Remarks on intrinsic isometries

by

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Abstract. A map \( f : A \to A' \) of metric spaces is said to be an intrinsic isometry if it preserves the length of every arc. It is shown in this note that the Euclidean \( n \)-space \( \mathbb{E}^n \) is intrinsically isometric to a subset \( \mathcal{A} \) of \( \mathbb{E}^{n+1} \) with arbitrarily small diameter \( \delta(\mathcal{A}) \). We also consider the intrinsic metric of a product of metric spaces.

1. Introduction. The notion of the intrinsic metric for metric spaces and related notions were introduced by K. Borsuk [1]. Let us say that a space \( A \) (with metric \( \delta \)) is geometrically acceptable (notation: \( A \in \text{GA} \)) if

(1.1) for every two points \( x, y \in A \) there exists an arc \( L \subset A \) with finite length such that \( x, y \in L \)

and

(1.2) for every point \( x \in A \) and for every \( \varepsilon > 0 \) there is a neighborhood \( U \) of \( x \) in \( A \) such that for every point \( y \in U \) there exists in \( A \) an arc \( L \) containing the two points \( x, y \) and such that the length \( |L| < \varepsilon \).

Then setting

(1.3) \( \delta_a(x, y) = \text{lower bound of the length of all arcs } L \subset A \text{ containing the two points } x, y \),

one gets a metric \( \delta_a \) in \( A \) called the intrinsic metric in \( A \). The topology in \( A \in \text{GA} \) induced by the metric \( \delta_a \) is the same as the topology induced by the metric \( \delta \).

A function \( f \) mapping a GA-space \( A \) onto another GA-space \( A' \) is said to be an intrinsic isometry provided

(1.4) \( \delta_a(x, y) = \delta_a(f(x), f(y)) \) for every \( x, y \in A \).

A map \( f \) is an intrinsic isometry if and only if it preserves the length of every arc. Every intrinsic isometry is a homeomorphism.

K. Borsuk has proved [1] that for every \( \varepsilon > 0 \) there exists an intrinsic isometry mapping the Euclidean \( n \)-space \( \mathbb{E}^n \) onto a subset \( \mathcal{A} \subset \mathbb{E}^{n+1} \) such that the diameter of \( \mathcal{A} \) (by the usual metric in \( \mathbb{E}^{n+1} \)) is less than \( \varepsilon \). We will prove the following
(1.5) Theorem. For every \( \varepsilon > 0 \) there exists a subset \( A \) of the \((n+1)\)-space \( E^{n+1} \) with a diameter less than \( \varepsilon \) which is intrinsically isometric to the Euclidean \( n \)-space \( E^n \).

Let \( A \) and \( B \) be \( GA \)-spaces with metrics \( q' \) and \( q'' \), respectively. Then the product \( A \times B \) with the metric \( q \) given by

\[
q((a_1, b_1), (a_2, b_2)) = \sqrt{(q'(a_1, a_2))^2 + (q''(b_1, b_2))^2}
\]

is a \( GA \)-space. The intrinsic \( q_{A \times B} \) in \( A \times B \) is given by

\[
q_{A \times B}((a_1, b_1), (a_2, b_2)) = \sqrt{(q'_A(a_1, a_2))^2 + (q''_B(b_1, b_2))^2}
\]

(see Theorem (3.7)). It follows that if \( f: A \to A' \) and \( g: B \to B' \) are intrinsic isometries of \( GA \)-spaces, then the map \( f \times g: A \times B \to A' \times B' \) is an intrinsic isometry.

2. The Euclidean \( n \)-space \( E^n \) is intrinsically isometric to a subset of \( E^{n+1} \) with a small diameter. We denote by \( I \) the set of all integers. Let \( F \) be the union of segments \( A_i \) in \( E^n (i \in I) \) such that the intersection \( A_i \cap A_{i+1} \) is a point and segments \( A_i \) and \( A_j \) are disjoint if \( |i-j| > 1 \). We say that the broken line \( F \) is obtained by the reflection of a straight line \( K \) relative to a family \( \{H_i\}_{i \in I} \) of hyperplanes \((n-1)\)-dimensional hyperplanes in \( E^n \) if the following conditions are satisfied:

\[
\begin{align*}
(1.1) \quad & A_0 \subset K, \\
(1.2) \quad & A_{i-1} \cap A_i \subset H_i, \\
(1.3) \quad & H_i \text{ is perpendicular to the bisectrix of the angle between the segments } A_{i-1} \text{ and } A_i, \\
(1.4) \quad & F \text{ is intrinsically isometric to } K.
\end{align*}
\]

For positive real numbers \( a \) and \( b \) we define

\[
p_i = \left( \frac{i}{|i|+1} a, (-1)^i b \right)
\]

(2.5) Lemma. For any positive real numbers \( a \) and \( b \) there exist positive real numbers \( \alpha \) and \( \beta \) such that for any straight line \( K' \) parallel to the segment \( A_0(a, b) \) where the distance of this segment from \( K' \) is less than \( \delta \) the broken line \( F' \) which is obtained by the reflection of \( K' \) relative to the family \( \{L_{i}(a, b)\}_{i \in I} \) has a diameter \( \delta(F') \) less than \( \delta \). The sets \( F' \) and \( f(F) \) are disjoint for any nontrivial translation \( f \) of the direction of the second axis.

Now we will prove the following

(2.6) Lemma. Let \( X \) be a subset of the Euclidean \( n \)-space \( E^n \) which does not intersect the set \( f(X) \) for any nontrivial translation \( f \) of the direction of some vector \( a \). Then for any \( \varepsilon > 0 \) there exists an embedding

\[
g: X \times E \to E^{n+1}
\]

which is an intrinsic isometry such that

\[
(2.7) \quad \delta(g(X \times E)) \leq \delta(X) + 2\varepsilon.
\]
(2.8) the sets $g(X \times E)$ and $f'(g(X \times E))$ are disjoint for any nontrivial translation $f'$ in the direction of some vector $\xi$.

Proof. We will consider $E^n$ as a subset of $E^{n+1}$. Let $P$ be a plane in $E^{n+1}$ parallel to the $(n+1)$-axis and to the vector $a$ such that the intersection $P \cap X$ is nonempty (any such plane intersects the set $X$ in at most one point). For $x$ and $y = \delta(x)$ we find positive real numbers $a$ and $b$ which satisfy the conditions of Lemma (2.5). Let $F(a, b) = \bigcup_{i \in I} A_i(a, b)$ be a broken line in $P$ defined as above in such a way that the straight line which contains the segment $A_i(a, b)$ intersects the set $X$ and is parallel to the $(n+1)$-axis of $E^{n+1}$. Let $L_i(a, b), i \in I$, be the straight line in the plane $P$ defined as above. By $H_i(a, b), i \in I$, we denote the $n$-dimensional hyperplane in $E^{n+1}$ with the image $L_i(a, b)$ under the orthogonal projection onto the plane $P$.

The straight line parallel to the $(n+1)$-axis which contains a point $x \in X$ we denote by $K(x)$. Let $F(x)$ be the broken line which is obtained by the reflection of the straight line $K(x)$ relative to the family $\{H_i(a, b)\}_{i \in I}$. Observe that $F(x)$ is contained in the plane $P(x) = E^{n+1}$ which is parallel to $P$ and which contains the line $K(x)$. We know that $P(x) \cap X = \{x\}$; thus $F(x)$ and $F'(x)$ are disjoint if $x$ and $x'$ are different points of $X$. By Lemma (2.5) the diameter $\delta(F(x))$ is less than $\varepsilon$ for every point $x \in X$. Thus the diameter of the set $Y = \bigcup F(x)$ is not greater than $\delta(X) + 2\varepsilon$.

Let $g_\varepsilon$ be the intrinsic isometry of $K(x)$ onto $F(x)$ which is the identity on the segment of $K(x)$ between the hyperplanes $H_0(a, b)$ and $H_1(a, b)$.

The embedding

$$g : X \times E \to Y \subset E^{n+1}$$

is an intrinsic isometry since $g$ restricted to any arc in $X \times E$ is a composition of the reflections relative to some hyperplanes $H_i(a, b)$.

Observe that for any nontrivial translation $f'$ in the direction of the bisectrix of the angle between the segments $A_{-1}(a, b)$ and $A_0(a, b)$ the sets $Y$ and $f'(Y)$ are disjoint.

From Lemma (2.6) one can obtain (by induction) Theorem (1.5).

It is easy to see [1] that there exists a smooth embedding $g : E^n \to Y \subset E^n$ which is an intrinsic isometry such that the diameter $\delta(Y)$ of $Y$ is small. There is no smooth intrinsic isometry $g : E^n \to E^{n+1}$ with the diameter $\delta(Y)$ finite, $n > 1$. This follows by the Hartman–Nirenberg theorem of [2] (see Theorem (5.3) Chapter VI, [3]). Let us formulate the following question:

(2.9) Is it true that there is no smooth intrinsic isometry $g : E^n \to E^n$ with the diameter $\delta(Y)$ finite if $m$ is less than $2n$?

3. The intrinsic metric of the product of metric spaces. By $R^+$ we denote the set of all nonnegative real numbers. We will consider functions $f : R^+ \times R^+ \to R^+$ which satisfy the following conditions:

(3.1) $f(s_1, t_1) + f(s_2, t_2) \geq f(s_1 + s_2, t_1 + t_2)$

(3.2) $f(s_1, t_1) \leq f(s_2, t_2)$ if $s_1 \leq s_2$ and $t_1 \leq t_2$.

(3.3) $f(s, at) = af(s, t)$.

(3.4) $f(s, t) = 0$ if and only if $s = 0 = t$.

Let $X$ and $Y$ be metric spaces with metrics $g'$ and $g''$ respectively. If $f : R^+ \times R^+ \to R^+$ is a function which satisfies conditions (3.1)-(3.4), then setting

$$\varrho((x_1, y_1), (x_2, y_2)) = f(g'(x_1, x_2), g''(y_1, y_2))$$

one gets a metric $\varrho$ on the product $X \times Y$ which induced the product topology. Observe that the function $f : R^+ \times R^+ \to R^+$ defined by

$$f(s, t) = \sqrt{s^2 + t^2}$$

satisfies conditions (3.1)-(3.4).

Now we will prove the following

(3.7) Theorem. Let $g'$ and $g''$ be the intrinsic metrics in $GA$-spaces $(X, g')$ and $(Y, g'')$ respectively. Let $g$ be the metric in the product $X \times Y$ given by (3.5), where $f : R^+ \times R^+ \to R^+$ is a function satisfying conditions (3.1)-(3.4). Then the intrinsic metric in the metric space $(X \times Y, g)$ is given by

$$\varrho_{x \times y}((x_1, y_1), (x_2, y_2)) = f(g(x_1, x_2), g(y_1, y_2)).$$

Proof. Let $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ be points in $X \times Y$. Let $L$ be an arc lying in $X \times Y$ and joining the points $z_1$ and $z_2$ with a parametric representation given by a homeomorphism $h : (0, 1) \to L$.

Let $h_x = p_x \circ h$ and $h_y = p_y \circ h$, where $p_x$ and $p_y$ are the natural projections of $X \times Y$ onto $X$ and $Y$ respectively. For every $\varepsilon > 0$ there exists a sequence $0 = t_0 < t_1 < \ldots < t_k = 1$ such that

$$\sum_{i=1}^{k} d_i^x > g_x(x_1, x_2) - \varepsilon \quad \text{and} \quad \sum_{i=1}^{k} d_i^y > (y_1, y_2) - \varepsilon$$

where

$$d_i^x = g'(h_x(t_i), h_x(t_{i-1})) \quad \text{and} \quad d_i^y = g''(h_y(t_i), h_y(t_{i-1})).$$
By conditions (3.1) and (3.2) we obtain
\[ |L| \geq \sum_{i=1}^{k} \frac{g(h(t_i), h(t_{i-1}))}{f(d', d'')} = \sum_{i=1}^{k} f(d', d'') \]
\[ \geq f\left( \sum_{i=1}^{k} d', \sum_{i=1}^{k} d'' \right) \geq f(\omega(x_1, x_2) - \varepsilon, \omega(y_1, y_2) - \varepsilon). \]

Since \( f \) is continuous, we obtain
\[ |L| \geq f(\omega(x_1, x_2), \omega(y_1, y_2)). \]

Thus
\[ |L| \geq f(\omega(x_1, x_2), \omega(y_1, y_2)). \quad (3.9) \]

For every \( \varepsilon > 0 \) there exists \( \omega \) joining \( x_1 \) and \( x_2 \) in \( Y \) joining \( y_1 \) and \( y_2 \) such that
\[ |L| + \varepsilon \geq |L'| \quad \text{and} \quad |L'| + \varepsilon \geq |L''. \]

Let the parametric representations
\[ h': (0, u) \to L' \quad \text{and} \quad h'': (0, v) \to L'' \]
be intrinsic isometries. Let
\[ \phi: (0, 1) \to (0, u) \times (0, v) \]
be a map given by \( \phi(t) = (t, u, v). \) Let \( \Phi = (h' \times h'') \circ \phi \), i.e.
\[ \Phi(t) = (h'(t, u), h''(t, v)). \]
For every \( \delta > 0 \) there exists a sequence \( t_0 < t_1 < \ldots < t_k = 1 \) such that
\[ |L| < \sum_{i=1}^{k} \frac{g(\Phi(t_i), \Phi(t_{i-1}))}{|L'| + \delta}. \]

For every \( \delta > 0 \). By conditions (3.2) and (3.10), it follows that
\[ |L| \leq \frac{1}{f(|L'|, |L'locked|)} \leq f(\omega(x_1, x_2) + \varepsilon, \omega(y_1, y_2) + \varepsilon). \]

Since \( \omega(x, y) \leq |L| \), we obtain
\[ \omega(x, y) \leq f(\omega(x_1, x_2) + \varepsilon, \omega(y_1, y_2) + \varepsilon) \]
for every \( \varepsilon > 0 \). Since \( f \) is continuous, we obtain
\[ |L'| \leq f(\omega(x_1, x_2), \omega(y_1, y_2)). \quad (3.11) \]

From Theorem (3.7) follows
(3.12) Corollary. If \( f': X' \to Y' \) and \( f'': X'' \to Y'' \) are intrinsic isometries of \( GA \)-spaces, then the map \( f' \times f'' \) is an intrinsic isometry.

References


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