \to f_*(S(X_*))$ implies that $f_*(S(X_*))$ is c.d. (see Remark 4.4). Thus $f_*(S(X_*))$ has tr Ind (see [9], Theorem 4.2). Consider now a closed subspace T of Y contained in $Y \setminus f_*(S(X_*))$. Since $f_*^{-1}(T) \subset X \setminus S(X_*)$, it follows from Theorem D that $\operatorname{Ind}[f_*^{-1}(T)] < \infty$. It follows from Theorem J that $\operatorname{Ind} T < \infty$. From Theorem D we infer that Y has tr Ind and thus our theorem is proved.

5.6. Remark. Arguing as in the above proof, one can prove that if f: $X \to Y$ is an open-and-σ-closed mapping of a metrizable space X onto a metrizable space Y, the subspace K of X_* is closed in X_* and K has tr Ind, then $f(K)$ has tr Ind.

5.7. Remark. The converses of Theorems 5.3 (ii) and 5.5 (ii) do not hold: the subspace X_* of X need not be S-w.i.d. (have tr Ind) in the case where the space Y is S-w.i.d. (has tr Ind). Indeed, let $Z = \bigoplus_{n=1}^{\infty} I^n$ and $Y = Z \cup \{a\}$ be the one-point compactification of Z; consider the space X, which is the Cartesian product $Y \times (I \cup \{2, 2\})$ with the points $(a, 1)$ and $(a, 2)$ identified, and the mapping f: $X \to Y$, which corresponds to the projection of $Y \times (I \cup \{2, 3\})$ onto Y. Clearly, the mapping f is open-and-closed, no fibre of the mapping f is dense-in-itself, the space Y is S-w.i.d. and has tr Ind, but $X_* = (Z \times \{2\}) \cup (Y \times \{3\})$ neither is S-w.i.d. nor has tr Ind.

However, we have the following theorem on the inverse invariance of S-w.i. dimensionality as well as the existence of tr Ind under open-and-closed mappings (cf. Theorems 3.7, 3.8).

5.8. THEOREM. *If f: $X \to Y$ is an open-and-closed mapping of a metrizable space X onto a metrizable space Y, all fibres of the mapping f are discrete and the space Y is S-w.i.d.* (*has* tr Ind), *then the space X is S-w.i.d.* (*has* tr Ind).

Proof. It is enough to give the details of the proof in the first case. As in the proof of Theorem 5.3, we represent Y as the union of two disjoint subsets Y_0 and Y_1 such that Y_0 is open discrete and the mapping $\tilde{f} = f_{Y_1}$: $f^{-1}(Y_1) \to Y_1$ is open-and-perfect. We let $X_1 = f^{-1}(Y_1)$. Since the subspace $S(Y_1)$ of Y_1 is compact (see Remark 2.3), the subspace $\tilde{f}^{-1}(S(Y_1))$ of X_1 is compact. It follows from Theorem 5.1(ii) that $\tilde{f}^{-1}(S(Y_1))$ is A-w.i.d. and thus is S-w.i.d. Consider now a closed subspace T of X_1 contained in $X_1 \setminus \tilde{f}^{-1}(S(Y_1))$. Since $\tilde{f}(T) \subset Y_1 \setminus S(Y_1)$, it follows from Theorem C that $\dim[\tilde{f}(T)] < \infty$. Since all fibres of the restriction $\tilde{f}|T$: $T \to \tilde{f}(T)$ are discrete, $\dim T < \infty$ (see [8], Theorem 4.3.6). From Theorem C we infer that X_1 is S-w.i.d. The subspace $X \setminus X_1$ of X being discrete, one easily checks that X is S-w.i.d.; thus the proof of the first case is concluded.

The proof of the second case, i.e. when Y has tr Ind, is similar. As $S(Y_1)$ is c.d., $\tilde{f}^{-1}(S(Y_1))$ is c.d. (see [9], Theorem 4.6 and either [4], Theorem 5.6 or [10], Section 7). Thus $\tilde{f}^{-1}(S(Y_1))$ has tr Ind (see [9], Theorem 4.2). One easily checks that every closed subspace T of X_1 contained in $X_1 \setminus \tilde{f}^{-1}(S(Y_1))$ is finite-dimensional, so that by virtue of Theorem D $\operatorname{tr}\operatorname{Ind} X_1$ is defined. To show that X has tr Ind it suffices to observe that $X \setminus X_1$ is discrete and to apply Theorem D once again.

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Confluent local expansions

by

Krzysztof Omiljanowski (Wrocław)

Abstract. Every confluent local expansion of an arcwise connected continuum onto itself is open.

In paper [6] Professor I. Rosenholtz has proved that every open local expansion of a continuum onto itself has a fixed point. The following question is asked in [2], Problem 3.1:

(∗) Does there exist a confluent local expansion of a continuum onto itself which is fixed point free?

This paper gives a partial answer to this question: every confluent local expansion of an arcwise connected continuum onto itself is open, and so it has a fixed point. This will follow from two results proved for locally one-to-one mappings. An example shows that this method cannot be extended to continua which are not arcwise connected.

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A continuum means a compact connected metric space. Let X and Y be metric spaces with metrics d_X and d_Y respectively. A continuous surjection $f\colon X \to Y$ is said to be

– a *local expansion* if for each $x \in X$ there is a neighbourhood U of x and a number $M > 1$ such that

$$d_Y(f(y), f(z)) \geqslant M \cdot d_X(y, z) \quad \text{for} \quad y, z \in U$$

(cf. [6]),

– *open* if the image of any open set in X is open in Y,

– *confluent* if for every continuum $Q \subset Y$ and for every component C of $f^{-1}(Q)$ we have $f(C) = Q$,

– *locally one-to-one* if for each point $x \in X$ there is an open neighbourhood U of x such that the restriction $f|_U$ is one-to-one.

Let us recall (see [1], VI, p. 214) that

(i) any open mapping of a compact space is confluent.