Decomposition spaces having arbitrarily small neighborhoods with 2-sphere boundaries II

by

Edythe P. Woodruff* (Princeton, N. J.)

Abstract. Let G be an usc decomposition of S^3, H be the collection of nondegenerate elements, and P be the natural projection of S^3 onto S^3/G. Suppose that each $g_0 \in H$ has an arbitrarily small neighborhood with a (possibly wild) 2-sphere boundary S and that S contains every $g \in H$ which it intersects. Then any one of the three following additional conditions implies that S^3/G is homeomorphic to S^3. (1) S is tame from one side. (2) The wildness from Ext S and Int S can be described by certain F_σ-sets F_1 and F_2, respectively, for which $P(F_1)$ and $P(F_2)$ are disjoint 0-dimensional sets in S^3/G. (3) G is a compact, 0-dimensional decomposition and the wildness from the two sides can be described by certain F_σ-sets F_1 and F_2 with the property that no $g \in H$ intersects both F_1 and F_2. Also proved is a theorem which states that S^3/G is homeomorphic to S^3 if each $g \in H$ lies in an arbitrarily small open neighborhood U such that $P(\text{Bd } U)$ is a 2-sphere and the new usc decomposition G' of S^3 whose $H' = \{g \in H\colon\ g \cap \text{Bd } U \neq \emptyset\}$ itself yields a decomposition space S^3/G' which is homeomorphic to S^3.

1. Introduction

1.1. Preliminaries. Definitions and notations are below in (1.2) and the motivation for this paper is given in (1.3). Section 2 concerns techniques used in later proofs. In Sections 3, 4, and 5 we state and prove our results.

1.2. Definitions and notations. All decompositions used are upper semicontinuous (usc) by a standard definition such as in Whyburn [Wh]. For a decomposition G of S^3, the set of nondegenerate elements is denoted by H, and the natural projection of S^3 onto S^3/G by P. A subset $A \subset S^3$ is called *G-saturated* if for $g \in G$, either $g \cap A = \emptyset$ or $g \subset A$. For a set $T \subset S^n$, let Sat T denote $\{p \in S^n\colon\ p$ is a point in some $g \in G$ which intersects $T\}$.

For any collection C of sets, $C^* = \{x \in C\}$.

A decomposition is called a *compact 0-dimensional decomposition* if $P(\text{Cl } H^*)$ is a compact 0-dimensional set.

A sequence $\{M_i\}$ of 3-manifolds with boundary in S^3 is called a *defining sequence* for a decomposition G of S^3 if and only if for each i, $M_{i+1} \subset \text{Int } M_i$, and the elements of H are the nondegenerate components of $\bigcap_{i=1}^{\infty} M_i$.

---

* Partially supported by NSF Grant MCS-7909542.

A *crumpled cube* is the closure of either component of the complement of a (possibly wild) 2-sphere in S^3.

Let A be an annulus bounded by 2-spheres S_1 and S_2. A homeomorphism ξ taking S_1 onto S_2 is called *admissible* if there exists a homotopy $H\colon S^2\times I\to A$ such that $H(S^2\times 0)=S_1$; $H(S^2\times 1)=S_2$; and for $x\in S^2$ if $H(x\times 0)=p\in S_1$, then $H(x\times 1)=\xi(p)\in S_2$.

1.3. Motivation. In "Decomposition spaces having arbitrarily small neighborhood with 2-sphere boundaries" [W] the author proved:

THEOREM. *Let G be an usc decomposition of S^3. Suppose that for any $g\in H$, and open set U containing g there is a crumpled cube X such that $g\subset$ Int $X\subset U$, and* Bd *X misses H^*. Then S^3/G is homeomorphic to S^3.*

Armentrout [A3] has shown that there is an usc decomposition G of S^3 such that S^3/G is not homeomorphic to S^3 although it shares the following property with decompositions satisfying the above theorem. For every $g\in G$ and open set U containing g there is a set V containing g in its interior and having the property that $P(\text{Bd } V)$ is a 2-sphere S in S^3/G. In this example, for each $g\in H$ Armentrout exhibits a set $P^{-1}(S)$ which is a 2-sphere unioned with a Cantor set of piercing arcs.

In the original version of this paper I conjectured that S^3/G would be homeomorphic to S^3 if every $g\in H$ has an arbitrarily small neighborhood V such that Bd V is a G-saturated 2-sphere. When Steve Armentrout considered this conjecture, he wrote [A4] in which he shows that the example in [A3] is a counterexample to this conjecture.

In this paper we present results which lie between the theorem [W] and Armentrout's example [A4]. We prove that for a decomposition satisfying the condition that every $g\in H$ has an arbitrarily small neighborhood V such that Bd V is a G-saturated 2-sphere the space S^3/G is homeomorphic to S^3 if one adds either (a) Bd V is tame from one side, or (b) the nondegenerate elements in Bd V have a certain nice placement with respect to any wildness in Bd V.

The last theorem in the paper replaces the assumption that Bd V is a 2-sphere by $P(\text{Bd } V)$ is a 2-sphere; and requires that for the new usc decomposition G' of S^3 into points and $\{g\in H\colon g\subset \text{Bd } V\}$, the decomposition space S^3/G' is homeomorphic to S^3.

The reader should notice that if the 2-sphere theorem in [W] applies to a decomposition G then $P(H)$ is 0-dimensional for that G. This is not necessarily true for the decompositions considered in this paper. Here, $P(H)$ can even be 3-dimensional.

2. Techniques and lemmas

2.1. Shrinkability. We will prove each of the theorems in this paper by showing that a given decomposition is shrinkable.

The basic concepts in the definition and theorem below were originally given by McAuley in [Mc1] and [Mc2]. Although there is a problem in [Mc2] involving a generalized definition of upper semicontinuity and shrinkability, his theorem is correct if the standard definition of usc is used. Concerning this, see the theorem by Reed in her thesis [R]. The version we use is from her work and provable using methods in [Mc1].

DEFINITION. Suppose that G is a decomposition of S^3. We say that H *is shrinkable in* S^3 if for each G-saturated open cover $\mathcal{U}$ of H^*, homeomorphism φ of S^3 onto S^3, and $\eta>0$; there exists a $\varphi(G)$-saturated open cover $\mathcal{W}$ of H^* that refines $\mathcal{U}$ and a homeomorphism f_η of S^3 onto S^3 such that (1) $f_\eta|S^3-\mathcal{U}^* = \varphi|S^3-\mathcal{U}^*$, (2) for each $g\in H$, Diam $f_\eta(g)<\eta$, and (3) for each $W\in\mathcal{W}$ there exists $U\in\mathcal{U}$ such that $\varphi(W)\cup f_\eta(W)\subset\varphi(U)$.

SHRINKABILITY THEOREM (McAuley, Reed). *If H is shrinkable in S^3, then S^3/G is homeomorphic to S^3.*

2.2. Existence of a shrinking homeomorphism. The following lemma uses a hypothesized local condition for a decomposition G to show that H is shrinkable.

LEMMA 2.1. *Suppose that G is an usc decomposition of S^3 such that S^3/G is 3-dimensional, for each $g_0\in G$ there are arbitrarily small G-saturated open sets X^1 and X^3 with $g_0\subset X^1$ and* Cl *$X^1\subset X^3$; and for each $\varepsilon>0$, each G-saturated closed set F, and each homeomorphism α of S^3 onto S^3 for which $g\in G$ and $g\subset F$ imply that* Diam *$\alpha(g)<\varepsilon$ there exists a homeomorphism h of S^3 onto S^3 such that*

(1) $h|\alpha(S^3-X^3)=\text{id}$,

(2) *if $g\in G$ and $g\subset X^1$, then* Diam *$h\alpha(g)<\varepsilon/2$, and*

(3) *if $g\in G$ and $g\subset F$,* Diam *$h\alpha(g)<\varepsilon$.*

Then G is shrinkable.

Proof. The Shrinkability Theorem hypotheses give us a G-saturated open cover $\mathcal{U}$ of H^*, a homeomorphism φ of S^3 onto S^3, and $\eta>0$. Choose ε_0 such that if p, $q\in S^3$, and the distance dist $(p,q)<\varepsilon_0/2$, then dist $(\varphi(p),\varphi(q))<\eta/2$. Denote the collection of "large" elements by $H_L = \{g\in H\colon \text{Diam } g\geqslant\varepsilon\}$.

From $\mathcal{U}\cup(S^3-H_L^*)$ choose a minimal finite subcover $\hat{\mathcal{U}}$ of S^3. Let $P(\hat{\mathcal{U}})$ denote the open cover $\{P(U)\colon U\in\hat{\mathcal{U}}\}$ of S^3/G. The decomposition space S^3/G is a compact metric space (although we do not yet know that it is homeomorphic to S^3). Hence, there is a Lebesgue number d for the cover $P(\hat{\mathcal{U}})$. Let $\mathcal{V}$ be any minimal finite open cover of S^3/G such that for each $V\in\mathcal{V}$, Diam $V<d/9$.

The decomposition space S^3/G is also 3-dimensional. Therefore, there is an open refinement $\mathcal{Y}$ of $\mathcal{V}$ such that $\mathcal{Y}$ splits as $\mathcal{Y}=\mathcal{Y}_a\cup\mathcal{Y}_b\cup\mathcal{Y}_c\cup\mathcal{Y}_d$ where the elements in each $\mathcal{Y}_k$ are pairwise disjoint [H, W, pp. 54, 55]. Let $\mathcal{Z}$ denote the collection $\{P^{-1}(Y)\colon Y\in\mathcal{Y}_a\}$. We note that if there is a $g\in H_L$ in Z_0, then Z_0 is in some member of $\mathcal{U}$. The cover $\mathcal{Z}$ is a G-saturated open cover of S^3.

For each $g\in H_L$ choose G-saturated open sets W_g^1 and W_g^3 such that $g\subset W_g^1$, Cl $W_g^1\subset W_g^3$, and $W_g^3\subset V$ for some $Z\in\mathcal{Z}$. From $\{W_g^1\colon g\in H_L\}$ choose a minimal finite subcover $\mathscr{W}^1$ of H_L. There is the corresponding cover $\mathscr{W}^3 = \{W_g^3$: there is a g such that $W_g^1 \in \mathscr{W}^1\}$.

Since the Shrinkability Theorem requires a cover $\mathscr{W}$ of all of H^*, we will augment $\mathscr{W}^1$. Let $\mathscr{W}' = \{Z \cap (\mathscr{U}^* - H_L^*)$: $Z \in \mathscr{Z}\}$. Let $\mathscr{W} = \mathscr{W}^1 \cup \mathscr{W}'$. It is a G-saturated open cover of H^*.

Order the elements of $\mathscr{W}$ so that those of $\mathscr{W}^1$ precede those of $\mathscr{W}'$.

Within $\mathscr{W}^1$ require the ordering to use all sets in $\mathscr{W}_a = \{W \in \mathscr{W}^1$: W corresponds to an element of $\mathscr{W}^3$ which is contained in $P^{-1}(\mathscr{Z}_a^*)\}$, then use all sets in $\mathscr{W}_b = \{W \in \mathscr{W}^1$: W corresponds to an element of $\mathscr{W}^3$ which is contained in $P^{-1}(\mathscr{Z}_b^* - \mathscr{Z}_a^*)\}$, then use all sets in $\mathscr{W}_c = \{W \in \mathscr{W}^1$: W corresponds to an element contained in $P^{-1}(\mathscr{Z}_c^* - \mathscr{Z}_a^* - \mathscr{Z}_b^*)\}$, and finally use all sets in $\mathscr{W}_d = \{W \in \mathscr{W}^1$: W corresponds to an element contained in $P^{-1}(\mathscr{Z}_d^* - \mathscr{Z}_a^* - \mathscr{Z}_b^* - \mathscr{Z}_c^*)\}$. Give each $W^1 \in \mathscr{W}^1$ a subscript i corresponding to the ordering. Assume there are m sets in W^1.

The shrinking homeomorphism f_η is the composition $\varphi f_m f_{m-1} \dots f_1$, where each f_i is the homeomorphism h hypothesized using

$$X^j = f_{i-1} \dots f_1(W_i^j) \quad \text{for} \quad j = 1, 3;$$

$$F = f_{i-1} \dots f_1\Big(\big(\bigcup_{k<i} \operatorname{Cl} W_k^1\big) \cup (S^3 - (\mathscr{W}^1)^*)\Big);$$

$\lambda = f_{i-1} \dots f_1$ for $i > 1$ and $\lambda = \mathrm{id}$ for $i = 1$; and ε be such that if $p, q \in S^3$ and $\operatorname{dist}(p, q) < \varepsilon$, then $\operatorname{dist}(\lambda(p), \lambda(q)) < \varepsilon_0$. (If the reader is comparing this proof with the [W] proof, note that we now use the same ε for each application of the hypothesized h.) Notice that when we shrink the image of a particular W_i^1, the nondegenerate elements in it shrink and in later work each set F limits the growth. Hence, we satisfy (2) of the shrinkability definition. Condition (1) of the definition is satisfied because f_i is fixed off $\mathscr{U}^*$. To see that we also have (3), we must consider the effect of the ordering. For any $W_i^3 \in \mathscr{W}^3$,

$$f_m f_{m-1} \dots f_1(W_i^3) = f_{m-1} \dots f_1(W_i^3)$$

if $W_i^3 \cap W_m^3 \neq \emptyset$, and

$$f_m f_{m-1} \dots f_1(W_i^3) \subset \big(f_{m-1} \dots f_1(W_i^3)\big) \cup \big(f_{m-1} \dots f_1(W_m^3)\big)$$

if $W_i^3 \cap W_m^3 \neq \emptyset$. Repeating this we see that

$$f_{m-1} \dots f_1(W_i^3) \subset \big(f_{m-2} \dots f_1(W_i^3)\big) \cup \big(f_{m-2} \dots f_1(W_{m-1}^3)\big)$$

if $W_i^3 \cap W_{m-1}^3 \neq \emptyset$, and

$$f_{m-1} \dots f_1(W_i^3) = f_{m-2} \dots f_1(W_i^3)$$

if $W_i^3 \cap W_{m-1}^3 \neq \emptyset$, while

$$f_{m-1} \dots f_1(W_m^3) \subset \big(f_{m-2} \dots f_1(W_m^3)\big) \cup \big(f_{m-2} \dots f_1(W_{m-1}^3)\big)$$

if $W_m^3 \cap W_{m-1}^3 \neq \emptyset$, and

$$f_{m-1} \dots f_1(W_m^3) = f_{m-2} \dots f_1(W_m^3)$$

if $W_m^3 \cap W_{m-1}^3 = \emptyset$. From this sort of analysis we find that $f_m f_{m-1} \dots f_1(W_i)$ is contained in the union of chains of the form $\{W_k\}$ where

(a) $\{W_k\}$ is a subsequence of $W_m^3, W_{m-1}^3, \dots, W_1^3$,
(b) $W_k \cap W_i = \emptyset$, and
(c) $W_{k-j} \cap W_{k-j-1} \neq \emptyset$.

In view of the order imposed on work, let us see how long a chain can be. If W_i intersects one set $\mathscr{W}_d$, then it may be possible for the chain to continue to grow within the particular $P^{-1}(Z)$ which contains W_i. But because the elements of $\mathscr{Z}_d$ are disjoint, this chain cannot intersect any other $P^{-1}(Z)$ corresponding to $\mathscr{Z}_d$. Next in the chain there can be W_i's which lie in $\mathscr{W}_c$, but all these must be in the same $P^{-1}(Z)$ corresponding one element of $\mathscr{Z}_c$. Similarly, we may have W_i's from $\mathscr{W}_b$ and then $\mathscr{W}_a$. It now easily follows from our use of the Lebesgue number that there is a $U \in \mathscr{U}$ such that W_i and $f_m f_{m-1} \dots f_1(W_i)$ lie in it. Hence, the McAuley–Reed Shrinkability Theorem is satisfied. ■

2.3. Tearing and resewing S^3. We wish to show that with certain added conditions the homeomorphism h in the Lemma 2.1 hypothesis follows if the other hypothesis conditions are given. In the standard method for showing the existence of a homeomorphism such as h, which shrinks some elements of H and controls the size of other ones, the desired h is shown to be the end of some isotopy. Instead, we use quite a different technique, which was first described in [W]. We tear out a crumpled cube in S^3 and shrink it. This shrinks the nondegenerate elements in the crumpled cube. Then we carefully sew the shrunk set and closure of the remainder of S^3 back together. Statements (A) and (B) below are for this resewing.

Lemma 2.2. *Suppose that G is an usc decomposition of S^3 such that for each $g_0 \in G$ and $i = 1, 2, 3$ there are arbitrarily small G-saturated open sets X^i with* $\operatorname{Bd} X^i$ *a G-saturated 2-sphere, $g_0 \subset X^1$,* $\operatorname{Cl} X^1 \subset X^2$, *and* $\operatorname{Cl} X^2 \subset X^3$. *Also, let $\varepsilon > 0$, homeomorphism α and F be given as in Lemma* 2.1. *Assume that statement* (A) *below implies statement* (B). *Then the homeomorphism h of Lemma* 2.1 *exists.*

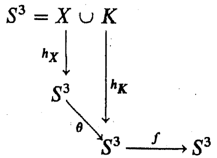
(A) *An annulus A is bounded by 2-spheres S and $\xi(S)$ where ξ is an admissible homeomorphism on S and there are 2-spheres Σ^1 and Σ^3 such that the component U of $S^3 - (\Sigma^1 \cup \Sigma^3)$ bounded by both Σ^1 and Σ^3 contains A. There is a decomposition G such that $H^* \subset S^3 -$* Int A, Bd A *is G-saturated, and ξ carries the decomposition of S onto the decomposition of $\xi(S)$. The set $F \subset S^3$ is closed and G-saturated, and $\delta > 0$. There are disjoint F_σ-sets F_1 and F_2 in S such that $F_1 \cup \xi(F_2) \cup$* Ext A *is* 1-ULC.

(B) *There is a map f of S^3 onto S^3 such that*
(1) $f | S^3 - U = \mathrm{id}$,

(2) *$f|S^3-A$ is a homeomorphism onto $S^3-f(A)$,*
(3) *$f|S=f\xi$ and $f\xi$ is a homeomorphism onto $f(A)$, and*
(4) *for $g\in G$ and $g\subset F$, $\operatorname{Diam} f(g)<3\delta/4+\varrho$; where $\varrho=\max\{\operatorname{Diam} g\colon g\subset F\}$.*

Proof. Since the homeomorphism h is for composition with the homeomorphism α, all work should be in $\alpha(S^3)$ with $\alpha(G)=\{\alpha(g)\colon g\in G\}$, $\alpha(F)$, etc. For simplicity we will work in S^3, and choose the appropriate corresponding distance $\hat{\varepsilon}$ for the given ε. Let $\hat{\varepsilon}$ be such that if $p, q\in S^3$ and $\operatorname{dist}(p, q)<\hat{\varepsilon}$, then $\operatorname{dist}(\alpha(p), \alpha(q))<\varepsilon$. Choose δ to be $\hat{\varepsilon}-\varrho$.

Consider S^3 to be the union of the crumpled cube $X=\operatorname{Cl} X^2$ and $K=\operatorname{Cl}(S^3-X)$. Maps which we will define are indicated in the diagram



The map h_X is a reembedding of X in $\alpha(S^3)$. It is given by the Hosay–Lininger Theorem [H], [L], which states: If C is a crumpled cube in $\alpha(S^3)$ and e is a positive number, then there exists a homeomorphism h from C into $\alpha(S^3)$ such that $\operatorname{Cl}(\alpha(S^3)-h(C))$ is a 3-cell and if $x\in C$ then the distance $\operatorname{dist}(x, h(x))<e$. When we apply the theorem, we require that h_X be the identity on X^1. (The method of proof in [L] implies that this is possible, since $\operatorname{Cl} X^1\subset X$.)

Let $\operatorname{dist}(\operatorname{Bd} X^1, \operatorname{Bd} X)$ be denoted by D. Let θ be a homeomorphism of $\alpha(S^3)$ onto itself taking $h_X(X)$ to a set of diameter less than the minimum of $D/2$ and $\varepsilon/2$ and not moving points in X^1. Note that now all nondegenerate elements that were in X are in $\theta h_X(X)$ and are small, and that $\theta h_X(X)\subset\operatorname{Int} X$.

We next apply the Hosay–Lininger Theorem to K to get the reembedding h_K of K in S^3. Choose e smaller than the minimum of $\operatorname{dist}(\operatorname{Bd}\alpha(X^2), \operatorname{Bd}\alpha(X^3))$, $\operatorname{dist}(K, \theta h_X(X))$, and $\delta/4$. We require that h_K be the identity on $\alpha(S^3-X^3)$. The first condition on the size of e is for condition (1) on h, the second guarantees that $\theta h_X(X)$ and $h_K(K)$ do not intersect, and the third condition is for condition (3) on h. Note that $\theta h_X(X)$ and $h_K(K)$ are disjoint crumpled cubes in S^3 and that the closure of the complement of each is a 3-cell. Denote $\operatorname{Cl}(\alpha(S^3)-\theta h_X(X)-h_K(K))$ by A.

We now show that Statement (A) is satisfied. The 2-sphere S is $\theta h_X(\operatorname{Bd} X)$. The homeomorphism ξ is $h_K h_X^{-1}\theta^{-1}$. Hence, $\xi(S)$ is $h_K(\operatorname{Bd} X)=h_K(\operatorname{Bd} K)$. The 2-spheres Σ^1 and Σ^3 are $\theta h_X(\operatorname{Bd}\alpha(X^1))$ and $h_K(\operatorname{Bd}\alpha(X^3))$, respectively. Let "$F$" in Statement (A) be $\theta h_X(F\cap X)\cup(F\cap K)$ where these use the set "F" of Lemma 2.1. We must show that A is an annulus, that $h_K h_X^{-1}\theta^{-1}$ is admissible, and that the F_σ-sets exist.

The boundary of A is the two copies of $\operatorname{Bd} X$. They are disjoint 2-spheres. Let T be a tame 2-sphere in A separating these boundary components. Consider a homeomorphism λ of $\operatorname{Cl}(\alpha(S^3)-\theta h_X(X))$ onto a polyhedral 3-cell P. Since T is tame in $\operatorname{Cl}(\alpha(S^3)-\theta h_X(X))$, it is bicollared in it. Its image is bicollared in P and, hence, is tame there. Since $\lambda\theta h_X(\operatorname{Bd} X)$ is $\operatorname{Bd} P$, it is tame. Hence $\lambda(T)$ and $\operatorname{Bd} P$ bound an annulus. The homeomorphism λ^{-1} must take this annulus back to an annulus. Similarly, the set bounded by T and $\theta h_K(\operatorname{Bd} K)$ is an annulus. Thus, A is the union of two annuli whose intersection is a 2-sphere in the boundary of each. This implies [K, p. 167] that A is an annulus.

Admissibility is concerned with preserving orientation. To guarantee that $h_K h_X^{-1}\theta^{-1}$ is admissible we made each of h_X, h_K, and θ have a neighborhood on which it is fixed. The following argument shows that this gives admissibility. Let δ_X and δ_K be tame 3-cells on which the reembeddings $\theta h_X(X)$ and $h_K(K)$, respectively, are fixed. We can make δ_K small enough that $\alpha(S^3)-\delta_K$ is a neighborhood of S. In $\alpha(S^3)-\delta_K-\operatorname{Int}\delta_X$ there are homotopies of $\operatorname{Bd} X$ to $\operatorname{Bd}\delta_X$ and of $\theta h_X(\operatorname{Bd} X)$ to $\operatorname{Bd}\delta_K$. Hence, in $\alpha(S^3)-\operatorname{Int}\delta_X-\operatorname{Int}\delta_K$ there is a homotopy of $\theta h_X(\operatorname{Bd} X)$ to $h_K(\operatorname{Bd} X)$. Since each of $\theta h_X(\operatorname{Bd} X)$ and $h_K(\operatorname{Bd} X)$ is tame on side containing A, the homotopy can be pushed into A.

Eaton's Mismatch Theorem [E] implies that, since X and K are crumpled cubes whose intersection is the boundary of each and their union is $\alpha(S^3)$, there exist disjoint 0-dimensional F_σ-sets F_1' and F_2' in $\operatorname{Bd} X$ such that $F_1'\cup\operatorname{Int} X$ and $F_2'\cup\operatorname{Int} K$ are 1-ULC. Hence $\theta h_X(F_1')$ and $h_K(F_2')$ are the necessary F_σ-sets.

We have shown that we have all the conditions in Statement (A). Since we are assuming that (A) implies (B), we may now use (B), which states that a certain map f exists. Define the map

$$h(x)=\begin{cases} fh_K & \text{for } x\in K,\\ f\theta h_X & \text{for } x\in X.\end{cases}$$

This map is the required homeomorphism h and completes the proof of Lemma 2.2. ■

2.4. An isotopy with control of nondegenerate element size. Construction of the map described in Statement (B) will involve pushing the boundaries of A together. In Theorem 1 we will push one boundary onto the other; in Theorem 2 we will move both boundaries and some points of each will be fixed. The paths of points during these motions can result in nondegenerate element growth. In Figure 1 we indicate an element g with subscripts denoting times 0, t, and 1 of some push. The possible growth is important because (1) a nondegenerate element from $\operatorname{Bd} A$ may grow larger and then be left in this state, and (2) though by the end of the motions each nondegenerate element in the image of $\operatorname{Bd} A$ may be an acceptable size, nondegenerate elements from S^3-A may grow, and then be left large, near the path of a nondegenerate element. The following lemma is for controlling the sizes of images of nondegenerate elements.

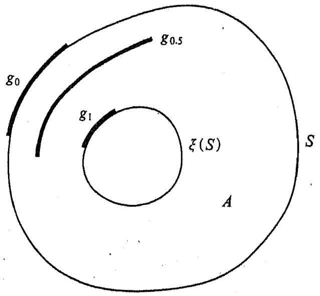


Fig.1

LEMMA 2.3. *Suppose that an annulus $A \subset S^3$ is bounded by (possibly wild) 2-spheres S_a and $S_b = \xi(S_a)$, where ξ is an admissible homeomorphism on S_a; and there exists an usc cellular decomposition G of $S^3 - \mathrm{Int}\ A$ such that $\mathrm{Bd}\ A$ is G-saturated and ξ carries the decomposition of S_a onto the decomposition of S_b. Let $\alpha > 0$. Assume that for each $g \in G$ which is in S_a, $\mathrm{Diam}\ \xi(g) < \mathrm{Diam}\ g + \alpha/2$. Let F_S be a compact G-saturated subset of S_a and assume that $F_S \subset O_F$, which is G-saturated and open in S_a. Then there is an isotopy $\mathscr{I}$: $S^2 \times I \to A$ such that $\mathscr{I}(S^2 \times 0) = S_a$ and $\mathscr{I}(S^2 \times 1) = S_b$; if for $x \in S^2$ the image $\mathscr{I}(x \times 0) = p \in S_a$, then $\mathscr{I}(x \times 1) = \xi(p) \in S_b$; and if for a subset $w \subset S^2$ we have $\mathscr{I}(w \times 0) = g \in G$ and $g \subset F_S$, then for any $t \in I$, it is true that $\mathrm{Diam}\ \mathscr{I}(w \times t) < \alpha + \max\{\mathrm{Diam}\ g: g \in G \text{ and } g \subset O_F\}$.*

Proof. From the homotopy in the definition of admissibility for ξ, one can get an isotopy $\mathscr{I}_1$: $S^2 \times I_1 \to A$ such that $\mathscr{I}_1(S^2 \times 0) = S_a$, $\mathscr{I}_1(S^2 \times 1) = S_b$; and for $x \in S^2$ if $\mathscr{I}_1(x \times 0) = p \in S_a$, then $\mathscr{I}_1(x \times 1) = \xi(p) \in S_b$. In this isotopy let us substitute the 2-sphere S_a for the abstract S^2.

This isotopy gives us no control of the size of nondegenerate elements. We will define another isotopy and then compose the two.

Let G_a be the usc decomposition of S_a which is the given decomposition G restricted to S_a. By Moore's Theorem [M], S_a/G_a is again a 2-sphere. Siebenmann's results in [S] used for $n = 2$, or Armentrout's methods in [A1] and [A2] applied to $n = 2$, imply that the decomposition G_a is shrinkable. Edwards and Glaser [E, G] show that this actually gives a pseudoisotopy of S_a with the following property. Let $\mathscr{V}$ be a G-saturated open cover of S_a. Then for any $g \in G$ there is a $V_g \in \mathscr{V}$ such that the pseudoisotopy on g is contained in V_g.

Choose a cover $\mathscr{V}$ of S_a such that no $V \in \mathscr{V}$ intersects both F_S and $S_a - O_F$ and that each V lies in the $\alpha/8$-neighborhood of some $g \in G$. Then the previous paragraph implies that there is a pseudoisotopy $\mathscr{I}_2$: $S^2 \times I_2 \to S_a$ such that if for $w \subset S^2$ we have $\mathscr{I}_2(w \times 0) = g$, then $\mathscr{I}_2(w \times 1)$ is a point; and if $w \subset F_S$, then for each $\tau \in I_2$, $\mathrm{Diam}\ \mathscr{I}_2(w \times \tau) < \alpha/4 + \max\{\mathrm{Diam}\ g\colon g \subset O_F\}$.

This isotopy $\mathscr{I}_2$ is indicated in Figure 2, where the image of some possible nondegenerate element is shown at the four times denoted by the subscripts. Our required isotopy $\mathscr{I}$ will be made by composing $\mathscr{I}_1$ with certain parts of $\mathscr{I}_2$. One can think of the 2-sphere S_a starting to move across A and as $\mathscr{I}_1$

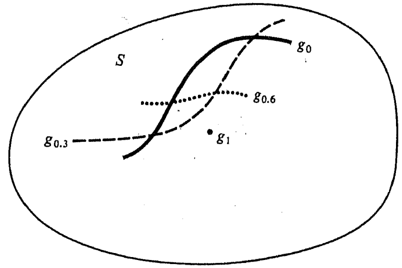


Fig. 2

moves it, it is being subjected to the shrinking pseudoisotopy $\mathscr{I}_2$. (Of course, we avoid using the final time for the pseudoisotopy.) Since close to the end of $\mathscr{I}_2$ all the nondegenerate elements are arbitrarily small, it must be possible for $\mathscr{I}_2$ to shrink elements fast enough that they do not grow too large from the possible growth caused by $\mathscr{I}_1$.

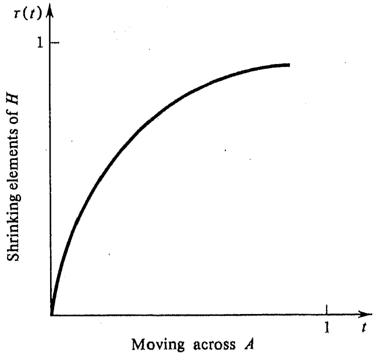


Fig. 3

As above, we denote times in I_1 by t and those in I_2 by τ. Let $\tau(t)$ be an increasing continuous 1-1 function such that $\tau(0) = 0$ and $\tau(1) = 1$. Such a function is in Figure 3. Now consider $\mathscr{I}_1(\mathscr{I}_2(S^2 \times I_2) \times I_1)$ to be the image of an isotopy. We can make τ be sufficiently large with respect to t that for each $g \subset F$

we have

$$\operatorname{Diam}\left(\mathscr{I}_1\left(\mathscr{I}_2(g\times\tau(t))\times t\right)\right) < \alpha/2+\max\{\operatorname{Diam} g\colon g\subset O_F\}$$

for each $t\in I_1$. Hence, we have a pseudoisotopy $\mathscr{I}_3\colon S^2\times I_1\to A$ defined by $\mathscr{I}_3(x,t)=\mathscr{I}_1\left(\mathscr{I}_2(x,\tau(t)),t\right)$ for each $t\in I_1$.

Analogous to the definition of $\mathscr{I}_2$ two paragraphs above, we can define an isotopy $\mathscr{I}_4$ using S_b, $\xi(F)$, and $\xi(O_F)$ in place of S_a, F, and O_F, respectively. (Recall that $\max\{\operatorname{Diam} g\colon g\subset\xi(O_F)\}=\alpha/2+\max\{\operatorname{Diam} g\colon g\subset O_F\}$.) $\mathscr{I}_1$ and $\mathscr{I}_4$ could then be used to define a pseudoisotopy. Then the first part of each of $\mathscr{I}_3$ and this new analogous pseudoisotopy could be pieced together to get an isotopy. Instead, we simply define

$$\mathscr{I}(x,t)=\begin{cases}\mathscr{I}_3(x,t) & \text{if } t\leqslant 0.5,\\ \mathscr{I}_1\left(\mathscr{I}_2(x,\tau(1-t)),t\right) & \text{if } t\geqslant 0.5.\end{cases}$$

In order for this to have all the desired properties near S_b, it may be necessary to require that τ be even larger with respect to t near 0. Once this is done we have the required isotopy. ∎

3. Decompositions in which the 2-sphere S is tame from one side

The decompositions in the theorem in this section have the property that each $g_0\in H$ lies in the interior of arbitrarily small either (1) (possibly wild) 3-cells with saturated boundary, or (2) crumpled cubes with saturated boundary and with an open 3-cell complement.

THEOREM 1. *Let G be an usc decomposition of S^3 with H the set of nondegenerate elements. Suppose that for any $g\in H$, and open set U containing g there is a crumpled cube X such that $g\subset\operatorname{Int} X\subset U$,* Bd *$X$ is G-saturated, and* Bd *X is tame from at least one side. Then S^3/G is homeomorphic to S^3.*

Proof. The proof will follow from the Shrinkability Theorem and Lemmas 2.1 and 2.2, after we show that our Theorem 1 hypotheses imply that Statement (A) implies Statement (B). For Lemma 2.1 notice that the 3-dimensionality of S^3/G follows from the G-saturated Bd X's.

For each $g_0\in H$ use the hypothesized crumpled cubes to get the X^i of Statement (A).

The decomposition G is obviously cellular. Hence, the corresponding decomposition in Lemma 2.3 is also cellular.

We wish to use Lemma 2.3 isotopy in the annulus A of Statement (A). For this, let $S_a=S$, the decomposition G be the same, $F_S=S\cap F$, and $\alpha=\delta/4$; for O_F we first choose an open (in S^3) saturated set Q_F containing F and containing no $g\in G$ such that Diam $g>\delta/4+\varrho$, and then let $O_F=Q_F\cap S$. We can assume that θ in the Lemma 2.2 proof is chosen so that Diam $\theta h_X(X)<\delta/8$, which certainly implies that Diam $\xi(g)<\alpha/2$ for each $g\in X$. With these conditions, the conclusion of Lemma 2.3 gives us the isotopy $\mathscr{I}$.

We collapse A by pushing its tame boundary, which we can assume is S_a, onto the wild one, S_b. This is done by using several pushes. Each push of S_a corresponds to changing the level t in the isotopy $\mathscr{I}$. For each push we choose an annular neighborhood of the image of S_a and let the push act on this neighborhood. These neighborhoods are contained in $U-S_b$, and for the last push S_b must be in the boundary of the chosen neighborhood. Except this S_b boundary, the annular neighborhoods always have tame boundaries. The push of S_a is extended linearly along paths given by $\mathscr{I}$ to the entire chosen annular neighborhood. We choose each neighborhood so that it contains no nondegenerate elements in the image of F with diameter more than $\delta/4+\varrho$. The length with respect to t of the push is chosen so that the new maximum size of nondegenerate elements in the image of G is $\delta/2+\max\{\operatorname{Diam} g\colon g\subset F\}$, or so that the push of S_a onto S_b is completed. Note that these choices of sizes are possible because we controlled the growth of nondegenerate elements during the isotopy. This completes the proof of Theorem 1. ∎

4. Decompositions in which there is a nice relationship between the wildness of the 2-sphere and the nondegenerate elements in it

In this section we will use a technique developed by Eaton [E] for collapsing an annulus. In that paper there were no nondegenerate elements. We use nondegenerate elements and alter the proof so that growth of nondegenerate element is controlled. In the annular collapse, what one might think of as the wildest points on each 2-sphere boundary are the fixed points. If one nondegenerate elements were to contain a fixed point in each annular boundary, these points would determine a lower boundary on the final size of this nondegenerate element. Hence, it is not surprising that the placement of nondegenerate elements in a wild 2-sphere boundary is a condition in the hypothesis.

THEOREM 2. *Let G be an usc decomposition of S^3. Suppose that for any $g\in H$ and open set U containing g there is a crumpled cube X such that $g\subset\operatorname{Int} X\subset U$, and* Bd *$X$ is G-saturated. Also, assume that there exist disjoint 0-dimensional F_σ-sets $F_1,F_2\subset$* Bd *X such that* (Ext $X)\cup F_1$ *and* (Int $X)\cup F_2$ *are* 1-ULC *and $P(F_1)$ and $P(F_2)$ are disjoint 0-dimensional sets in $P($*Bd *$X)$. Then S^3/G is homeomorphic to S^3.*

Proof. As in Theorem 1 proof, we will use the Shrinkability Theorem and Lemmas 2.1 and 2.2. Again, the G-saturated Bd X's imply that S^3/G is 3-dimensional. We must now show that the hypotheses concerning the F_σ-sets are sufficient added conditions for a proof that Statement (A) implies Statement (B).

This proof depends on Lemma 4.1 below. That lemma is a modification of Eaton's "Main Lemma" in [E]. Before we consider our modification, let us briefly describe his method. In that paper, a 3-cell, which is a subset of a partially collapsed annulus, is further collapsed. The boundary of that C is the union of two disks with disjoint interiors. His proof involves pushing the disks towards one another and fully together on a grid. The resulting set after the collapse is a finite collection of 3-cells, each having a smaller size. The [E] map is defined with sufficient care that infinite iteration of the lemma fully collapses C, identifies the disks, and is a homeomorphism on the remainder of S^3.

In [W] modifications were made involving the presence of an usc decomposition of S^3 with C totally missing the nondegenerate elements. That modified map was defined so that there was control of the size of nondegenerate elements. Now h in Lemma 2.1 is a map on the open annulus of Statement (A), and this annulus contains nondegenerate elements in its boundary. In our proof we must control the size of those nondegenerate elements and of the nondegenerate elements in the set F.

Most of the size control is in Lemma 4.1 below, which starts the collapse by taking the open annulus onto a 2-sphere plus a finite collection of open 3-cells. There is control of sizes and placement of nondegenerate elements. The size of each 3-cell which intersects F is small. In the proof of the theorem this lemma is applied only once. It is then possible to complete the proof by collapsing each 3-cell using an infinite iteration of a map very similar to Eaton's original one. In the infinite iteration the only size control needed for the nondegenerate element is that the image of the collapse of a 3-cell C_i be in C_i.

After the proof of Lemma 4.1 we will quickly complete the proof of this theorem.

LEMMA 4.1. *Assume Statement* (A) *is true. Also assume that S is a G-saturated 2-sphere, and that for the F_σ-sets in Statement* (A), *the projections $P(F_1)$ and $P(F_2)$ are disjoint 0-dimensional sets in $P(S)$. Then there exist a cellular subdivision $\{P(\mathrm{Cl}\, D_1), \ldots, P(\mathrm{Cl}\, D_n)\}$ of $P(S)$ with each D_i a G-saturated open disk in S and a map f of S^3 onto S^3 such that*

(1) $f|S^3-U = \mathrm{id}$,

(2) $f|S^3-A$ *is a homeomorphism onto* $S^3-f(A)$,

(3) *both* $f|S$ *and* $f|\xi(S)$ *are homeomorphisms*,

(4) $(\bigcup \mathrm{Sat\, Bd}\, D_i)\cap(F_1\cup F_2) = \emptyset$,

(5) $f(S)\cap f\xi(S) = f(\bigcup \mathrm{Sat\, Bd}\, D_i)$,

(6) $f|\bigcup \mathrm{Sat\, Bd}\, D_i = f\xi|\bigcup \mathrm{Sat\, Bd}\, D_i$,

(7) $\mathrm{Cl}\big((f(D_i))\cup f\xi(D_i)\big)$ *bounds an open 3-cell C_i in $f(A)$ such that $P(\mathrm{Cl}\, C_i)$ is a 3-cell in S^3/G*,

(8) *for $g\in H$ and $g\subset F$, $\mathrm{Diam}\, f(g) < \delta/4+\varrho$, where $\varrho = \max\{\mathrm{Diam}\, g\colon g\in G$ and $g\subset F\}$*,

(9) *if $\mathrm{Cl}\, D_i\cap F\neq\emptyset$, then $\mathrm{Diam}\, f(D_i) < 3\delta/32+\varrho$ and $\mathrm{Diam}\, C_i < \delta/8+\varrho$.*

Proof of Lemma 4.1. Notice that at the conclusion of this Lemma 4.1 proof there is a summary of sizes chosen throughout the proof.

The proof will be presented by specifying changes in the [E] Main Lemma proof.

The reader familiar with the proof in [W] will recall that [W] Lemma 4 was concerned with using admissibility of ξ to start the collapsing. If one did not use admissibility there, then the orientation of the J might not be correct. In the

present proof admissibility is used in the formulation of the isotopy $\mathscr{I}$, so we do not need to be further concerned with the orientation. Hence, we proceed immediately with the analogue of [W] Lemma 3.

If we were to simply apply the [E] method with no concern about nondegenerate elements, some might be stretched from S to $\xi(S)$. To avoid this problem in our proof, whenever we push a point, we will push the entire nondegenerate element in which it occurs. Hence, several conclusion involve saturations. Also, instead of trying to attain a given mesh, we will choose a local subdivision size related to nondegenerate elements and the set F. Control of growth of nondegenerate elements will be similar to that in Theorem 1 of this paper.

Instead of Eaton's projection map called "P" we will need the control available from Lemma 2.3. Since the controlled growth involves the size of elements in a neighborhood of F_S, we choose a particular neighborhood for F. For this, choose G-saturated open sets Q_F and $\tilde{Q}_F$ such that $F\subset Q_F$, $\mathrm{Cl}\, Q_F\subset\tilde{Q}_F$, and $\max\{\mathrm{Diam}\, g\colon g\subset\tilde{Q}_F\} < \delta/32+\varrho$. Apply Lemma 2.3 using S_a and S_b for S and $\xi(S)$, respectively; ξ as the admissible homeomorphism; $\alpha = \delta/32$; and $F_S = \mathrm{Sat\, Cl}\,\tilde{Q}_F\cap S$ and $O_F = Q_F\cap S$. Again we can assume that work already done has shrunk $\xi(S)$ so small that for each $g\in G$ in S, $\mathrm{Diam}\,\xi(g) < \mathrm{Diam}\, g+\alpha/2$. Hence, we have the isotopy $\mathscr{I}\colon S^2\times I\to A$ such that $\mathscr{I}(S^2\times 0) = S$ and $\mathscr{I}(S^2\times 1) = \xi(S)$; $\mathscr{I}$ carries decomposition elements in S to corresponding ones in $\xi(S)$, and at any level $t\in I$ no nondegenerate element in F_S has grown larger than $\delta/32+$ $+\max\{\mathrm{Diam}\, g\colon g\subset Q_F\} = \delta/16+\varrho$. For the abstract S^2 in $\mathscr{I}$, we use the 2-sphere $S\subset \mathrm{Bd}\, A$.

The details of the [E] proof start in his **P2**. Replace [E] **P2** by this paragraph. In the decomposition space the set $P(S)$ is a 2-sphere by Moore's theorem [M]. Imitating the set G in **P2** of [E], choose a nonempty grid $\mathscr{G}$ of simple closed curves in $P(S)$ missing $P(F_1\cup F_2)$. The open disks in $S-P^{-1}(\mathscr{G})$ form the collection $\{D_1, \ldots, D_n\}$ of our Lemma 4.1 conclusion. We further require that $\mathscr{G}$ be chosen so that (a) no $\mathrm{Cl}\, P(D_i)$ intersects both $P(F)$ and $P(S-O_F)$, and (b) for each $P(D_i)\subset P(O_F)$ there is a $g\in G$ such that $P(D_i)$ is contained in P (the $\delta/32$-neighborhood of g). The isotopy $\mathscr{I}$ serves the purpose of Eaton's P^{-1} and levels of the isotopy are analogous to his $L(t)$. The set $P^{-1}(\mathscr{G})$ is tame because it misses the F_σ-set F_1, and $\xi P^{-1}(\mathscr{G})$ is tame because it misses $\xi(F_2)$. The set $\mathscr{I}\big((P^{-1}(\mathscr{G}))\times I\big)$ is tame, because the ends of the isotopy on $P^{-1}(\mathscr{G})$ are tame and the isotopy is on all of S.

In our modification of [E] **P3** it is the tame set $\xi P^{-1}(\mathscr{G})$ that we push, and the push is along $\mathscr{I}\big((P^{-1}(\mathscr{G}))\times I\big)$ from $\xi(S)$, which corresponds to $t = 1$ to S, corresponding to $t = 0$. We will again use the composition of a finite collection of maps $\{\alpha_k\}$ of S^3 onto S^3. In [E] the purpose of these was for cross sectional control; we achieve our cross sectional control by our choices of $\mathscr{I}$ and $\mathscr{G}$. The control we exercise by using the map sequence is concerned with the size to which nondegenerate elements grow.

In our modified P4 we use $1 = t_0 > t_1 > \ldots > t_n = 0$ in I of the isotopy for the analogue of the partition of his segment ab. We will specify the choice of t_j's below.

Define the projection map α_j of $T_j = \mathscr{I}(P^{-1}(\mathscr{G}) \times [t_{j-1}, t_j])$ onto $G_j = \mathscr{I}(P^{-1}(\mathscr{G}) \times t_j)$ by the condition that $\alpha_j(x) = \mathscr{I}(y \times t_j)$ where $y \in S$ such that $x \in \mathscr{I}(y \times [t_{j-1}, t_j])$. Our next task is the extension of α_j to all of S^3. We will use two (not necessarily disjoint) subsets of $P^{-1}(\mathscr{G})$. For each of these two we will find a homeomorphism extending α_j in a neighborhood of that subset. The union of the two subsets will be $P^{-1}(\mathscr{G})$. The two homeomorphisms will be combined using a partition of unity to get the extension of α_j to all of S^3.

The two subsets of $\mathscr{G}$ will be $\mathscr{G}_F$ and $\mathscr{G}_L$. By using F and the choice of the grid $\mathscr{G}$, we can choose $\mathscr{G}_F$ so that $P^{-1}(\mathscr{G}_F)$ contains no large nondegenerate elements. The other set, $P^{-1}(\mathscr{G}_L)$ may contain large ones. To choose these sets, we first consider the grid $\mathscr{G}$ to be the union of open arcs and vertices (in the obvious way). Let a vertex be in $\mathscr{G}_F$ if it is the end point of an open arc which intersects $P(F)$. Let an open arc be in $\mathscr{G}_F$ if it has an end point in $\mathscr{G}_F$. Let a vertex be in $\mathscr{G}_L$ if it does not lie in $P(F)$. Let an open arc be in $\mathscr{G}_L$ if both its end points are in $\mathscr{G}_L$. Note that $\mathscr{G}_F \cup \mathscr{G}_L = \mathscr{G}$ and $\mathscr{G}_F \cap \mathscr{G}_L$ is empty only if either $\mathscr{G}_F$ or $\mathscr{G}_L$ is empty.

We first extend the homeomorphism $\alpha_j | \xi P^{-1}(\mathscr{G}_F)$. Assume that t_{j-1} and the extension of $\alpha_{j-1} | \xi P^{-1}(\mathscr{G}_F)$ have been chosen. (Use $\alpha_0 = \mathrm{id}$ and $t_0 = 1$.) Choose a regular neighborhood $N_{j-1} \subset N_{j-2}$ of $\mathscr{G}_F$ in $P(S)$ sufficiently close to $\mathscr{G}_F$ to ensure that $\alpha_{j-1} \ldots \alpha_1 \xi P^{-1}(N_{j-1})$ lies in a thin tubular neighborhood of $G_{j-1,F} = \mathscr{I}(P^{-1}(\mathscr{G}_F) \times t_{j-1})$. (Require that $\alpha_1 \xi P^{-1}(N_0)$ also lie in Q_F.) Also choose N_{j-1} sufficiently close that $P^{-1}(N_{j-1})$ contain no nondegenerate element as large as $5\delta/64 + \varrho$. (Note that under the isotopy, images of nondegenerate elements in $\mathscr{I}(P^{-1}(\mathscr{G}_F) \times I)$ are smaller than this. The Epsilonics Summary may be helpful here.) Although we have not yet chosen t_j, we can state the next steps and then specify t_j. Define $T_{j,F}$ by replacing $\mathscr{G}$ by its subset $\mathscr{G}_F$ in the definition T_j. Select a small neighborhood $V_{j,F}$ of $T_{j,F} - G_{j,F}$ in $\alpha_{j-1} \ldots \alpha_1 \xi P^{-1}(N_{j-1})$ such that $V_{j,F}$ misses $\mathscr{I}(S \times [t_j, t_n])$. We now specify t_j: choose it as far as possible from t_{j-1}, but such that there exists an extension of $\alpha_j | T_{j,F}$ to $V_{j,F}$ causing no nondegenerate element to grow more than $11\delta/128 + \varrho$. A further condition on the length of $[t_{j-1}, t_j]$ is specified five paragraphs below. We denote this extension by $\alpha_{j,F}$.

We next extend the homeomorphism $\alpha_j | \xi P^{-1}(\mathscr{G}_L)$. Choose the sets $P^{-1}(N_{j-1,L})$ and $V_{j,L}$ so that their closures miss F. Define $\alpha_{j,L}$ using all the steps in the last paragraph except those concerning the sizes of nondegenerate elements and the choice of t_j.

Combine $\alpha_{j,F}$ and $\alpha_{j,L}$ using a partition of unity on the closure of each open arc which lies in both $\mathscr{G}_F$ and $\mathscr{G}_L$. This gives us the map $\alpha = \alpha_n \ldots \alpha_j \ldots \alpha_1$, which causes no nondegenerate element from F to grow larger than $3\delta/32 + \varrho$. The map α squeezes A to a finite collection of $\alpha(G)$-saturations of open 3-cells with the cross-sections controlled for those 3-cells intersecting F. A typical open 3-cell C_i

has its boundary in (Sat Bd D_i)$\cup \alpha\xi$(Sat Bd D_i). The remainder of [E] **P5** can be kept with only some obvious changes in words.

This completes Step 1. For each C_i which does not intersect F, the map $\alpha | C_i$ is the desired map $f | C_i$. For each C_i which does intersect F we must perform an analogue of [E] Step 2.

P 6 becomes: We squeeze the cross-sectionally controlled open cells C_i from Step 1 to thin cells. We find a finite collection $\{U_i\}$ of disjoint open sets in U such that (Sat Cl C_i) $-$ Sat Bd $D_i \subset U_i$, and we squeeze each open cell which intersects F individually, moving only points in U_i. For convenience, we drop the subscripts on these sets, and use the notation and hypotheses of this lemma with Sat Cl C_i and Sat Cl D_i replacing A and S, respectively, and with the additional requirement that the cross sectional diameter of open 3-cell C_i now denoted by C with respect to open 2-cell D_i now denoted by D be less than $3\delta/32 + \varrho$. Instead of again applying Lemma 2.3 in the proof below we continue to use the isotopy $\mathscr{I}$ from above.

Concerns about nondegenerate elements and effects of previous modifications cause many changes in **P7**. We will choose a finite collection $\{C_0, C_1, \ldots, C_n\}$ of open 3-cells such that $\bigcup C_i \subset C \subset \bigcup \mathrm{Cl}\, C_i$. The diameter of each C_i is less than $\delta/8 + \varrho$, and the thickness is less than $\delta/32$, where thickness means max $\{\mathrm{Diam}\,((\mathscr{I}(x \times I)) \cap C_i \colon x \in C\}$. These $\{C_i\}$ are in a linear order and Int (Bd $C_{i-1} \cap$ Bd C_i) is an open 2-disk, which we denote by H_i. The G-saturations of the closures of $D, H_1, \ldots, H_n$ are disjoint and $D \subset$ Bd C_0. We can assume that $P(\xi^{-1}(\bigcup \mathrm{Sat\ Bd}\ H_i)) \cap P(F_1 \cup F_2) = \emptyset$, because $P(F_1 \cup F_2)$ is 0-dimensional. Hence, Sat Cl H_i and ξ^{-1}(Sat Bd H_i) are tame. We also assume that each H_i lies in a $\delta/64$-neighborhood of a constant t-level of $\mathscr{I}$. To attain this last condition we must know that Sat Bd H_i not extend too far from some such constant t-level. This we now get by putting the further restriction on the size of $[t_{j-1}, t_j]$ in the above modification of **P 4**: Make each $[t_{j-1}, t_j]$ sufficiently small that we can achieve this condition on H_i.

In **P 8** through **P 19** we repeatedly modify by (a) replacing 3-cells, 2-cells, and annuli by the corresponding open sets; and (b) instead of pushing certain arcs and simple closed curves, we push the corresponding analogues, which are G-saturations of particular sets.

When our analogue of [E] **P 17** flops a disk down, the homeomorphism q acts on a set which has a controlled diameter, and this controls the nondegenerate element growth. The points in certain G-saturations of boundaries flop to prescribed points, and the remainder of Z is required to go homeomorphically onto the corresponding set in Z'. We add a requirement that if $p \in C$, then $q(p)$ lies in the $\delta/32$-neighborhood of $\mathscr{I}((P^{-1}(q)) \times I)$.

Each time we push a set (as in **P 4** or similar later pushes **P 12** and **P 19**) the set is tame. Either it starts exactly on its isotopy path or very close to it. The isotopy path is tame. If we did start on it, we can follow it. If we started very near it, we can in a short portion of the push get onto it. Hence, the growth of nondegenerate elements near the push is the same as that in the modification of **P4**.

As in [E], the Parts 1 and 2 are repeated a finite number of times. The controls of sizes turn out to need the same numbers used above. They do not change with the order of the iteration stage.

The following summary completes the proof of this lemma.

Epsilonics summary. We list here only conditions concerning sizes. Of course, the other conditions in the text must be simultaneously satisfied. The "d_n" distance notations are introduced below to help the reader follow this summary. Recall that $\varrho = \max\{\text{Diam } g\colon g \in H \text{ and } g \subset F\}$. In this summary we use "g" for $g \in H$ and "nondegenerate element" when we mean an image of some g. The "$\delta/32$" used very often below is not adjusted for each use, but rather is sufficiently small to be used in every one of the places.

1. Choose an open set $\tilde{Q}_F$ such that $g \subset \tilde{Q}_F$ implies that Diam $g < \delta/32+\varrho = d_1$.

2. Choose an open Q_F such that Cl $Q_F \subset \tilde{Q}_F$. Then for $g \subset Q_F$, again Diam $g < d_1$.

3. Choose $\mathscr{I}$ such that for every $t \in I$, $g \subset Q_F$, Diam $\mathscr{I}(g \times t) < \delta/32+d_1 = \delta/16+\varrho = d_3$.

4. Choose $\mathscr{G}$ such that if $D_i \subset Q_F$, then D_i is contained in a sufficiently small neighborhood of some $g \subset Q_F \cap$(Image of S) that if $D_i \cap F \neq \emptyset$, then for $t \in I$, Diam $\mathscr{I}(D_i \times t) < \delta/32+d_3 = 3\delta/32+\varrho = d_4$.

5. Move points in the modified **P4** so that no nondegenerate element from F grows during this work to more than $\delta/32+$ (the diameter of nondegenerate elements which it is trailing) $= \delta/32+d_3 = 3\delta/32+\varrho = d_5$.

6. In the modified **P7** choose the thickness of C_i as measured along $\mathscr{I}$ to be less than $\delta/32$. Hence, Diam $C_i < \delta/32+d_4 = \delta/8+\varrho = d_6$.

7. Flop in the modified **P17** so that no nondegenerate element grows to more than $\delta/32+d_6 = 5\delta/32+\varrho = d_7$.

In the finite iteration of **P4** through **P20**, note that:

a. There is no change at all in the objects involved in our choices (1), (2), (3), and (4).

b. It is possible to again satisfy (5), (6), and (7).

This completes the proof of Lemma 4.1. ∎

For the completion of the proof that Statement (A) implies (B) we must fully collapse each Sat Cl C_i from Lemma 4.1 Conclusion (7) onto an image of its Sat Cl D_i. As in our modification of [E] **P6**, we find a finite collection $\{U_i\}$ of disjoint open sets in U such that (Sat Cl C_i) $-$ Sat Bd $D_i \subset U_i$. We further require that if C_i intersects F, then Diam $U_i < \delta/4+\varrho$. We will squeeze each C_i individually, moving only points in U_i.

The choice of Q_F, Lemma 4.1 Conclusion (8), and the conditions on U_i

together give us the Statement (B) Conclusion (4). Hence, we need no further control on the size of any nondegenerate elements.

In order to accomplish a partial collapse of each C_i we again slightly modify the original [E] Main Lemma. This modification is necessary because it is possible that Cl C_i is not a 3-cell, since Sat Bd D_i may non-locally connected. This problem can be handled as it was in Lemma 4.1 above. Other than Sat Bd D_i we will not need to ever consider G-saturations in this modification. No other changes in the [E] lemma are needed. Note that we do use the [E] mesh requirement on the subdivision $\{D_1, \ldots, D_n\}$; denote that mesh size now as β instead of the ε in [E].

In the first application of this slightly modified [E] lemma to each C_i, the size of $\beta > 0$ is arbitrary. We iterate the use of the lemma with $\beta/2^j$ for the jth repeated application.

This completes the proof that Statement (A) implies (B) and of Theorem 2. ∎

The following lemma will lead to a corollary of Theorem 2.

Lemma 4.2. *Let G_S be a compact 0-dimensional use decomposition of a 2-sphere S. Let $F_1 \cup F_2$ be a 0-dimensional subset of S. Then $P(F_1 \cup F_2)$ is a 0-dimensional subset of $P(S)$.*

Proof. Since G_S is a compact 0-dimensional decomposition of S, there is a defining sequence $\{A_i\}_{i \in \omega}$ for G_S.

Let $x \in P(F_1 \cup F_2)$ and U be an open set containing x. We will consider two cases.

First suppose that x is such that $P^{-1}(x) \cap \text{Cl } H^* = \emptyset$. The set $P^{-1}(x)$ is an element of G; it is either an element of H or a point in (Cl H^*) $-$ H^*. It lies in the interior of every A_i. Choose an n such that the component of A_{n-1} containing $P^{-1}(x)$ lies in $P^{-1}(U)$. Let K denote the component of A_n containing $P^{-1}(x)$. Let $d = \text{dist}(\text{Bd } A_n, (\text{Bd } A_{n-1} \cup \text{Bd } A_{n+1}))$. Note that the d-neighborhood of Bd K does not intersects H^*. Using 0-dimensionality of $F_1 \cup F_2$, we find for each $p \in (\text{Bd } K) \cap (F_1 \cup F_2)$ an open set V_p containing p, such that Diam $V_p < d$ and Bd V_p misses $F_1 \cup F_2$. Let $W = K \cup (\bigcup_p V_p)$. It is an open set whose boundary misses $F_1 \cup F_2$ and whose boundary also misses H^*. Hence, $P(W) \subset U$ is an open set such that $x \in P(W)$ and Bd $P(W)$ misses $P(F_1 \cup F_2)$. This completes the first case.

For the second case we suppose that $P^{-1}(x)$ does not intersect Cl H^*. Since by hypothesis, Cl H^* is compact, there is an open set $V \subset P^{-1}(U)$ containing $P^{-1}(x)$ and missing Cl H^*. Also, $P^{-1}(x)$ is a single point, which is in $F_1 \cup F_2$. Using 0-dimensionality of $F_1 \cup F_2$, we find an open set $W \subset V$ such that $P^{-1}(x) \subset W$ and Bd W misses $F_1 \cup F_2$. Hence, for this case we have the required open set $P(W) \subset U$ such that $x \in P(W)$ and Bd $P(W)$ misses $P(F_1 \cup F_2)$.

Hence, in both cases we have shown that $P(F_1 \cup F_2)$ is 0-dimensional. ∎

Corollary. *Let G be a compact 0-dimensional usc decomposition of S^3.*

Suppose that for any $g \in H$ and open set U containing g there is a crumpled cube X such that $g \subset \operatorname{Int} X \subset U$, and $\operatorname{Bd} X$ is saturated. Also, assume that there exist disjoint 0-dimensional F_σ-sets $F_1, F_2 \subset \operatorname{Bd} X$ such that $(\operatorname{Ext} X) \cup F_1$ and $(\operatorname{Int} X) \cup F_2$ are 1-ULC; and there is no $g \in G$ which intersects both F_1 and F_2. Then S^3/G is homeomorphic to S^3.

Proof. This is immediate from Theorem 2 and Lemma 4.2.

5. Situations in which the decomposition of the 2-sphere S is shrinkable

THEOREM 3. *Let G be an usc decomposition of S^3. Assume that for each $g \in H$ and open set U containing g there is an open set X such that $g \subset X \subset U$; the set $P(\operatorname{Bd} X)$ is a 2-sphere; and for the new usc decomposition*

$$G_g = \{g \in G\colon g \subset \operatorname{Sat} \operatorname{Bd} X\} \cup \{p \in S^3\colon p \notin \operatorname{Sat} \operatorname{Bd} X\}$$

of S^3, the decomposition space S^3/G_g is homeomorphic to S^3. Then S^3/G is homeomorphic to S^3.

Remark. Notice that the hypotheses do not require that $\operatorname{Bd} X$ be G-saturated or be a 2-sphere. The condition that $P(\operatorname{Bd} X)$ is a 2-sphere would follow if $\operatorname{Bd} X$ were a G-saturated 2-sphere.

Proof. We will show that Theorem 3 hypotheses yield Lemma 2.1 hypothesis, including the homeomorphism h. Then Lemma 2.1 and the Shrinkability Theorem complete the proof.

The existence for each $g \in H$ of the sequence of nested open sets X^i, $i = 1, 2, 3$, is immediate from the hypotheses of Theorem 3. Hence, Statement (A) is true.

Recall that in [W] there are certain 2-spheres which totally miss the nondegenerate elements. We will apply the [W] Lemma 1 to the image of $\operatorname{Bd} X^2$ in S^3/G_g. Then we will approximate the resulting composite map of S^3 by a homeomorphism, which will be the required f. To satisfy the required epsilonics, we must do the work carefully.

From Armentrout [A1], [A2] and Price [P] we know that because S^3/G_g is homeomorphic to S^3 and each element of the decomposition is cellular, it must be true that the decomposition is shrinkable. This shrinkability can be described as follows. Let $\mathscr{V}$ be a saturated open cover of S^3. Then there is a pseudoisotopy $h_t\colon S^3 \to S^3$, $t \in [0,1]$ such that

(a) $h_0 = \mathrm{id}$,
(b) $h_1 = P_g$, where P_g is the projection map of S^3 onto S^3/G_g, and
(c) for each $g' \in G_g$ there is a $V_{g'} \in \mathscr{V}$ such that for all t, $h_t(g') \subset V_{g'}$.

For our use, we will choose $\mathscr{V}$ to be an open cover which is G-saturated with respect to the original decomposition G, and also has the properties

(i) if for any $g' \in H_g$ an element V of $\mathscr{V}$ contains g', then $V \subset X^3$, and
(ii) if an element V of $\mathscr{V}$ intersects F, then there is a nondegenerate element $g' \in H_g$ such that $g' \subset F$ and V is in the $(\delta/8)$-neighborhood of g', where $\delta = \varepsilon - \max\{\operatorname{Diam} g\colon g \in H \text{ and } g \subset F\}$.

Note that because each V is G-saturated with respect to G, we actually have condition (ii) with respect to G. This implies that $P_g(F)$ contains no $P_g(g)$ for $g \in G$ with diameter more than $\varepsilon - \delta/4$.

We now apply [W] Lemma 1. For "ε" in that hypothesis use $\min\{\operatorname{dist}(P_g(\operatorname{Bd} X^2), P_g(\operatorname{Bd} X^3)), \varepsilon/4, \text{ and } \delta/4\}$. Hence, we get a homeomorphism f such that

(1) $f | S^3 - P_g(\operatorname{Bd} X^3) = \mathrm{id}$,
(2) if $g' \in G$ and $g' \subset X^1$, then $\operatorname{Diam} f P_g(g') < \varepsilon/4$, and
(3) if $g' \in G$ and $g' \subset F$, then $\operatorname{Diam} f P_g(g') < \varepsilon - \delta/2$.

The map $f P_g$ on S^3 is cellular. Hence, it is shrinkable and can be approximated arbitrarily closely by homeomorphisms. This shows that the required homeomorphism h exists. The 3-dimensionality of S^3/G follows from the hypotheses on $\operatorname{Bd} X$. This completes the proof that Statement (A) implies Statement (B), and completes the proof of Theorem 3. ■

COROLLARY 3A. *Let G be an usc decomposition of S^3. Assume that for each $g \in H$ and open U containing g there is an open set X such that $g \subset X \subset U$; the set $P(\operatorname{Bd} X)$ is a 2-sphere; each element of $\operatorname{Sat} \operatorname{Bd} X$ is a tame polyhedron, and there are only countably many nondegenerate elements in $\operatorname{Bd} X$. Then S^3/G is homeomorphic to S^3.*

Proof. This is immediate from the Starbird–Woodruff result [S, W].

COROLLARY 3B. *Let G be an usc decomposition of S^3. Assume that each $g \in H$ is a tame polyhedron and that H contains at most a countable number of arcs (but any number of other polyhedra). Also assume that for each $g \in H$ and open set U containing g there is a crumpled cube X such that $g \subset \operatorname{Int} X \subset U$ and $\operatorname{Bd} X$ is saturated. Then S^3/G is homeomorphic to S^3.*

Proof. This follows from Corollary 3A, since the conditions here imply that H_A is countable.

Remark. In Corollaries 3A and 3B the tameness condition can be replaced by assuming that each $g \in H$ has a mapping cylinder neighborhood in S^3. This follows from Theorem 2 in [W1].

References

[A1] S. Armentrout, *Cellular decompositions of 3-manifolds that yield 3-manifolds*, Memoir 107, Amer. Math. Soc. (1971).
[A2] — *Concerning cellular decompositions of 3-manifolds with boundary*, Trans. Amer. Math. Soc. 137 (1969), pp. 231–236.
[A3] — *A three-dimensional spheroidal space which is not a sphere*, Fund. Math. 68 (1970), pp. 183–186. MR 42 # 5239.
[A4] — *Saturated 2-spheres in Bing's Straight Line Segment Example*, to appear in Proceedings of the Texas Topology Conference (1980).

[E] W. T. Eaton, *The sum of solid spheres*, Michigan Math. J. 19 (1972), pp. 193–207.
[E, G] R. D. Edwards and L. C. Glaser, *A method of shrinking decompositions of certain manifolds*, Trans. Amer. Math. Soc. 165 (1972), pp. 45–56.
[H] N. Hosay, *Erratum to "The sum of a cube and a crumpled cube is S^3"*, Notices Amer. Math. Soc. 11 (1964), p. 152.
[H, W] W. Hurewicz and H. Wallman, *Dimension Theory*, Princeton University Press, Princeton, N. J., 1948.
[K] L. V. Keldys, *Topological Imbeddings in Euclidean Space*, Proc. Steklov Inst. Math. 81 (1966); English transl., Amer. Math. Soc., Providence, R. I., 1968. MR 34 # 6745; 38 # 696.
[L] L. L. Lininger, *Some results on crumpled cubes*, Trans. Amer. Math. Soc. 118 (1965), pp. 534–549. MR 31 # 2717.
[Mc1] L. F. McAuley, *Some upper semi-continuous decompositions of E^3 into E^3*, Ann. of Math. (2) 73 (1961), pp. 437–457. MR 23 # A3554.
[Mc2] — *Upper semicontinuous decompositions of E^3 into E^3 and generalizations to metric spaces*, Topology of 3-Manifolds and Related Topics (Proc. Univ. of Georgia Inst., 1961), Prentice-Hall, Englewood Cliffs, N. J., 1962, 21–26. MR 25 # 4502.
[M] R. L. Moore, *Concerning upper semi-continuous collections of continua*, Trans. Amer. Math. Soc. 27 (1925), pp. 416–428.
[P] T. M. Price, *Decompositions of S^3 and pseudo-isotopies*, Trans. Amer. Math. Soc. 140 (1969), pp. 295–299.
[R] M. J. Reed, *Decomposition spaces and separation properties*, Doctoral Dissertation, SUNY, Binghamton, 1971.
[S] L. C. Siebenmann, *Approximating cellular maps by homeomorphisms*, Topology 11 (1972), pp. 271–294.
[S, W] M. Starbird and E. Woodruff, *Decompositions of E^3 with countably many non-degenerate elements*, Geometric Topology, edited by J. C. Cantrell, Academic Press, 1979, pp. 239–252.
[Wh] G. T. Whyburn, *Analytic Topology*, Amer. Math. Soc. Colloq. Publ., vol. 28, Amer. Math. Soc., Providence, R. I., 1942.
[W] E. P. Woodruff, *Decomposition spaces having arbitrarily small neighborhoods with 2-sphere boundaries*, Trans. Amer. Math. Soc. 232 (1977), pp. 195–204.
[W1] — *Decomposition of E^3 into cellular sets*, Geometric Topology edited by J. C. Cantrell, Academic Press, 1979, 253–257.

THE INSTITUTE FOR ADVANCED STUDY
Princeton, New Jersey 08540
and
TRENTON STATE COLLEGE
Trenton, New Jersey 08625

Accepté par la Rédaction 6. 6. 1981

ω-Trees in stationary logic

by

A. Baudisch, D. G. Seese and **H. P. Tuschik** (Berlin)

Abstract. It is proved that for all trees $\underset{\sim}{A}$, $\underset{\sim}{B}$ of height at most ω $\underset{\sim}{A} \equiv \underset{\sim}{B}(Q_1)$ implies $\underset{\sim}{A} \equiv \underset{\sim}{B}(aa)$. Moreover all such trees are finitely determinate and the theory of the class of all trees of height at most ω in stationary logic is decidable.

Preliminaries. The study of stationary logic $L_{\omega\omega}(aa)$ was begun by J. Barwise, M. Kaufmann and M. Makkai [1], following a suggestion of S. Shelah [8]. In their paper Barwise, Kaufmann and Makkai proved Completeness, Compactness, Downward–Löwenheim–Skolem-Theorem and Omitting Types theorems for stationary logic. The quantifier Q_1 "there exist uncountably many" is definable in stationary logic. Thus $L_{\omega\omega}(Q_1)$ is a sublogic of $L_{\omega\omega}(aa)$. We assume the reader familiar with stationary logic.

Throughout this paper L denotes an elementary language for partially ordered structures with finitely many individual constants and predicates eventually.

Structures for L are denoted by $\underset{\sim}{A}$, $\underset{\sim}{B}$, etc. and their universes $|\underset{\sim}{A}|$, $|\underset{\sim}{B}|$, etc. by the corresponding capital letters A, B, etc. For a set M let $P_{\omega_1}(M)$ denote the set of all countable subsets of M.

A set $\underline{A} \subseteq P_{\omega_1}(A)$ is unbounded if every $B \in P_{\omega_1}(A)$ is a subset of some $C \in \underline{A}$. $\underline{A}$ is closed if the union of each increasing sequence $B_0 \subseteq B_1 \subseteq \ldots \subseteq B_n \subseteq \ldots$ of elements of $\underline{A}$ is again an element of $\underline{A}$.

Closed and unbounded (cub) subsets of $P_{\omega_1}(A)$, $P_{\omega_1}(B)$, etc. are denoted by $\underline{A}$, $\underline{B}$, etc.

To get $L(aa)$ we expand L by adding countably many set variables $X_1, X_2, \ldots$, the $\in$ symbol and a new quantifier aa. Formulas of $L(aa)$ are formed as usual with the new formation rule:

if φ is a formula of $L(aa)$ so is $(aaX)\varphi$ for each set variable X.

For an L-structure $\underset{\sim}{A}$, $\underset{\sim}{A} \models (aaX\ \varphi(X))$ holds iff there is a cub collection $\underline{A} \subseteq P_{\omega_1}(A)$ such that for all $B \in \underline{A}\ \underset{\sim}{A} \models \varphi(B)$ hold. Let K be a class of structures for L then $\mathrm{Th}_{aa}(K)$, $\mathrm{Th}_1(K)$ denote the theory of K in the language $L(aa)$, $L(Q_1)$ respectively. In case that K has only one element $\underset{\sim}{A}$ we write $\mathrm{Th}_{aa}(\underset{\sim}{A})$, $\mathrm{Th}_1(\underset{\sim}{A})$ instead of $\mathrm{Th}_{aa}(K)$, $\mathrm{Th}_1(K)$ respectively.