$< r$. Hence $C \cap U_q \neq \emptyset$. Let $u \in K$ such that $G(u) \cap U_q \neq \emptyset$, then $d(u, G(u)) < \varepsilon$. This contradiction completes the proof.

Problem 3.1 would have an affirmative answer if the following problem, due to Maćkowiak (see [9]), has an affirmative answer.

3.3. Problem. Do arc-like continua have the fixed-point property for upper semi-continuous refluent set valued functions?

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Metrizability of certain quotient spaces

by

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Abstract. The metrizability of certain sequential spaces can be characterized by whether or not they contain two canonical subspaces.

Introduction. Let us begin with the following well known examples. These example will play an important role in this paper. Let α be an infinite cardinal number. Let S_α be the space obtained from the topological sum of α convergent sequences by identifying all the limit points. S_ω is especially called sequential fan. We also need another canonical example S_2. That is, $S_2 = (N \times N) \cup \cup N \cup \{0\}$, N is the set of integers, with each point of $N \times N$ an isolated point. A basis of neighborhoods of $n \in N$ consists of all sets of the form $\{n\} \cup \{(m, n); m \geqslant m_0\}$. And U is a neighborhood of 0 if and only if $0 \in U$ and U is a neighborhood of all but finitely many $n \in N$.

We recall some basic definitions. Let X be a space and Let $\mathfrak{A}$ be a cover (not necessarily closed or open) of X. Then X has the *weak topology* with respect to $\mathfrak{A}$, if $F \subset X$ is closed in X whenever $F \cap A$ is closed in A for each $A \in \mathfrak{A}$. Of course we can replace "closed" by "open". A space X is *sequential* (resp. a *k-space*), if X has the weak topology with respect to the cover consisting of all compact metric subsets (resp. compact subsets). As is well known, a sequential space (resp. k-space) is characterized as a quotient image of a metric space [5] (resp. locally compact space [2]). A space X is a *k_ω-space* [14], if it has the weak topology with respect to a countable cover consisting of compact subsets of X. A space X is *Fréchet* (resp. *strongly Fréchet* [21], E. Michael [15] calls it countably bi-sequential) if whenever $x \in \bar{A}$ (resp. $x \in \bar{A}_n$ with $A_{n+1} \subset A_n$), there exist $x_n \in A$ (resp. $x_n \in A_n$) such that $x_n \to x$. We shall remark that S_ω is a Fréchet k_ω-space which is not strongly Fréchet, and that S_2 is a non-Fréchet, k_ω-space.

Now, S_ω (resp. S_2) is helpful in analyzing the gap of Fréchet spaces and strongly Fréchet spaces [22; 16 (b)] (resp. gap of sequential spaces and Fréchet spaces [6; Proposition 7.3]). A. V. Arhangel'skiĭ and S. P. Franklin [1] introduced the sequential order $\sigma(X)$ of a space X. For a hereditarily normal sequential space X, V. Kannan [11] gave a characterization of $\sigma(X)$ by whether or not X contains spaces S_n defined inductively, and showed that such a space X

is Fréchet if and only if it contains no closed copy of S_2. In connection with the study of products of k-spaces, sequential spaces, and spaces of countable tightness, spaces S_ω, S_{ω_1}, and $S_{2\omega}$ play important parts in [8], [10], [25], and [27].

S. P. Franklin and B. V. Smith Thomas [7] gave the following metrization: Every k_ω-space X with metrizable "pieces" is metrizable if and only if it contains no copy of S_ω and no S_2. This result is precisely a case where X is the quotient image of a locally compact, separable metric space. As generalizations of this case, we shall consider certain quotient spaces of metric spaces, and as related spaces, spaces having the weak topology with respect to a certain point-countable cover, and CW-complexes. In this paper, we give some metrizations of these spaces by whether or not they contain S_ω and S_2.

We assume all spaces to be *Hausdorff*, and all maps continuous and onto.

1. A-spaces, and S_ω. E. Michael [16] introduced the notion of A-spaces, inner-closed A-spaces, strict A-spaces etc., and characterized spaces X with the property that each map onto X belonging to some class $\mathfrak{C}_1$ must belong to some class $\mathfrak{C}_2$. In E. Michael, R. C. Olson and F. Siwiec [17], these spaces are investigated detailedly. A space X is an *A-space*, if whenever $\{A_n;\ n\in N\}$ is a decreasing sequence with each $\overline{A_n-\{x\}}\ni x$ (simply, $(A_n)\downarrow x$), then there exist $B_n\subset A_n$ such that $B=\bigcup \bar{B}_n$ is not closed in X. If the B_n are closed (resp. singletons), then such a space is *inner-closed A* (resp. *inner-one A*). If $x\in\bar{B}-B$, then X is a *strict A-space*. It is easy to show that S_2 is an A-space (indeed, strict A-space), but S_ω is not A.

THEOREM 1.1. *Let X be a sequential space. Then X is an A-space if and only if it contains no closed copy of S_ω.*

Proof. "Only if". Since every closed subset of an A-space is A, if X contains a closed copy of S_ω, then S_ω is an A-space. But S_ω is not A. This is a contradiction. Hence X contains no closed copy of S_ω.

"If". First we shall prove that if $(A_n)\downarrow x$ with each A_n closed, then $\{A_n;\ n\in N\}$ is not hereditarily closure preserving. To show this, suppose that $\{A_n;\ n\in N\}$ is hereditarily closure preserving. Since $x\in\overline{A_n-\{x\}}$, x is not isolated in a closed subset A_n. Then, since A_n is sequential, there exists a convergent sequence $\{x_{ni};\ i\in N\}$ in $A_n-\{x\}$ with $x_{ni}\to x$. For each $n\in N$, let $C_n=\{x_{ni};\ i\in N\}\cup\{x\}$, $Y=\bigcup C_n$ and let X_0 be the topological sum of C_n's. Let $f\colon X_0\to Y$ be the obvious map. Then, since $\{C_n;\ n\in N\}$ is hereditarily closure preserving, f is a closed map of a metric space X_0 onto a Fréchet space Y. Since Y does not contain a closed copy of S_ω, by [8; Lemma 2] $\partial f^{-1}(x)$ is compact. However, $\partial f^{-1}(x)$ is not compact. This is a contradiction. Hence $\{A_n;\ n\in N\}$ is not hereditarily closure preserving if $(A_n)\downarrow x$ with each A_n closed.

Next we prove X is an A-space. To show this, let $(A_n)\downarrow x$. Then $(\bar{A}_n)\downarrow x$, hence by the above there exist subsets B_n of $\bar{A}_n$ such that $B=\bigcup\bar{B}_n$ is not closed in X. Since X is sequential, there exist $b\notin B$ and $b_n\in\bar{A}_n$ such that $b_n\to b$ with $b_n\neq b$. Thus there exist neighborhoods V_n of b_n with $\bar{V}_n\notin b$. If $C_n=A_n\cap V_n$, then $\bigcup C_n-\bigcup\bar{C}_n\ni b$. This implies that there exist subsets $C_n\subset A_n$ such that $\bigcup\bar{C}_n$ is not closed in X. Thus X is an A-space.

By the following example, the sequentialness of X of Theorem 1.1. is essential.

EXAMPLE 1.2. A paracompact k-space which contains no copy of S_ω and no S_2, but is not an A-space.

Proof. Let X be the space obtained from the topological sum of ω ordinal space $[0,\omega_1]$ by identifying all the first uncountable ordinal numbers ω_1. Since no sequence converges to ω_1, it is easy to check that X is a k-space which contains no copy of S_ω and no S_2. But X is not an A-space.

The following lemma due to [17] will be useful.

LEMMA 1.3. (i) *A regular space X is strongly Fréchet if and only if X is a Fréchet A-space.*

(ii) *Suppose X is a regular sequential space. If X is an A-space (resp. inner-closed A-space), then every subset of X is an A-space (resp. inner-one A-space).*

Proof. (i) is Proposition 8.1 in [17].

(ii) Since X is sequential, as is well known, if $x\in\bar{A}$ then $x\in\bar{C}$ for some countable $C\subset A$ (cf. [15; Propositions 8.3 & 8.5]). Then, by [17; Proposition 5.1], X is a strict A. Hence every subset of X is an A-space. The parenthetic part follows from [17; Proposition 5.4].

COROLLARY 1.4. *Let X be a regular Fréchet space. Suppose that X has the weak topology with respect to a point-countable cover $\mathfrak{C}$ consisting of compact subsets (for example, X is a k_ω-space). Then X is locally compact if and only if it contains no closed copy of S_ω.*

Proof. The "only if" part is obvious.

"If". By Theorem 1.1, X is an A-space. Thus X is strongly Fréchet by Lemma 1.3 (i). Suppose that X is not locally compact. Then there exists a point $x_0\in X$ such that the closure of any neighborhood of x_0 is not compact. Let $\{C^*\in\mathfrak{C};\ x_0\in C^*\}=\{C_1^*,C_2^*,\ldots\}$ and $X_n=\bigcup_{i=1}^{n}C_i^*$ for $n\in N$. Then $(X-X_n)\downarrow x_0$, hence there exist $x_n\in X-X_n$ with $x_n\to x_0$. Let $K=\{x_n;\ n\in N\}\cup\{x_0\}$ and $\{C\in\mathfrak{C};\ C\cap K\neq\varnothing\}=\{C_1,C_2,\ldots\}$, and let $Y_n=\bigcup_{i=1}^{n}C_i$ for $n\in N$. Assume $K\not\subset Y_n$ for $n\in N$, so there is $D=\{y_n;\ n\in N\}$ with $y_n\in K-Y_n$. Since $D\cap C$ is at most finite for each $C\in\mathfrak{C}$, D is discrete in X, hence in K, a contradiction. Thus K is contained in a finite union of elements of $\mathfrak{C}$. But each element of $\mathfrak{C}$ is closed, so there exists $C_{i_0}^*$ such that $C_{i_0}^*$ meets infinitely many elements of K. This is a contradiction. Hence X is locally compact.

THEOREM 1.5. *Let X be a regular sequential space. Then the following are equivalent.*

(a) *X contains of copy of S_ω.*

(b) *X contains no closed copy of S_ω.*

(c) *X is an A-space.*

(d) *Every Fréchet subspace of X is strongly Fréchet.*

Proof. (a) → (b) is clear. (b) → (c) follows from Theorem 1.1. (c) → (d) follows from Lemma 1.3. (d) → (a) is obvious.

LEMMA 1.6. *Let X have the weak topology with respect to a cover $\mathfrak{A}$ consisting of strongly Fréchet subspaces. If for each $x \in X$, $\{A \in \mathfrak{A};\ x \in A\}$ is finite (resp. countable), then X contains no copy of S_ω (resp. no copy of S_{ω_1}).*

Proof. Since the parenthetic part is proved similarly, so suppose that X contains a copy Y of S_ω. Let $Y = \{x_0\} \cup \bigcup_{i=1}^{\infty} \{x_{in};\ n \in N\}$ with $x_{in} \to x_0$ for each $i \in N$. Let $C_i = \{x_{in};\ n \in N\} \cup \{x_0\}$ for $i \in N$. Then Y has the weak topology with respect to $\{C_i;\ i \in N\}$. On the other hand, since each C_i is closed in X, each C_i has the weak topology with respect to $\mathfrak{A} \cap C_i = \{A \cap C_i;\ A \in \mathfrak{A}\}$. Thus Y has the weak topology with respect to a cover $\{A \cap C_i;\ A \in \mathfrak{A},\ i \in N\}$. But each element of the cover of Y is contained in an element of a cover $\mathfrak{A} \cap Y$ of Y. Therefore Y has the weak topology with respect to $\mathfrak{A} \cap Y$. Let $\mathfrak{B} = \mathfrak{A} \cap Y$ and $\mathfrak{B}_0 = \{B \in \mathfrak{B};\ x_0 \in B\}$. Then $\mathfrak{B}_0$ is finite. Since each element of $\mathfrak{B}_0$ is strongly Fréchet, it contains no copy of S_ω. Thus there exists C_{n_0} such that $C_{n_0} \cap B$ is finite for each $B \in \mathfrak{B}_0$. Let $S = C_{n_0} - \{x_0\}$. Then $S \cap B$ is closed in B for every $B \in \mathfrak{B} - \mathfrak{B}_0$. To show this, suppose that $S \cap B_0$ is not closed in B_0 for some $B_0 \in \mathfrak{B} - \mathfrak{B}_0$. If $S \cap B_0$ has an accumulation point a_0 in B_0, then $a_0 = x_0$ so that $B_0 \in \mathfrak{B}_0$. This is a contradiction. Then $S \cap B_0$ has no accumulation point in B_0. Thus $S \cap B_0$ is closed in B_0. This is a contradiction. Hence $S \cap B$ is closed in B for every $B \in \mathfrak{B} - \mathfrak{B}_0$. In the sequel, $S \cap B$ is closed in B for every $B \in \mathfrak{B}$. Since Y has the weak topology with respect to $\mathfrak{B}$, this shows that S is closed in Y. However, S does not contain the limit point $x_0 \in Y$. This is a contradiction. Hence X contains no copy of S_ω.

THEOREM 1.7. *Let $f\colon X \to Y$ and X be metric.*

(i) *Suppose that f is quotient. If f is compact, i.e., every $f^{-1}(y)$ is compact and Y is regular (resp. f is an s-map, i.e., every $f^{-1}(y)$ is separable), then Y contains no copy of S_ω* (resp. *under* (CH)[1] *no of S_{ω_1}). Moreover if X is locally compact metric, then the parenthetic part holds without* (CH).

(ii) *Suppose that f is closed. Then every $\partial f^{-1}(y)$ is compact (resp. Lindelöf) if and only if Y contains no copy of S_ω (resp. no S_{ω_1}).*

Proof. (i) Let f be compact. Let S be any Fréchet subspace of Y. Then S has the weak topology with respect to the cover $\{C_\gamma;\ \gamma \in \Gamma\}$ consisting of all compact

(1) (CH) can be omitted.

metric subspaces. Since each C_γ is closed in Y, each $f|f^{-1}(C_\gamma)$ is quotient. Hence $g = f|f^{-1}(S)$ is a quotient compact map onto a Fréchet space S. Thus by [5; Theorem 2.3] and [14; Proposition 3.2], g is bi-quotient, so that S has a point-countable base by [4; Theorem 1.1]. Suppose now that Y contains a copy S of S_ω. Then, by the above S has a point-countable base, a contradiction. Hence Y contains no copy of S_ω.

Next, let f be an s-map. Suppose that Y contains a copy Y_0 of S_{ω_1}. Since Y_0 is Fréchet, $h = f|f^{-1}(Y_0)$ is a quotient s-map. Let $\mathfrak{B}$ be a σ-locally finite base of $X_0 = f^{-1}(Y_0)$. Since h is a quotient s-map, Y_0 has the weak topology with respect to a point-countable cover $\mathfrak{G} = h(\mathfrak{B})$. Let x_0 be non-isolated in Y_0. Let U be an open subset of Y_0 with $x_0 \in U$. Then U has the weak topology with respect to $\mathfrak{G}' = \{G \in \mathfrak{G};\ G \subset U\}$, because $f|f^{-1}(U)$ is quotient and $\{B \in \mathfrak{B};\ B \subset f^{-1}(U)\}$ is a base for $f^{-1}(U)$. Suppose that $x_0 \in \overline{U - \mathrm{St}(x_0, \mathfrak{G}')}^{\,U}$. Since U is Fréchet, there exist $x_n \in U - \mathrm{St}(x_0, \mathfrak{G}')$ with $x_n \to x_0$. Let $A = \{x_n;\ n \in N\}$. Then $A \cap G$ is closed for each $G \in \mathfrak{G}'$. Thus A is closed in U, a contradiction. Hence, $x_0 \in \operatorname{int}\mathrm{St}(x_0, \mathfrak{G}') \subset U$. This implies that $\{\operatorname{int}\mathrm{St}(x_0, \mathfrak{H});\ \mathfrak{H} \subset \mathfrak{G}\}$ is a local base of x_0 with cardinality $\leqslant 2^\omega$ in Y_0. This is a contradiction under (CH). Hence Y has no copy of S_{ω_1} under (CH). When X is moreover locally compact, X has the weak topology with respect to a locally finite closed cover $\mathfrak{F}$ consisting of compact metric subspaces. Thus Y has the weak topology with respect to a point-countable cover $f(\mathfrak{F})$ consisting of compact metric subspaces. Thus by Lemma 1.6, Y contains no copy of S_{ω_1}.

(ii) In view of [8; Lemma 2] we have the "if" part.

"Only if". If every $\partial f^{-1}(y)$ is compact, then Y is metric. So this part is clear. Let every $\partial f^{-1}(y)$ is Lindelöf, and $\mathfrak{F}$ be a σ-locally finite closed k-network of X. Recall that a closed k-network is a closed cover such that if $C \subset U$ with C compact and U open, there exists a finite subcover $\mathfrak{C}$ with $C \subset \bigcup \mathfrak{C} \subset U$. Then $f(\mathfrak{F})$ is a closed k-network of Y. Indeed, let $C \subset U$ with C compact and U open in Y. By [12; Corollary 1.2], C is the image of some compact subset K of X. Thus there is a finite subcover $\mathfrak{F}'$ of $\mathfrak{F}$ with $K \subset \bigcup \mathfrak{F}' \subset f^{-1}(U)$, hence $C \subset \bigcup f(\mathfrak{F}') \subset U$. Then $f(\mathfrak{F})$ is a closed k-network. Since every $\partial f^{-1}(y)$ is Lindelöf, as in the proof of [12; Corollary 1.2] we can assume that $f^{-1}(y)$ is Lindelöf. Then $f(\mathfrak{F})$ is point-countable. Hence Y has a point-countable closed k-network. Thus Y contains no copy of S_{ω_1} by [26; Proposition 1].

We remark that the converse of Theorem 1.7 (i) is not valid. Indeed, let Y be a regular separable first countable, non-metric space. Then by [5; Corollary 1.13], Y is the quotient image of a locally compact metric space. Since Y is first countable, it contains no copy of S_ω and no S_{ω_1}. But since Y has no point-countable base, by [14; Proposition 3.3(d)] and [4; Theorem 1.1] there is no quotient map $f\colon X \to Y$ with X metric and each $\partial f^{-1}(y)$ separable.

LEMMA 1.8. *Let X be a* CW-*complex due to Whitehead, and let $\{e_\gamma;\ \gamma\}$ be the cells of X.*

(i) *Then X is a sequential space having the weak topology with respect to $\{\bar{e}_\gamma;\ \gamma\}$, where $\bar{e}_\gamma = \mathrm{cl}\, e_\gamma$.*

(ii) ([27; Lemma 2.2]). *If X contains no closed copy of S_ω (resp. S_{ω_1}), then each $\{\gamma;\ \bar{e}_\gamma \ni x\}$ is finite (resp. countable).*

From Lemmas 1.6 and 1.8, we have

THEOREM 1.9. *Let X be a CW-complex with the cells $\{e_\gamma; \gamma\}$. Then X contains no copy of S_ω (resp. no S_{ω_1}) is and only if each $\{\gamma;\ \bar{e}_\gamma \ni x\}$ is finite (resp. countable).*

By Theorems 1.1 and 1.9 together with Lemma 1.8(i), we have

COROLLARY 1.10. *Let X be a CW-complex with the cells $\{e_\gamma;\ \gamma\}$. Then X is an A-space if and only if each $\{\gamma;\ \bar{e}_\gamma \ni x\}$ is finite.*

2. Fréchet spaces, and S_2.

THEOREM 2.1. *Let a regular space X have the weak topology with respect to a point-countable cover $\mathfrak{A}$. Suppose that* (a) *or* (b) *below holds. Then X is Fréchet if and only if it contains no copy of S_2.*

(a) *Each finite union of elements of $\mathfrak{A}$ is Fréchet, or a sequential space in which every point is G_δ.*

(b) *X is sequential and each countable union of elements of $\mathfrak{A}$ is a space in which every point is G_δ.*

Proof. Case (a). The "only if" part is obvious. So we shall prove the "if" part. Since X has the weak topology with respect to $\mathfrak{A}$, it has the weak topology with respect to the collection $\mathfrak{A}^*$ of all finite union of elements of $\mathfrak{A}$. Moreover each of these unions is sequential. Thus X is sequential. Suppose that X is not Fréchet. Thus, following the proof of [6; Proposition 7.3], we can choose a countable subset $X_* = \{x_0\} \cup \{x_i; i \in N\} \cup \{x_{ij};\ i, j \in N\}$ of X such that $x_i \in G_i$ for some pairwise disjoint open subsets G_i, and $x_{ij} \to x_i$, $x_i \to x_0$, also no sequence of x_{ij}'s converges to x_0. Thus X_* is a copy of S_2, if X_* is sequential; that is, every subset U of X_* is open in X_* whenever each sequence converging to a point in U is eventually in U.

Now, let $\{A \in \mathfrak{A};\ A \cap X_* \neq \emptyset\} = \{A_i; i \in N\}$, and let $X_n = \bigcup_{i=1}^{n} A_i$ for $n \in N$. Let us put $C_0 = \{x_0\} \cup \{x_i; i \in N\}$, $C_i = \{x_i\} \cup \{x_{ij}; j \in N\}$ for $i \in N$. Since $\mathfrak{A}$ is point-countable, by the proof of Corollary 1.4, each C_i, $i \in \omega$, is contained in some element of $\mathfrak{A}^*$. Thus there exists X_{n_0} with $C_0 \subset X_{n_0}$. Suppose that $\{i \in N;\ X_{n_0} \cap C_i$ is infinite$\}$ is not finite. Then there exists an infinite subset $X_0 = \{x_0\} \cup \{x_{i_k};\ \ k \in N\} \cup \{x_{i_k j_q};\ q \in N\}$ of X_{n_0} such that $X_0 \subset X_*$. If X_{n_0} is Fréchet, then x_0 is a limit point of some $x_{i_k j_q}$-s. This is a contradiction. So we assume that X_{n_0} is a sequential space in which every point is G_δ. Since x_0 is a G_δ-set in X_{n_0}, there exists a decreasing sequence $\{V_i; i \in N\}$ od open subsets of X_{n_0} with $\mathrm{cl}_{X_{n_0}} V_{i+1} \subset V_i$ and $x_0 = \bigcap V_i$. Since each V_i contains x_0, we can assume that for each i_k, $\{x_{i_k}\} \cup \{x_{i_k j_q}; q \in N\}$ is contained in V_{i_k}. Hence it follows that X_0

is closed in X_{n_0}. Since X_{n_0} is sequential, so is X_0. Hence X_0 is a copy of S_2. Thus X contains a copy of S_2, a contradiction. Therefore $X_{n_0} \cap C_{m_0}$ is at most finite for some C_{m_0}. Since C_{m_0} is contained in some X_{n_1} $(n_1 > n_0)$, we may assume that $C_{m_0} \subset X_{n_1} - X_{n_0}$. By induction, there exists an infinite subset $\{m_0, m_1, \ldots\}$ of N such that $C_{m_k} \subset X_{n_{k+1}} - X_{n_k}$ $(n_{k+1} > n_k)$. Let $Y = C_0 \cup \bigcup C_{m_k}$ and $\mathfrak{A}_0 = \mathfrak{A} - \{A_i;\ i \in N\}$. Then $Y \cap X_n$ is closed in X_n for $n \in N$, also $Y \cap A = \emptyset$ if $A \in \mathfrak{A}_0$. Since X has the weak topology with respect to $\{X_n;\ n \in N\} \cup \mathfrak{A}_0$, this shows that Y is closed in X. Thus Y is sequential. Then Y is a copy of S_2, hence X contains a copy of S_2, a contradiction. Therefore X must be Fréchet.

Case (b). The notation used here is the same as in case (a). Let $Z = \bigcup \{A \in \mathfrak{A};\ A \cap X_* \neq \emptyset\}$ assuming X is not Fréchet. Since $X_* \subset Z$ and $x_0 \in X_*$ is a G_δ-set in Z, by the same way as in (a), we can assume that X_* is closed in Z. But $A \cap X_* = \emptyset$ if $A \in \mathfrak{A}_0$. Then X_* is closed in X, because X has the weak topology with respect to $\{Z\} \cup \mathfrak{A}_0$. Since X is sequential, so is X_*. Then X_* is a copy of S_2, hence X contains a copy of S_2, a contradiction. Therefore X must be Fréchet.

COROLLARY 2.2. *Let a regular space X have the weak topology with respect to a point-countable cover $\mathfrak{A}$. Suppose that each element of $\mathfrak{A}$ is closed and that each element is Fréchet or a sequential space in which every point is G_δ. Then X is Fréchet if and only if X contains no copy of S_2.*

Proof. The "only if" part is obvious, so we prove the "if" part. Suppose that X is not Fréchet. For $n \in N$, let X_n be the subsets defined in the proof of Theorem 2.1. Then each X_n is Fréchet, or a sequential space in which every point is G_δ, or $F_1 \cup F_2$, where F_1 is closed and Fréchet, F_2 is closed and a sequential space in which every point is G_δ. From the proof given there, we have a contradiction. Thus X is Fréchet.

From the proof of Theorem 2.1, we also have

THEOREM 2.3. *Let X be a regular sequential space in which every point is G_δ. Then the following are equivalent.*

(a) *X contains no copy of S_2.*

(b) *X contains no closed copy of S_2.*

(c) *X is Fréchet.*

The following example shows that the condition "each point of X of the previous theorem is G_δ" is essential.

EXAMPLE 2.4. A compact sequential space which contains no copy of S_2 and no S_ω, but it is not Fréchet.

Proof. Let X be the sequential space, non-Fréchet compact space constructed by S. P. Franklin in [6; Example 7.1]. Then [20; Theorem 10] showed that X contains no copy of S_2. But we shall give an indirect proof here. Since X is compact and sequential, by Lemma 1.3(ii), every subspace of X is inner-one A. However S_2 nor S_ω is not inner-one A, so that X contains no copy of S_2 and no S_ω.

3. Strongly Fréchet spaces, and products of spaces of countable tightness.

THEOREM 3.1. *Let X be a sequential space. If X is a regular space in which every point is G_δ, or hereditarily normal, then the following are equivalent.*

(a) *X contains no copy of S_ω and no S_2.*

(b) *X contains no closed copy of S_ω and no closed copy of S_2.*

(c) *X is strongly Fréchet.*

Proof. (a) → (b) and (c) → (a) are obvious.

(b) → (c). Since X contains no closed copy of S_ω, X is an A-space by Theorem 1.1. If each point of X is G_δ, since X contains no closed copy of S_2, then X is Fréchet by Theorem 2.3. If X is hereditarily normal, X is also Fréchet by [11; Corollary 2.3]. Hence X is strongly Fréchet by Lemma 1.3(i).

Recall that a space X has *countable tightness*, $t(X) \leqslant \omega$, if $x \in \bar{A}$ in X, then $x \in \bar{C}$ for some countable $C \subset A$. It is well known that every sequential space has countable tightness.

The following theorem gives a necessary condition for the product to have countable tightness.

THEOREM 3.2. (CH). *Let $f\colon X \to Y$ be a closed map with X paracompact and sequential, and let Z satisfy one of the properties below. It $t(Y\times Z) \leqslant \omega$, then either every $\partial f^{-1}(y)$ is Lindelöf or Z is strongly Fréchet.*

(a) *Regular Fréchet space.*

(b) *Regular sequential space in which every point is G_δ.*

(c) *Hereditarily normal, sequential space.*

Proof. Suppose that some $\partial f^{-1}(y)$ is not Lindelöf. Since $\partial f^{-1}(y)$ is paracompact, $\partial f^{-1}(y)$ has a closed discrete subset of cardinality ω_1. Thus, since X is collectionwise normal and Y is sequential, Y contains a closed copy of S_{ω_1} $(= S_{2\omega})$ by [27; Lemma 1.5]. Since $t(Y\times Z) \leqslant \omega$, so $t(S_{2\omega}\times Z) \leqslant \omega$, hence every k_ω-subspace of Z is locally compact by [27; Proposition 1.1(2)]. Thus Z contains no copy of S_ω and no S_2. Thus, if Z satisfies (a), Z must be strongly Fréchet by Theorem 1.5. If Z satisfies (b) or (c), then Z is also strongly Fréchet by Theorem 3.1.

THEOREM 3.3. *Let X be a regular Fréchet space, and let Y be a non-discrete first countable space. Then the following are equivalent.*

(a) *$X \times Y$ contains no copy of S_ω.*

(b) *$X \times Y$ contains no copy of S_2.*

(c) *X is strongly Fréchet.*

Proof. (a) → (c). Since $X \times Y$ contains no copy of S_ω, neither does X, hence X is an A-space by Theorem 1.1. Thus X is strongly Fréchet by Lemma 1.3(i).

(c) → (a) & (b). Since X is strongly Fréchet and Y is first countable, $X \times Y$ is strongly Fréchet by [15; Proposition 4.D.4]. Hence we have the implication.

(b) → (c). Since Y is not discrete, there is a sequence $\{y_n; n\in N\}$ in Y with $y_n \to y_0$ and $y_n \neq y_0$. Let $C_0 = \{y_n; n\in N\} \cup \{y_0\}$. Suppose now that X is not an A-space. Then X contains a copy of S_ω by Theorem 1.1. Hence $X \times Y$ contains a copy of $S_\omega \times C_0$. But, $S_\omega \times C_0$ is a sequential space in which every point is G_δ, and it contains no copy of S_2. Hence $S_\omega \times C_0$ is Fréchet by Theorem 2.3. However, since S_ω is not strongly Fréchet, by the proof of [15; Proposition 4.D.5], $S_\omega \times C_0$ is not Fréchet. This is a contradiction. Thus X is an A-space. Hence X is strongly Fréchet by Lemma 1.3(i).

4. Metrizability of certain sequential spaces.

LEMMA 4.1. *Let X be a regular space having the weak topology with respect to a point-countable closed cover $\mathscr{F}$ consisting of metric subspaces. Then X is a locally metric space with a point-countable base if and only if X contains no copy of S_ω and no S_2.*

Proof. We prove only the "if" part. Since X is a sequential space which contains no copy of S_ω and no S_2, by Theorems 1.1 and 2.1, X is a Fréchet and A-space. Thus X is strongly Fréchet space by Lemma 1.3(i). Hence, as in the proof of Corollary 1.4, X is locally metric. Let X_0 be the topological sum of $\mathfrak{F}$ and $f\colon X_0 \to X$ be the obvious map. Then f is quotient s-map of a metric space X_0. Thus X has a point-countable base by [4; Theorem 2.2].

THEOREM 4.2. *Let a regular space X have the weak topology with respect to a closed cover $\mathfrak{A}$ consisting of metric subspaces. If* (a) *of* (b) *below holds, then X is metrizable if and only if X contains no copy of S_ω and S_2.*

(a) *$\mathfrak{A}$ is star-countable.* (b) *X is paracompact and $\mathfrak{A}$ is point-countable.*

Proof. By Lemma 4.1, X is locally metric. Thus to show X is metric, it suffices to prove X is paracompact for case (a). Let $\mathfrak{A} = \{A_\beta; \beta\in B\}$, and let $\beta \sim \beta'$ if $\mathrm{St}^n(A_\beta, \mathfrak{U}) \supset A_{\beta'}$, for some $n\in N$. Then by this equivalent relation $\sim$, the set B can be decomposed as $\sum_{\gamma\in\Gamma} B_\gamma$. Let $X_\gamma = \bigcup \{A_\beta; \beta\in B_\gamma\}$ for each $\gamma\in\Gamma$. Then $X_\gamma \cap A$ is empty of A for each $A\in\mathfrak{A}$, so each X_γ is open and closed in X. While each X_γ has the weak topology with respect to $\mathfrak{A}_\gamma = \{A_\beta; \beta\in B_\gamma\}$. Since $\mathfrak{A}_\gamma$ are assumed to be an increasing countable closed covering of X_γ, X_γ has the weak topology with respect to $\mathfrak{A}_\gamma$ in the sense of K. Morita [18]. Thus each X_γ is paracompact by Theorem 4 in [18]. Hence X is paracompact.

By the following example due to R. W. Heath (for example, see [3; Example 5.4.B]), the closedness of $\mathfrak{A}$ in case (a) (resp. the paracompactess of X in case (b)) is essential.

EXAMPLE 4.3. A regular non-metric space X which has the weak topology with respect to a countable open cover (resp. point-finite open and closed cover) consisting of metric subspaces, and X contains no copy of S_ω and no S_2.

Proof. Let X be the subset of the plane defined by the condition $y \geqslant 0$.

Define a topology on X as follows: Let each point above the x-axis be isolated and take as a base at a point $(x, 0)$ the family of all segments starting at $(x, 0)$ which form with the x-axis an angle of 90° if x is rational and an angle of 45° if x is irrational.

Then X is a regular space which is not normal by the Baire category theorem. Since X is first countable, X contains no copy of S_ω and S_2. Let R; $Q = \{q_n;\ n \in N\}$ be the set of real numbers; rational numbers respectively. For $n \in N$, let $X_n = (X-R) \cup (R - \{q_j;\ j > n\})$ (resp. for $x \in R$, let F_x be the line starting at $(x, 0)$ which forms with the x-axis an angle of 90° if x is rational and an angle of 45° if x is irrational. Then $\{X_n;\ n \in N\}$ (resp. $\{\{y\};\ y$ is a point above the x-axis$\} \cup \{F_x;\ x \in R\}$) is a countable open cover (resp. point-finite open and closed cover) of X, so that X has the weak topology with respect to these covers. Since each F_x is obviously metrizable, we only prove that each X_n is metrizable. Each X_n is regular and $X_n = X_0 \cup P_n$, P_n is a finite subset $\{q_j;\ j \leqslant n\}$ and $X_0 = (X-R) \cup (R-Q)$. Then, since X_0 is paracompact, X_n is also paracompact. While, X_n is locally metrizable. Hence X_n is metrizable.

As a generalization of $\aleph_0$-spaces due to E. Michael [13], P. O'Meara [19] introduced the notion of $\aleph$-spaces. An *$\aleph$-space* is a space with a σ-locally finite closed k-network.

THEOREM 4.4. *Let X have one of the properties listed below. Then X is metrizable if and only if X contains no copy of S_ω and no S_2.*

(a) *Regular sequential, $\aleph$-space.*

(b) CW-*complex.*

(c) *Regular space which is the quotient s-image of a locally separable, metric space.*

Proof. The "only if" part is clear, so we prove the "if" part. Suppose that X satisfies (a) or (b). Then X is a sequential space in which every point is G_δ. Since X contains no copy of S_ω and no S_2, X is strongly Fréchet by Theorem 3.1. Thus (a) or (b) implies that X is metrizable by [24; Lemma 2.1] or [23; Lemma 4.3] respectively.

Case (c). Suppose that $f\colon Y \to X$ is a quotient s-map with Y locally separable, metric. Then Y has the weak topology with respect to a locally finite closed cover $\mathfrak{F}$ consisting of separable metric subspaces. Since f is quotient and every $f^{-1}(x)$ is Lindelöf, X has the weak topology with respect to a point-countable cover $f(\mathfrak{F})$. Moreover each element of $f(\mathfrak{F})$ is hereditarily Lindelöf, hence every countable union of elements of $f(\mathfrak{F})$ is a space in which every point is G_δ. Thus, since X is sequential, X is Fréchet by Theorem 2.1. While, X is an A-space by Theorem 1.1. Hence X is strongly Fréchet by Lemma 1.3(i). Thus X has a point-countable base by [15; Theorem 9.8]. Hence X is locally separable, metric space by [4; Corollary 1].

From the proof of case (c) of the previous theorem, and Theorem 1.7(i), we have

COROLLARY 4.5. *Let a regular space X be the quotient s-image of a metric space. If each point of X is G_δ, then X has a point-countable base, or contains a copy of S_2 or S_ω. When X is the quotient compact image of a metric space, we can omit "or S_ω".*

As an application of case (c) of Theorem 4.4, we have the following theorem in terms of weak topologies. Compare with Theorem 4.2, where each element of $\mathfrak{A}$ is assumed to be closed.

THEOREM 4.6. *Let a regular space X have the weak topology with respect to a point-countable cover $\mathfrak{A}$ consisting of locally separable, metric subspaces. Then X is metric, or contains a copy of S_2 or S_ω. When $\mathfrak{U}$ is point-finite, we can omit "or S_ω".*

We shall remark that, by Example 4.3, the separability of each element of $\mathfrak{A}$ is essential even if $\mathfrak{A}$ is countable or point-finite.

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