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A sum theorem for A-weakly infinite-dimensional spaces

by

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Abstract. In this note we shall establish a hereditarily closure-preserving sum theorem for A-weakly infinite-dimensional spaces. The applications of this theorem to the closed mappings defined on A-weakly infinite-dimensional spaces are given in [5].

Our terminology and notation follow [2]. Let us recall that a normal space X is said to be *A-weakly infinite-dimensional* (abbrev. A-w.i.d.) if for every sequence $(A_1, B_1), (A_2, B_2), \ldots$ of pairs of disjoint closed subsets of X there exists a sequence $L_1, L_2, \ldots$ of closed subsets of X such that, for each positive integer i, the set L_i is a partition between A_i and B_i in X (meaning that there exist disjoint open subsets U_i, V_i of X such that $A_i \subset U_i$, $B_i \subset V_i$ and $X \setminus L_i = U_i \cup V_i$), and $\bigcap_{i=1}^{\infty} L_i = \emptyset$. It is manifest that every closed subspace of an A-w.i.d. space is A-w.i.d.

We begin with the following obvious lemma (cf. the proof of Lemma 1.2.9 in [2]).

LEMMA 1. *Let F be a closed subspace of a hereditarily normal space X and A, B a pair of disjoint closed subsets of X. For every partition L between $A \cap F$ and $B \cap F$ in F with $F \setminus L = G \cup H$, where disjoint open subsets G, H of F are such that $A \cap F \subset G$ and $B \cap F \subset H$, there exists a partition L' between A and B in X with $X \setminus L' = M \cup N$, where disjoint open subsets M, N of X are such that $A \subset M$, $B \subset N$, $M \cap F = G$ and $N \cap F = H$.*

The next lemma deals with countable families of partitions.

LEMMA 2. *Let F be a closed subspace of a hereditarily normal A-w.i.d. space X and $(A_1, B_1), (A_2, B_2), \ldots$ a sequence of pairs of disjoint closed subsets of X. For every sequence $L_1, L_2, \ldots$, where L_i is a partition between $A_i \cap F$ and $B_i \cap F$ in F for $i = 1, 2, \ldots$, such that $\bigcap_{i=1}^{\infty} L_i = \emptyset$, there exists a sequence*

$L'_1, L'_2, \ldots$, *where* L'_i *is a partition between* A_i *and* B_i *in* X *for* $i = 1, 2, \ldots$, *with the properties*:

(i) $L'_i \cap F = L_i$ *for* $i = 1, 2, \ldots$;

(ii) $\bigcap_{i=1}^{\infty} L'_i = \emptyset$.

Proof. By Lemma 1, for each positive integer i, there exists a partition S_i between A_i and B_i in X such that $S_i \cap F = L_i$. By hypothesis, $F \cap \bigcap_{i=1}^{\infty} S_i = \emptyset$; by the normality of X there exists a closed G_δ-set P such that $F \subset P \subset X \setminus \bigcap_{i=1}^{\infty} S_i$. Let $X \setminus P = \bigcup_{j=1}^{\infty} Z_j$, where Z_j is closed in X for $j = 1, 2, \ldots$

We decompose the set of positive integers into disjoint infinite subsets $N_1, N_2, \ldots$ (cf. Levšenko [4]). For each positive integer j, the subspace Z_j is A-w.i.d., and thus there exists a sequence $\{J_k\}_{k \in N_j}$, where J_k is a partition between $A_k \cap Z_j$ and $B_k \cap Z_j$ in Z_j for each $k \in N_j$, with the property that $\bigcap_{k \in N_j} J_k = \emptyset$. For each positive integer j, the subspace $P \cup Z_j$ is the discrete union of the subspaces P and Z_j, and hence $(S_k \cap P) \cup J_k$ is a partition between $A_k \cap (P \cup Z_j)$ and $B_k \cap (P \cup Z_j)$ in $P \cup Z_j$ for each $k \in N_j$. By Lemma 1, for each positive integer j and each $k \in N_j$, there exists a partition L'_k between A_k and B_k in X such that

$$L'_k \cap (P \cup Z_j) = (S_k \cap P) \cup J_k.$$

It follows that $L'_i \cap F = L_i$ for $i = 1, 2, \ldots$, so that we have (i). Clearly, $\bigcap_{i=1}^{\infty} L'_i = \emptyset$ and the lemma is proved.

We now pass to our sum theorem. In this theorem the notion of a hereditarily closure-preserving family appears; let us recall that a family $\{A_s\}_{s \in S}$ of subsets of a topological space X is said to be *hereditarily closure-preserving* if for every family $\{B_s\}_{s \in S}$, where $B_s \subset A_s$ for every $s \in S$, we have $\overline{\bigcup_{s \in S} B_s} = \bigcup_{s \in S} \bar{B}_s$. Let us note that every locally finite family is hereditarily closure-preserving.

THEOREM 1. *If a hereditarily normal space* X *can be represented as the union of a hereditarily closure-preserving family* $\{F_s\}_{s \in S}$ *of closed* A-*w.i.d. subspaces, then the space* X *is* A-w.i.d.

Proof. Let $(A_1, B_1), (A_2, B_2), \ldots$ be a sequence of pairs of disjoint closed subsets of X. We have to show that there exists a sequence $L_1, L_2, \ldots$, where L_i is a partition between A_i and B_i in X for $i = 1, 2, \ldots$, such that $\bigcap_{i=1}^{\infty} L_i = \emptyset$.

For each $T \subset S$, we let $X_T = \bigcup_{s \in T} F_s$. We consider the set $\mathscr{S}$ consisting of pairs $(T, \{(U_{T,i}, V_{T,i})\}_{i=1}^{\infty})$, where $T \subset S$ and the sequence $(U_{T,1}, V_{T,1})$, $(U_{T,2}, V_{T,2}), \ldots$ of pairs of open subsets of X_T satisfies the following conditions:

(1) $$U_{T,i} \cap V_{T,i} = \emptyset \quad \text{for} \quad i = 1, 2, \ldots;$$

(2) $$A_i \cap X_T \subset U_{T,i} \quad \text{and} \quad B_i \cap X_T \subset V_{T,i} \quad \text{for} \quad i = 1, 2, \ldots;$$

(3) $$\bigcup_{i=1}^{\infty} (U_{T,i} \cup V_{T,i}) = X_T.$$

We now define an order $\leqslant$ in $\mathscr{S}$ as follows: $(T, \{(U_{T,i}, V_{T,i})\}_{i=1}^{\infty}) \leqslant (T', \{(U_{T',i}, V_{T',i})\}_{i=1}^{\infty})$ if and only if $T \subset T'$, $U_{T',i} \cap X_T = U_{T,i}$ and $V_{T',i} \cap X_T = V_{T,i}$ for $i = 1, 2, \ldots$ If a subset $\mathscr{A}$ of the set $\mathscr{S}$ ordered by $\leqslant$ is linearly ordered, then the pair

$$(T(\mathscr{A}), \{(U_{T(\mathscr{A}),i}, V_{T(\mathscr{A}),i})\}_{i=1}^{\infty}),$$

where

$$T(\mathscr{A}) = \bigcup \{T\colon (T, \{(U_{T,i}, V_{T,i})\}_{i=1}^{\infty}) \in \mathscr{A}\},$$

$$U_{T(\mathscr{A}),i} = \bigcup \{U_{T,i}\colon (T, \{(U_{T,i}, V_{T,i})\}_{i=1}^{\infty} \in \mathscr{A}\}$$

and

$$V_{T(\mathscr{A}),i} = \bigcup \{V_{T,i}\colon (T, \{(U_{T,i}, V_{T,i})\}_{i=1}^{\infty}) \in \mathscr{A}\} \quad \text{for} \quad i = 1, 2, \ldots,$$

satisfies conditions (1)–(3) with $T = T(\mathscr{A})$. Thus, by the Kuratowski–Zorn lemma, there exists an element $(T_0, \{(U_{T_0,i}, V_{T_0,i})\}_{i=1}^{\infty})$, maximal in $\mathscr{S}$ ordered by $\leqslant$.

It follows from the maximality of $(T_0, \{(U_{T_0,i}, V_{T_0,i})\}_{i=1}^{\infty})$ and Lemma 2 that $X_{T_0} = X$, and thus we conclude the proof by letting $L_i = X_{T_0} \setminus (U_{T_0,i} \cup V_{T_0,i})$ for $i = 1, 2, \ldots$

The first result along the lines of Theorem 1 is due to Levšenko [4], who proved, under the additional assumption that the space X is countably paracompact, shown to be redundant by van Douwen [1], the *contable sum theorem*: *if a normal space* X *can be represented as the union of a sequence* $F_1, F_2, \ldots$ *of closed* A-*w.i.d. subspaces, then the space* X *is* A-*w.i.d.* Let us add that the following *locally countable sum theorem* was proved by Leĭbo [3]: *if a normal weakly paracompact space* X *can be represented as the union of a locally countable family of closed* A-*w.i.d. subspaces, then the space* X *is* A-*w.i.d.* The assumption that the space X is weakly paracompact, however, permits us to reduce the case of locally countable families to the case of countable families.

We close this note with observing that Theorem 1, along with the countable sum theorem, yields the following corollary. Let us recall that a family $\{A_s\}_{s \in S}$ of subsets of a topological space X is said to be *σ-hereditarily closure-preserving* if the family $\{A_s\}_{s \in S}$ can be represented as the union of countably many subfamilies each of which is a hereditarily closure-preserving family of subsets of X.

Corollary 1. *If a hereditarily normal space X can be represented as the union of a σ-hereditarily closure-preserving family $\{A_s\}_{s\in S}$ of F_σ-sets in X and A_s is A-w.i.d. for each $s\in S$, then the space X is A-w.i.d.*

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Some theorems on invariance of infinite dimension under open and closed mappings

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Abstract. We discuss here the invariance and the inverse invariance of three classes of spaces: A-weakly infinite-dimensional spaces, S-weakly infinite-dimensional spaces and spaces which have the transfinite dimension trInd under three classes of mappings: open mappings with finite fibres, closed mappings with finite fibres and open-and-closed mappings with no fibres dense-in-themselves; finite-dimensional spaces, countable-dimensional spaces and strongly countable-dimensional spaces also occasionally appear in our paper.

Our terminology and notation follow [7] and [8]. We shall quote from [7] and [8] all the necessary theorems of general topology and dimension theory. Similarly, we shall quote from [9] when the transfinite dimension trInd is concerned.

1. Definitions. We start with the definitions of the most important notions to be used in the sequel.

1.1. Definition. Let X be a space and A, B a pair of disjoint closed subsets of X; a closed subset L of X is said to be a *partition between A and B* if there exist open subsets U, V of X which satisfy the conditions

$$A\subset U,\quad B\subset V,\quad U\cap V=\emptyset\quad\text{and}\quad X\setminus L=U\cup V.$$

1.2. Definition. A normal space X is said to be *A-weakly infinite-dimensional* (abbrev. A-w.i.d.) if for every sequence $(A_1, B_1), (A_2, B_2), \ldots$ of pairs of disjoint closed subsets of X there exists a sequence $L_1, L_2, \ldots$, where L_i is a partition between A_i and B_i for $i=1, 2, \ldots$, with the property that $\bigcap_{i=1}^{\infty} L_i=\emptyset$.

1.3. Definition. A Tychonoff space X is said to be *S-weakly infinite-dimensional* (abbrev. S-w.i.d.) if for every sequence $(A_1, B_1), (A_2, B_2), \ldots$ of pairs of disjoint functionally closed subsets of X there exists a sequence $L_1, L_2, \ldots$ where the functionally closed subset L_i of X is a partition between