< \varepsilon. Hence C \cap U_\varepsilon \neq \emptyset. Let u \in K such that G(u) \cap U_\varepsilon \neq \emptyset, then d(u, G(u)) < \varepsilon. This contradiction completes the proof.

Problem 3.1 would have an affirmative answer if the following problem, due to Mackowiak (see [9]), has an affirmative answer.

3.3. Problem. Do arc-like continua have the fixed-point property for upper semi-continuous refulent set valued functions?

References


Metrizability of certain quotient spaces

by

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Abstract. The metrizability of certain sequential spaces can be characterized by whether or not they contain two canonical subspaces.

Introduction. Let us begin with the following well known examples. These example will play an important role in this paper. Let \( \varepsilon \) be an infinite cardinal number. Let \( S_\varepsilon \) be the space obtained from the topological sum of \( \varepsilon \) convergent sequences by identifying all the limit points. \( S_\varepsilon \) is especially called sequential fan. We also need another canonical example \( S_2 \). That is, \( S_2 = (N \times N) \cup \{ \alpha \} \), \( N \) is the set of integers, with each point of \( N \times N \) an isolated point. A basis of neighborhood of \( \alpha \in N \) consists of all sets of the form \([n] \cup \{(m, n); m \geq m_0\} \). And \( U \) is a neighborhood of \( 0 \) if and only if the \( U \) and \( \alpha \) is a neighborhood of almost all \( n \in N \).

We recall some basic definitions. Let \( X \) be a space and \( \mathfrak{B} \) be a cover (not necessarily closed or open) of \( X \). Then \( X \) has the weak topology with respect to \( \mathfrak{B} \), if \( F \subset X \) is closed in \( X \) whenever \( F \cap A \) is closed in \( A \) for each \( A \in \mathfrak{B} \). Of course we can replace “closed” by “open”. A space \( X \) is sequential (resp. a k-space, if \( X \) has the weak topology with respect to the cover consisting of all compact metric subsets (resp. compact subsets). As is well known, a sequential space (resp. k-space) is characterized as a quotient image of a metric space [5] (resp. locally compact space [2]). A space \( X \) is a Fréchet k-space [14], if it has the weak topology with respect to a countable cover consisting of compact subsets of \( X \). A space \( X \) is a Fréchet space [21]. E. Michael [15] calls it countably bi-sequential (i.e. sequential with \( \alpha \) in \( A \) and \( A_{\alpha+} \subset A_\alpha \), there exist \( x, y \in A \) (resp. \( x, y \in A_\alpha \)) such that \( x, y \rightarrow x \). We shall remark that \( S_\varepsilon \) is a Fréchet k-space which is not strongly Fréchet, and that \( S_2 \) is not non-Fréchet, k-space.

Now, \( S_\varepsilon \) is helpful in analyzing the gap of Fréchet spaces and strongly Fréchet spaces [22; 16(b)] (resp. gap of sequential spaces and Fréchet spaces [6; Proposition 7.3]). A. V. Arhangelskii and S. P. Franklin [1] introduced the sequential order \( \sigma(X) \) of a space \( X \). For a hereditarily normal sequential space \( X \), V. Kannan [11] gave a characterization of \( \sigma(X) \) by whether or not \( X \) contains spaces \( S_\varepsilon \) defined inductively, and showed that such a space \( X \)
is Fréchet if and only if it contains no closed copy of $S_2$. In connection with the study of products of $k$-spaces, spacios, and spaces of countable tightness, spaces $S_n$, $S_{n+1}$, and $S_{n+2}$ play important parts in [3], [10], [25], and [27].

S. P. Franklin and B. V. Smith Thomas [7] gave the following metrization:

Every $k_n$-space with metrizable “pieces” is metrizable if and only if it contains no copy of $S_n$ or $S_{n+1}$. This result is precisely a case where $X$ is the quotient image of a locally compact, separable metric space. As generalizations of this case, we shall consider certain quotient spaces of metric spaces, and as related spaces, spaces having the weak topology with respect to a certain point-countable cover, and CW-complexes. In this paper, we give some metrizations of these spaces by whether or not they contain $S_n$ and $S_{n+1}$.

We assume all spaces to be Hausdorff, and all maps continuous and onto.

1. **A-spaces, and $S_n$.** E. Michael [16] introduced the notion of $A$-spaces, inner-closed $A$-spaces, strict $A$-spaces etc., and characterized spaces $X$ with the property that each map onto $X$ belonging to some class $\mathcal{E}_n$, must belong to some class $\mathcal{E}_{n+1}$. In E. Michael, R. C. Olson and F. Siwiec [17], these spaces are investigated detailedy. A space $X$ is an $A$-space, if whenever $\{A_n; n \in N\}$ is a decreasing sequence with $A_n \supseteq \{x\}$ for each $n \in N$, then there exist $B_n \subseteq A_n$ such that $B_n \supseteq \{x\}$, and $X \subseteq \{y\}$ is not closed in $X$. If the $B_n$ are closed (resp. singletons), then such a space is inner-closed $A$ (resp. inner-one $A$). If $x \in B$, then $X$ is a strict $A$-space. It is easy to show that $S_2$ is an $A$-space (indeed, strict $A$-space), but $S_3$ is not a $A$.

**Theorem 1.1.** Let $X$ be a sequential space. Then $X$ is an $A$-space if and only if it contains no closed copy of $S_n$.

**Proof.** "If". Since every closed subset of an $A$-space is $A$, if $X$ contains a closed copy of $S_n$, then $S_n$ is an $A$-space, but $S_n$ is not $A$. This is a contradiction. Hence $X$ contains no closed copy of $S_n$.

"If". First we shall prove that if $(A_n \downarrow x$ with each $A_n$ closed, then $(A_n; n \in N)$ is not hereditarily closure preserving. To show this, suppose that $(A_n; n \in N)$ is hereditarily closure preserving. Since $x \in A_n \supseteq \{x\}$, $x$ is not isolated in a closed subset $A_n$. Then, since $A_n$ is sequential, there exists a convergent sequence $(x_m; m \in N)$ in $A_n \supseteq \{x\}$ with $x_m \to x$. For each $n \in N$, let $C_n = \{x_m; m \in N\} \cup \{x\}$ and let $X_n$ be the topological sum of $C_n$'s. Then; $X_0 = Y$ be the obvious map. Since, then $P_n \in N$ is hereditarily closure preserving, $f$ is a closed map of a metric space $X_0$ onto a Fréchet space $Y$. Since $Y$ does not contain a closed copy of $S_n$, by [3; Lemma 2] $f^{-1}(X)$ is compact. However, $f^{-1}(x)$ is not compact. This is a contradiction. Hence $(A_n; n \in N)$ is not hereditarily closure preserving if $(A_n \downarrow x$ with each $A_n$ closed.

Next we prove $X$ is an $A$-space. To show this, let $(A_n \downarrow x$. Then $(A_n \downarrow x$, hence by the above there exist subsets $B_n$ of $A_n$ such that $B_n \supseteq \{x\}$, and $X \subseteq \{y\}$ is not closed in $X$. Since $X$ is sequential, there exist $b \in B$ and $b \in A_n$, such that $b_n \to b$ with $b_n \neq b$. Thus there exist neighborhoods $V_n$ of $b_n$ with $V_n \cap b$. If $C_n = A_n \cap V_n$, then $C_n \cap b$. This implies that there exists subsets $C_n \subseteq A_n$ such that $C_n \cap b$. Thus $X$ is an $A$-space.

By the following example, the sequentialness of $X$ of Theorem 1.1 is essential.

**Example 1.2.** A paracompact $k$-space which contains no copy of $S_n$ and no $S_{n+1}$, but is not an $A$-space.

**Proof.** Let $X$ be the space obtained from the topological sum of $\omega$ ordinal space $[0, \omega_1]$ by identifying all the first uncountable ordinal numbers $\omega_1$. Since no sequence converges to $\omega_1$, it is easy to check that $X$ is a $k$-space which contains no copy of $S_n$ and no $S_{n+1}$, but $X$ is not an $A$-space.

The following lemma due to [17] will be useful.

**Lemma 1.3.** (i) A regular space $X$ is strongly Fréchet if and only if $X$ is a Fréchet space.

(ii) Suppose $X$ is a regular sequential space. If $X$ is an $A$-space (resp. inner-closed $A$-space), then every subset of $X$ is an $A$-space (resp. inner-one $A$-space).

**Proof.** (i) is Proposition 8.1 in [17].

(ii) Since $X$ is sequential, as is well known, if $x \in A$ then $x \in C$ for some countable $C \subseteq A$ (cf. [15; Propositions 8.3 & 8.5]). Then, by [17; Proposition 5.1], $X$ is a strict $A$. Hence every subset of $X$ is an $A$-space. The parenthesis part follows from [17; Proposition 5.4].

**Corollary 1.4.** Let $X$ be a regular Fréchet space. Suppose that $X$ has the weak topology with respect to a point-countable cover $\mathcal{C}$ consisting of compact subsets (for example, $X$ is a $k$-space). Then $X$ is locally compact if and only if it contains no closed copy of $S_n$.

**Proof.** The "only if" part is obvious.

"If". By Theorem 1.1, $X$ is an $A$-space. Thus $X$ is strongly Fréchet by Lemma 1.3(i). Suppose that $X$ is not locally compact. Then there exists a point $x_0 \in X$ such that the closure of any neighborhood of $x_0$ is not compact. Let $\{C_n; n \in N\}$ be an open cover of $X$ and $x_n \in C_n$ for $n \in N$. Then $(X \setminus \{x_0\}) \cup \{x_0\}$, hence there exist $x_0 \in X \setminus \{x_0\}$ with $x_0 \to x_0$. Let $K = \{x_n; n \in N\}$ and $(K \cap \{x_0\}) \cup \{x_0\}$ for $n \in N$. Assume $K \subseteq \{x_0\}$ for $n \in N$, then there is $D = \{x_n; n \in N\}$ with $y \in K \setminus X_0$. Since $D \cap C$ is at most finite for each $C \cap D$ is discrete in $X$, hence in $K$, a contradiction. Thus $K$ is contained in a finite union of elements of $C$. But each element of $C$ is closed, so there exists $C_n$ such that $C_n$ meets infinitely many elements of $C$. This is a contradiction. Hence $X$ is locally compact.
Theorem 1.5. Let X be a regular sequential space. Then the following are equivalent:
(a) X contains a copy of Sωω.
(b) X contains no closed copy of Sωω.
(c) X is an A-space.
(d) Every Fréchet subset of X is strongly Fréchet.

Proof. (a) → (b) is clear. (b) → (c) follows from Theorem 1.1. (c) → (d) follows from Lemma 1.3. (d) → (a) is obvious.

Lemma 1.6. Let X have the weak topology with respect to a cover $\mathcal{U}$ consisting of strongly Fréchet subspaces. If for each $x \in X$, $\{A \in \mathcal{U}; x \in A\}$ is finite (resp. countable), then X contains no copy of $S_\omega$ (resp. no copy of $S_{\omega_1}$).

Proof. Since the parenthetic part is proved similarly, so suppose that X contains a copy $Y$ of $S_\omega$. Let $Y = \{x_0\} \cup \bigcup (x_n; n \in N)$ with $x_n \to x_0$ for each $i \in N$. Let $C_i = \{x_k; n \in N\} \cup \{x_k; k \in N\}$ for $i \in N$. Then Y has the weak topology with respect to $\{C_i; i \in N\}$. On the other hand, since each $C_i$ is closed in X, each $C_i$ has the weak topology with respect to $\mathcal{U} \cap C_i = \{A \cap C_i; A \in \mathcal{U}\}$. But each element of the cover of Y is contained in an element of the cover $\mathcal{U} \cap Y$ of X. Therefore, Y has the weak topology with respect to $\mathcal{U} \cap Y$. Let $\mathcal{U}_1 = \mathcal{U} \cap Y$ and $\mathcal{B}_1 = \{B \in \mathcal{U}_1; x_0 \in B\}$. Then $\mathcal{B}_1$ is finite. Since each element of $\mathcal{B}_1$ is strongly Fréchet, it contains no copy of $S_\omega$. Thus there exists $C_{x_0}$ such that $C_{x_0} \cap B$ is finite for each $B \in \mathcal{B}_1$. Let $S = C_{x_0} \setminus \{x_0\}$. Then $S \cap B$ is closed in B for every $B \in \mathcal{B}_1$. To show this, suppose that $S \cap B$ is not closed in B for some $B \in \mathcal{B}_1$. If $S \cap B$ has an accumulation point $a_0$ in $B_0$, then $a_0 = x_0$ so that $B_0$ is the contradiction. Hence $S \cap B$ is closed in B for every $B \in \mathcal{B}_1$. In the sequel, $S \cap B$ is closed in B for every $B \in \mathcal{B}_1$. Since Y has the weak topology with respect to $\mathcal{B}_1$, this shows that $S$ is closed in Y. However, S does not contain the limit point $x_0 \in Y$. This is a contradiction. Hence X contains no copy of $S_\omega$.

Theorem 1.7. Let $f: X \to Y$ be metric.

(i) Suppose that f is quotient. If $f$ is compact, i.e., every $f^{-1}(y)$ is compact and Y is regular (resp. is a $\sigma$, i.e., every $f^{-1}(y)$ is separable), then Y contains no copy of $S_{\omega_1}$ (resp. under (CH) no of $S_{\omega_1}$). Moreover if X is locally compact metric, then the parenthetic part holds without (CH).

(ii) Suppose that f is closed. Then every $f^{-1}(y)$ is compact (resp. Lindelöf) if and only if $Y$ contains no copy of $S_\omega$ (resp. no $S_{\omega_1}$).

Proof. (i) Let f be compact. Let S be any Fréchet subspace of Y. Then S has the weak topology with respect to the cover $\{C_i; i \in I\}$ consisting of all compact metric subspaces. Since each $C_i$ is closed in Y, each $f^{-1}(C_i)$ is quotient. Hence $g = f^{-1}(S)$ is a quotient compact map onto a Fréchet space S. Thus by [3; Theorem 2.3] and [14; Proposition 3.2], g is bi-quotient, so that S has a point-countable base by [4; Theorem 1.1]. Suppose now that Y contains a copy $S$ of $S_{\omega_1}$. Then, by the above S has a point-countable base, a contradiction. Hence Y contains no copy of $S_{\omega_1}$.

Next, let f be a $\sigma$-map. Suppose that $Y$ contains a copy $S_\omega$ of $S_{\omega_1}$. Since $S_\omega$ is Fréchet, $h = f^{-1}(S_\omega)$ is a quotient $\sigma$-map. Let $\mathcal{B}$ be a $\sigma$-locally finite base of $X$ and $f^{-1}(S_\omega)$ is a quotient $\sigma$-map. Let $Y$ be an open subset of $Y$ with $x_0 \in U$. Then U has the weak topology with respect to $\mathcal{B}$, so that $f^{-1}(U)$ is quotient and $f^{-1}(U)$ is a base for $f^{-1}(U)$). Suppose that $x_0 \in U \cap f^{-1}(U) \cap f^{-1}(U) \cap f^{-1}(U)$. Since U is Fréchet, there exist $x_n \in U \cap f^{-1}(U) \cap f^{-1}(U) \cap f^{-1}(U)$. Then $A \cap S$ is closed for each $A \in \mathcal{B}$. Thus $A$ is closed in U, a contradiction. Hence, $x_0 \in U \cap f^{-1}(U) \cap f^{-1}(U)$. This implies that $\mathcal{B}$ is a local base of $x_0$ with cardinality of $\mathcal{B}$ in $Y$. This is a contradiction under (CH). Hence Y has no copy of $S_{\omega_1}$ under (CH). When X is moreover locally compact, X has the weak topology with respect to a locally finite closed cover $\mathcal{G}$ consisting of compact metric subspaces. Thus Y has the weak topology with respect to a point-countable cover $\mathcal{F}$ consisting of compact metric subspaces. Thus by Lemma 1.6, Y contains no copy of $S_{\omega_1}$.

(ii) In view of [8; Lemma 2] we have the “if” part.

*Only if.* If every $f^{-1}(y)$ is compact, then Y is metric. So this part is clear. Let every $f^{-1}(y)$ is Lindelöf, and $\mathcal{B}$ be a $\sigma$-locally finite closed $k$-network of X. Recall that a closed k-network is a closed cover such that if $C \subseteq U$ with C compact and U open, there exists a finite subcover $\mathcal{E} \subseteq C \subseteq \bigcup \mathcal{E} \subseteq U$. Then $f^{-1}(y)$ is a closed k-network of Y. Indeed, let $C \subseteq U$ with C compact and U open in Y. By [12; Corollary 1.2], C is the image of some compact subset $\mathcal{K}$ of X. Thus there is a finite subcover $\mathcal{E}$ of $\mathcal{B}$ with $\mathcal{K} \subseteq \bigcup \mathcal{E} \subseteq f^{-1}(y)$. Hence $\mathcal{E}$ is a closed k-network. Since every $f^{-1}(y)$ is Lindelöf, as in the proof of [12; Corollary 1.2] we can assume that $f^{-1}(y)$ is Lindelöf. Then $f^{-1}(y)$ is point-countable. Hence Y has a point-countable closed k-network. Thus Y contains no copy of $S_{\omega_1}$ by [26; Proposition 1].

We remark that the converse of Theorem 1.7(i) is not valid. Indeed, let $Y$ be a regular separable first countable, non-metric space. Then by [3; Corollary 1.13], Y is the quotient image of a locally compact metric space. Since Y is first countable, it contains no copy of $S_\omega$ and no $S_{\omega_1}$. But since Y has no point-countable base, by [14; Proposition 3.3(d)] and [4; Theorem 1.1] there is no quotient map $f: X \to Y$ with X metric and each $f^{-1}(y)$ separable.

Lemma 1.8. Let X be a CW-complex due to Whitehead, and let $\{v_i; y\}$ be the cells of X.
(i) Then \( X \) is a sequential space having the weak topology with respect to
\( \{ \bar{z}_i; \gamma \} \), where \( \bar{z}_i = \text{cl}_{\gamma} z_i \).

(ii) ([27; Lemma 2.2]): If \( X \) contains no closed copy of \( S_n \) (resp. \( S_{\omega} \)), then
each \( \{ \gamma; \bar{z}_i \not\in x \} \) is finite (resp. countable).

From Lemmas 1.6 and 1.8, we have

**Theorem 1.9.** Let \( X \) be a CW-complex with the cells \( \{ e_i; \gamma \} \). Then \( X \) contains
no copy of \( S_n \) (resp. \( S_{\omega} \)) if and only if each \( \gamma; \bar{z}_i \not\in x \) is finite (resp. countable).

By Theorems 1.1 and 1.9 together with Lemma 1.8(i), we have

**Corollary 1.10.** Let \( X \) be a CW-complex with the cells \( \{ e_i; \gamma \} \). Then \( X \) is an
A-space if and only if each \( \gamma; \bar{z}_i \not\in x \) is finite.

### 2. Fréchet spaces, and \( S \).

**Theorem 2.1.** Let a regular space \( X \) have the weak topology with respect to a
point-countable cover \( \mathfrak{U} \). Suppose that (a) or (b) below holds. Then \( X \) is Fréchet if
and only if it contains no copy of \( S \).

(a) Each finite union of elements of \( \mathfrak{U} \) is Fréchet, or a sequential space in
which every point is \( G_\delta \).

(b) \( X \) is sequential and each countable union of elements of \( \mathfrak{U} \) is a space in
which every point is \( G_\delta \).

**Proof.** Case (a). The "only if" part is obvious. So we shall prove the "if" part. Since \( X \) has the weak topology with respect to \( \mathfrak{U} \), it has the weak topology with respect to the collection \( \mathfrak{U}' \) of all finite unions of elements of \( \mathfrak{U} \). Moreover
each of these unions is sequential. Thus \( X \) is sequential. Suppose that \( X \) is not
Fréchet. Thus, following the proof of [8; Proposition 7.3], we can choose a
countable subset \( X_\omega = \{ x_0 \} \cup \{ x_i; i \in N \} \cup \{ x_i; j \in N \} \) of \( X \) such that \( x_i \in G_i \)
for some pairwise disjoint open subsets \( G_i \), and \( x_i \to x_0 \) \( x_i \to x_0 \) also no
sequence of \( x_i \)'s converges to \( x_0 \). Thus \( X_\omega \) is a copy of \( S \). If \( X_\omega \) is sequential;
that is, every subset \( U \) of \( X_\omega \) is open in \( X_\omega \) whenever each sequence converging to a
point in \( U \) is eventually in \( U \).

Now, let \( \{ A \in \mathfrak{U}; A \cap X_\omega \neq \emptyset \} = \{ A_i; i \in N \} \), and let \( X_\omega = \bigcup_{i=1}^n A_i \) for \( i \in N \).

Let us put \( C_0 = \{ x_0 \} \cup \{ x_i; i \in N \} \), \( C_i = \{ x_i \} \cup \{ x_j; j \in N \} \) for \( i \in N \). Since \( \mathfrak{U} \) is
point-countable, by the proof of Corollary 1.4, each \( C_i \subseteq \omega \), is contained in
some element of \( \mathfrak{U}' \). Thus there exists \( X_{\omega} \) in \( C_\omega \). Suppose that \( \{ i \in N; X_{\omega} \cap \omega \) is infinite \) is not finite. Then there exists an infinite subset \( X_0 = \{ x_0 \} \cup \{ x_i; k \in N \} \cup \{ x_i; q \in N \} \) of \( X_\omega \) such that \( X_0 \subseteq X_\omega \). If \( X_\omega \) is Fréchet, then \( x_0 \) is a limit point of some \( x_i \). This is a contradiction. So we
assume that \( X_\omega \) is a sequential space in which every point is \( G_\delta \). Since \( x_0 \) is a \( G_\delta \)
set in \( X_\omega \), there exists a decreasing sequence \( \{ V_i; i \in N \} \) of open subsets of \( X_\omega \) with \( \text{cl}_{X_\omega} V_i \subseteq V_i \) and \( x_0 \in \bigcap V_i \). Since each \( V_i \) contains \( x_0 \), we can assume that
for each \( i; \{ x_i \} \cup \{ x_i; q \in \omega \} \) is contained in \( V_i \). Hence it follows that \( X_0 \)
is closed in \( X_\omega \). Since \( X_\omega \) is sequential, so is \( X_\omega \). Hence \( X_\omega \) is a copy of \( S_\omega \). Thus\( X_\omega \) contains a copy of \( S_\omega \), a contradiction. Therefore \( X_\omega \cap \omega \) is at most finite
for some \( C_\omega \). Since \( C_\omega \) is contained in some \( X_\omega \) (\( n_0 > n_0 \)), we may assume that \( C_\omega \subseteq X_\omega \). By induction, there exists an infinite subset \( \{ m_0; m_1; \ldots \} \)
of \( N \) such that \( C_\omega \subseteq X_{n_0} \subseteq X_{m_0} \) (\( n_0 > n_0 \)). Let \( Y = C_\omega \cup \omega \left( \text{if } A \in X_\omega \right) \) and \( \gamma_0 \) such that \( C_\omega \subseteq X_{n_0} \subseteq X_{m_0} \) (\( n_0 > n_0 \)). Let \( Y = C_\omega \cup \omega \left( \text{if } A \in X_\omega \right) \) and \( \gamma_0 \).

Then \( Y \cap X_\omega \) is closed in \( X_\omega \), for \( x \in N \), also \( Y \cap A = Y \) if \( A \in X_\omega \). Since \( X \) has the weak topology with respect to \( \{ X_\omega; n \in N \} \cup \omega \), this
shows that \( Y \) is closed in \( X \). Thus \( Y \) is sequential. Then \( Y \) is a copy of \( S_\omega \), hence \( X \) contains a copy of \( S_\omega \), a contradiction. Therefore \( X \) must be Fréchet.

**Case (b).** The notation used here is the same as in case (a). Let \( Z = \bigcup \{ \{ A \cap X_\omega \neq \emptyset \} \) assuming \( X \) is not Fréchet. Since \( X_\omega \subseteq Z \) and \( x_0 \in X_\omega \) is a \( G_\delta \)-set in \( Z \), by the same way as in (a), we can assume that \( X_\omega \) is closed in \( Z \). But \( A \cap X_\omega \neq \emptyset \) \( A \subseteq \omega \). Then \( X_\omega \) is closed in \( X \), because \( X_\omega \) has the weak topology with respect to \( \{ X_\omega; n \in N \} \cup \omega \), since \( X \) is sequential, so is \( X_\omega \).

Then \( X_\omega \) is a copy of \( S_\omega \), hence \( X \) contains a copy of \( S_\omega \), a contradiction.

Therefore \( X \) must be Fréchet.

**Corollary 2.2.** Let a regular space \( X \) have the weak topology with respect to a
point-countable cover \( \mathfrak{U} \). Suppose that each element of \( \mathfrak{U} \) is closed and that each
element is Fréchet or a sequential space in which every point is \( G_\delta \). Then \( X \) is
Fréchet if and only if \( X \) contains no copy of \( S_\omega \).

**Proof.** The "only if" part is obvious, so we prove the "if" part. Suppose that \( X \) is not Fréchet. For \( n \in N \), let \( X_n \) be the subsets defined in the proof of
Theorem 2.1. Then each \( X_n \) is Fréchet, or a sequential space in which every
point is \( G_\delta \) or \( P_1 \) or \( P_2 \), where \( P_1 \) or \( P_2 \) is a sequential space in which every point is \( G_\delta \). From the proof given there, we have
a contradiction. Thus \( X \) is Fréchet.

From the proof of Theorem 2.1, we also have

**Theorem 2.3.** Let \( X \) be a regular sequential space in which every point is \( G_\delta \), then
the following are equivalent.

(a) \( X \) contains no copy of \( S_\omega \).

(b) \( X \) contains no closed copy of \( S_\omega \).

(c) \( X \) is Fréchet.

The following example shows that the condition "each point of \( X \) of the
previous theorem is \( G_\delta \)" is essential.

**Example 2.4.** A compact sequential space which contains no copy of \( S_\omega \) and
no \( S_\omega \) but it is not Fréchet.

**Proof.** Let \( X \) be the sequential, non-Fréchet compact space
constructed by S. P. Franklin in [6; Example 7.1]. Then \( [20; \text{Theorem } 10] \)
showed that \( X \) contains no copy of \( S_\omega \). But we shall give an indirect proof here.

Since \( X \) is compact and sequential, by Lemma 1.3(ii), every subspace of \( X \) is
inner-one A. However S₂ or S₃ is not inner-one A, so that X contains no copy of S₂ or no S₃.


**Theorem 3.1.** Let X be a sequential space. If X is a regular space in which every point is G₃, or hereditarily normal, then the following are equivalent.

(a) X contains no copy of S₃ or no S₂.
(b) X contains no closed copy of S₃ or no closed copy of S₂.
(c) X is strongly Fréchet.

*Proof.* (a) ⇒ (b) and (c) ⇒ (a) are obvious.

(b) ⇒ (c). Since X contains no closed copy of S₃, X is an A-space by Theorem 1.1. If each point of X is G₃, since X contains no closed copy of S₂, then X is Fréchet by Theorem 2.3. If X is hereditarily normal, X is also Fréchet by [11; Corollary 2.3]. Hence X is strongly Fréchet by Lemma 1.3(i).

Recall that a space X has countable tightness, t(X) ≤ ω₁ if x ∈ 4, then there exists a countable C ⊆ A. It is well known that every sequential space has countable tightness.

The following theorem gives a necessary condition for the product to have countable tightness.

**Theorem 3.2. (CH).** Let f: X → Y be a closed map with X paracompact and sequential, and let Z satisfy one of the properties below. If t(Y × Z) ≤ ω₁, then either every f⁻¹(y) in Lindelöf or Z is strongly Fréchet.

(a) Regular Fréchet space.
(b) Regular sequential space in which every point is G₃.
(c) Hereditarily normal, sequential space.

*Proof.* Suppose that some f⁻¹(y) is not Lindelöf. Since f⁻¹(y) is paracompact f⁻¹(y) has a closed discrete subset of cardinality α₁. Thus, since X is collectionwise normal and Y is sequential, Y contains a closed copy of S₃ or S₃ by [27; Lemma 1.5]. Since t(Y × Z) ≤ ω₁, so t(4×Z) ≤ ω₁, hence every kₐ-subspace of Z is locally compact by [27; Proposition 1.1(2)]. Thus Z contains no copy of S₃ or no S₃. Then, since Z satisfies (a), Z must be strongly Fréchet by Theorem 1.5. If Z satisfies (b) or (c), then Z is also strongly Fréchet by Theorem 3.1.

**Theorem 3.3.** Let X be a regular Fréchet space, and let Y be a non-discrete first countable space. Then the following are equivalent.

(a) X × Y contains no copy of S₃.
(b) X × Y contains no copy of S₂.
(c) X is strongly Fréchet.

*Proof.* (a) ⇒ (c). Since X × Y contains no copy of S₃, neither does X, hence X is an A-space by Theorem 1.1. Thus X is strongly Fréchet by Lemma 1.3(i).

(c) ⇒ (a) & (b). Since X is strongly Fréchet and Y is first countable, X × Y is strongly Fréchet by [15; Proposition 4.D.4]. Hence we have the implication.

(b) ⇒ (c). Since Y is not discrete, there is a sequence {yₙ: n ∈ N} in Y with yₙ → y₀ and yₙ ≠ y₀. Let C₀ = {yₙ: n ∈ N} ∪ {y₀}. Suppose now that X is not an A-space. Then X contains a copy of S₂ by Theorem 1.1. Hence X × Y contains a copy of S₃ or S₂. But, S₃ × C₀ is a sequential space in which every point is G₃, and it contains no copy of S₂. Hence S₃ × C₀ is Fréchet by Theorem 2.3. However, since S₃ is not strongly Fréchet, the proof of [15; Proposition 4.D.5], S₃ × C₀ is not Fréchet. This is a contradiction. Thus X is an A-space. Hence X is strongly Fréchet by Lemma 1.3(i).

4. Metrizability of certain sequential spaces.

**Lemma 4.1.** Let X be a regular space having the weak topology with respect to a point-countable closed cover F consisting of metric subspaces. Then X is a locally metric space with a point-countable base if and only if X contains no copy of S₃ and no S₂.

*Proof.* We prove only the "if" part. Since X is a sequential space which contains no copy of S₃ and no S₂, by Theorems 1.1 and 2.1, X is a Fréchet and A-space. Thus X is strongly Fréchet space by Lemma 1.3(i). Hence, as in the proof of Corollary 1.4, X is locally metric. Let X₀ be the topological sum of all f: X₀ → X be the obvious map. Then f is quotient map of a metric space X₀. Thus X has a point-countable base by [4; Theorem 2.2].

**Theorem 4.2.** Let a regular space X have the weak topology with respect to a closed cover F consisting of metric subspaces. If (a) below holds, then X is metrizable if and only if X contains no copy of S₃ and no S₂.

(a) F is star-countable. (b) X is paracompact and F is point-countable.

*Proof.* By Lemma 4.1, X is locally metric. Thus to show X is metric, it suffices to prove X is paracompact for case (a). Let F = {Aᵢ: β ∈ B}, and let β ~ β' if f(Aᵢ, β) ∼ Aᵢ, for some α, β ∈ F. Then by this equivalent relation ~, the set B can be decomposed as X[n]. Let Xᵦ = ∪ {Aᵢ: β ∈ Bᵦ} for each γ ∈ Γ. Then Xᵦ = Aᵦ is open for each Aᵦ ∈ F, so each Xᵦ is open and closed in X. Each Xᵦ has the weak topology with respect to 0F, F = {Aᵦ: β ∈ Bᵦ}. Since Xᵦ are assumed to be an increasing countable closed covering of X, X has the weak topology with respect to 0F, in the sense of G. Morita [18]. Thus each Xᵦ is paracompact by Theorem 4 in [18]; hence X is paracompact.

By the following example due to R. W. Heath (for example, see [3; Example 5.4.B]), the closedness of F in case (a) (resp. the paracompactness of X in case (b)) is essential.

**Example 4.3.** A regular non-metric space X which has the weak topology with respect to a countable open cover (resp. point-finite open and closed cover) consisting of metric subspaces, and X contains no copy of S₃ and no S₂.

*Proof.* Let X be the subset of the plane defined by the condition y ≥ 0.
Define a topology on \( X \) as follows: Let each point above the x-axis be isolated and take as a base at a point \((x, 0)\) the family of all segments starting at \((x, 0)\) which form with the x-axis an angle of 90° if \(x\) is rational and an angle of 45° if \(x\) is irrational.

Then \( X \) is a regular space which is not normal by the Baire category theorem. Since \( X \) is first countable, \( X \) contains no copy of \( S_\omega \) and \( S_\omega \). Let \( R: Q = \{q_n, n \in N\} \) be the set of real numbers; rational numbers respectively. For \( n \in N \), let \( X_n = (X - R) \cup (R - \{q_n, j > n\}) \) (resp. for \( x \in R \), let \( F_x \) be the line starting at \((x, 0)\) which forms with the x-axis an angle of 90° if \(x\) is rational and an angle of 45° if \(x\) is irrational. Then \( X_n \) is a point above the x-axis) \( \cup \{F_x, x \in R\} \) is a countable open cover (resp. point-finite open and closed cover) of \( X \), so that \( X \) has the weak topology with respect to these covers. Since each \( F_x \) is obviously metrizable, we only prove that each \( X_n \) is metrizable. Each \( X_n \) regular and \( X_n = X_0 \cup F_x \). \( F_x \) is a finite subset \( \{q_j, j < n\} \) and \( X_0 = (X - R) \cup (R - \emptyset) \). Then, since \( X_0 \) is paracompact, \( X_0 \) is also paracompact. While, \( X_0 \) is locally metrizable. Hence \( X_0 \) is metrizable.

As a generalization of \( N_\omega \)-spaces due to E. Michael [13], P. O'Meara [19] introduced the notion of \( N \)-spaces. An \( N \)-space is a space with a \( \sigma \)-locally finite closed k-network.

**Theorem 4.4.** Let \( X \) have one of the properties listed below. Then \( X \) is metrizable if and only if \( X \) contains no copy of \( S_\omega \) and no \( S_\omega \).

- (a) Regular sequential, \( N \)-space.
- (b) CW-complex.
- (c) Regular space which is the quotient s-image of a locally separable, metric space.

**Proof.** The “only if” part is clear, so we prove the “if” part. Suppose that \( X \) satisfies (a) or (b). Then \( X \) is a sequential space in which every point is \( G_\delta \). Since \( X \) contains no copy of \( S_\omega \) and no \( S_\omega \), \( X \) is strongly Fréchet by Theorem 3.1. Thus (a) or (b) implies that \( X \) is metrizable by [24; Lemma 2.1] or [23; Lemma 4.3] respectively.

**Case (c).** Suppose that \( f: Y \to X \) is a quotient s-map with \( Y \) locally separable, metric. Then \( Y \) has the weak topology with respect to a locally finite closed cover \( \{\} \) consisting of separable metric subspaces. Since \( f \) is quotient and every \( f^{-1}(x) \) is Lindelöf, \( X \) has the weak topology with respect to a point-countable cover \( \{\} \). Moreover each element of \( f(\{\}) \) is hereditarily Lindelöf, hence every countable union of elements of \( f(\{\}) \) is a space in which every point is \( G_\delta \). Thus, since \( X \) is sequential, \( X \) is Fréchet by Theorem 2.1. While, \( X \) is an A-space by Theorem 1.1. Hence \( X \) is strongly Fréchet by Lemma 1.3(a). Thus \( X \) has a point-countable base by [15; Theorem 9.8]. Hence \( X \) is locally separable, metric space by [4; Corollary 1].

From the proof of case (c) of the previous theorem, and Theorem 1.7, we have

**Corollary 4.5.** Let a regular space \( X \) be the quotient s-image of a metric space. If each point of \( X \) is \( G_\delta \), then \( X \) has a point-countable base, or contains a copy of \( S_\omega \) or \( S_\omega \). When \( X \) is the quotient compact image of a metric space, we can omit “or \( S_\omega \)’’.

As an application of case (c) of Theorem 4.4, we have the following theorem in terms of weak topologies. Compare with Theorem 4.2, where each element of \( \mathfrak{W} \) is assumed to be closed.

**Theorem 4.6.** Let a regular space \( X \) have the weak topology with respect to a point-countable cover \( \mathfrak{W} \) consisting of locally separable, metric subspaces. Then \( X \) is metric, or contains a copy of \( S_\omega \) or \( S_\omega \). When \( X \) is point-finite, we can omit “or \( S_\omega \)”.

We shall remark that, by Example 4.3, the separability of each element of \( \mathfrak{W} \) is essential even if \( \mathfrak{W} \) is countable or point-finite.

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