we infer that $\dim X \leqslant 1$. Applying the characterization of tree-like continua from [2] we conclude that X is tree-like. This completes the proof.

It follows from the theorem that h.i. continua with trivial shape must be tree-like. In this form the theorem was discovered by the second author. Continua with trivial shape may be characterized as those which are the limits of inverse sequences of absolute retracts.

References

[1] M. Brown, *On the inverse limit of euclidean n-spheres*, Trans. Amer. Math. Soc. 96 (1960), pp. 129–134.
[2] J. H. Case and R. E. Chamberlin, *Characterizations of tree-like continua*, Pacific J. Math. 10 (1960), pp. 73–84.
[3] J. Krasinkiewicz, *Mapping properties of hereditarily indecomposable continua*, Houston J. Math. (to appear).
[4] K. Kuratowski, *Topology*, vol. 2, New York–London–Warszawa 1968.

INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
00-950 Warszawa

AUBURN UNIVERSITY
Auburn, Alabama 36849

Accepté par la Rédaction le 17. 7. 1981

The L^t-theory of profinite abelian groups

by

Peter H. Schmitt (Heidelberg)

Abstract. The concept of an algebraically complete topological abelian (ACTA-) group was introduced by J. Flum and M. Ziegler in their monograph on the topological first-order language L^t ([5] below). We determine the structure of saturated ACTA-groups and give cardinal invariants for their L^t-equivalence. We show that the profinite abelian (PFA-) groups constitute a subclass of the ACTA-groups. We axiomatize the L^t-theory of PFA-groups and show its decidability.

The topological logic L^t, recently introduced by Sgro, turned out to be a surprisingly good analog of first-order logic in the context of topological structures. A detailed description of L^t will be presented in §1 below.

In [5] Flum and Ziegler introduced the concept of an algebraically complete topological group. They proved that a topological abelian group is algebraically complete if and only if it is L^t-equivalent to a direct sum of abelian groups with discrete topologies. From this they inferred decidability of the L^t-theory of this class of groups. In § 2 we will determine the structure of saturated algebraically complete topological abelian groups and give cardinal invariants for L^t-equivalence. In § 3 we show that profinite abelian groups are in fact algebraically complete and we give axioms for the L^t-theory of this class of topological groups and prove its decidability. Our approach also yields a new proof of the decidability and axiomatizability results contained in [1].

We should like to thank Martin Ziegler for pointing out a mistake in the original proof of Corollary 3.6.

§ 1. Prerequisites

A. The topological logic L^t. We will present the first-order topological logic L^t in a form specifically adapted to the discussion of first-order properties of topological groups.

Let LG be the usual first-order language of group theory (written additively) and let LG^{II} be the extension of LG to the following weak second-order logic:

1. Syntax: Conventional second-order logic with second-order variables $X, Y, \ldots$, second-order constants and the binary relation symbol $\in$. The class of formulas is closed under second-order quantification.

2. Semantics: a structure for LG^{II} is a group G with a family $\mathscr{B}$ of subsets of G.

3. Interpretation: $\in$ represents membership. Second-order variables range over $\mathscr{B}$.

DEFINITION 1.1. An occurrence of a second-order variable X in an LG^{II}-formula φ is said to be *positive* (*negative*) if it is governed by an even (odd) number of negation symbols. (In this connection we take the propositional connectives to be $\neg$, $\wedge$, $\vee$ only; thus $\varphi \to \psi$ abbreviates $\neg\varphi \vee \psi$.)

L^t is a sublogic of LG^{II} with the same semantics but a restricted syntax:

DEFINITION 1.2. A formula φ of LG^{II} belongs to L^t iff for each subformula $\exists X\psi$ (resp. $\forall X\psi$) of φ all occurrences of X in ψ are negative (resp. positive).

EXAMPLE. The following three L^t-sentences

1. $\forall X\ \forall Y\ \exists Z\ \forall x(x\in Z \to x\in X \wedge x\in Y)$,
2. $\forall X(O\in X)$,
3. $\forall X\ \exists Y\,\forall x, y(x, y\in Y \to x-y\in X)$

assert that the family $\mathscr{B}$ constitutes a neighbourhood basis at 0 for a topology τ on G such that $\langle G, \tau\rangle$ is a topological group.

Elementary equivalence with respect to L^t is denoted by $\equiv_t$. The following two theorems are easily verified (see e.g. [5]).

THEOREM 1.3. *If $\mathscr{B}_1, \mathscr{B}_2$ are neighbourhood bases for the same topology on the topological group G then $\langle G, \mathscr{B}_1\rangle \equiv_t \langle G, \mathscr{B}_2\rangle$.*

THEOREM 1.4. *The logic L^t satisfies the Compactness Theorem.*

DEFINITION 1.5. Let $\langle G, \mathscr{B}\rangle$ be a structure for L^t. A type Γ over $\langle G, \mathscr{B}\rangle$ is a set of formulas of L^t involving a fixed finite set of first and second-order variables $x_1, \ldots, x_n, X_1, \ldots, X_m$ such that

1. all constants occurring in formulas in Γ denote elements of G or sets in $\mathscr{B}$,
2. the variables X_i occur only negatively in formulas of Γ,
3. Γ is finitely satisfiable in $\langle G, \mathscr{B}\rangle$ using elements $a_1, \ldots, a_n \in G$ and sets $A_1, \ldots, A_m \in \mathscr{B}$.

DEFINITION 1.6. 1. For $\varkappa$ an infinite cardinal $\langle G, \mathscr{B}\rangle$ is *$\varkappa$-saturated* if each type over $\langle G, \mathscr{B}\rangle$ which involves fewer than $\varkappa$ constants in $G \cup \mathscr{B}$ is realized in $\langle G, \mathscr{B}\rangle$.

2. $\langle G, \mathscr{B}\rangle$ is *saturated* iff $\langle G, \mathscr{B}\rangle$ is $\varkappa$-saturated for $\varkappa = \operatorname{card}(G\cup\mathscr{B})$.

3. G is *$\varkappa$-saturated* (*saturated*) if there is a neighbourhood basis $\mathscr{B}$ for G such that $\langle G, \mathscr{B}\rangle$ is $\varkappa$-saturated (saturated).

THEOREM 1.7. *If G_0, G_1 are saturated topological groups of the same cardinality and $G_0 \equiv_t G_1$ then G_0 is topologically isomorphic with G_1.*

THEOREM 1.8. *Let $\varkappa$ be a regular cardinal. Then each L^t-structure is L^t-equivalent to a $\varkappa^+$-saturated structure of cardinality at most $2^\varkappa$.*

The proofs of these theorems follow the routine used in first-order logic and may be found in [5].

Saturating a topological group actually simplifies its topological structure.

LEMMA 1.9. *If G is a $\varkappa$-saturated topological group then the intersection of fewer than $\varkappa$ open sets is again open.*

Proof. Fix a neighbourhood basis $\mathscr{B}$ of G at 0 so that $\langle G, \mathscr{B}\rangle$ is $\varkappa$-saturated. Our assertion reduces easily to the following: if $\{U_\alpha\colon \alpha<\lambda\}$ is a family of sets belonging to $\mathscr{B}$ and $\lambda<\varkappa$ then $\bigcap\{U_\alpha\colon \alpha<\lambda\}$ is again a neighbourhood of 0. Consider the set $\Gamma = \{\forall x(x\in X \to x\in U_\alpha)\colon \alpha<\lambda\}$ with the free second order variable X. Γ is a type over $\langle G, \mathscr{B}\rangle$. Letting $U\in\mathscr{B}$ realize Γ our claim follows.

LEMMA 1.10. *If G is an $\aleph_1$-saturated topological group then there is a neighbourhood basis for G at 0 consisting exclusively of open subgroups of G.*

Proof. For any given neighbourhood A of 0 in G we choose open sets B_n such that $B_{n+1} \subseteq B_n$, $B_0 \subseteq A$ and $B_{n+1} - B_{n+1} \subseteq B_n$ for all n. Let $A_1 = \bigcap\{B_n\colon n\in\omega\}$, then A_1 is a subgroup of G containing A which by Lemma 1.9 is open.

B. Abelian groups. In notation concerning abelian groups we follow [6]. Let G be a (non-topological) abelian group. For all primes p and natural numbers n the following are called the *Szmielew-invariants* of G:

$$\gamma_p(G) = \inf\{\dim p^n G[p]\colon n\in\omega\},$$
$$\beta_p(G) = \inf\{\dim p^n G/p^{n+1} G\colon n\in\omega\},$$
$$\alpha_{p,n}(G) = \dim p^{n-1} G[p]/p^n G[p],$$
$$\delta(G) = \begin{cases} 0 & \text{if } G \text{ is bounded,}\\ \infty & \text{otherwise.}\end{cases}$$

We do not distinguish between different infinite cardinalities, for a Szmielew-invariant ι $\iota(G)$ is either a finite number or the symbol ∞.

Each Szmielew-invariant can be described by a set of elementary sentences, which we list for later reference:

$$\chi_1(p, n, k) = \exists x_1 \ldots x_k \big(\textstyle\bigwedge\!\!\bigwedge \{p^{n+1} x_i = 0\colon 1 \leqslant i \leqslant k\} \wedge \operatorname{indep}(p^n x_1, \ldots, p^n x_k)\big)$$
$$= \dim p^n G[p] \geqslant k,$$
$$\chi_2(p, n, k) = \exists x_1 \ldots x_k \big(\operatorname{indep}(\operatorname{mod} p^{n+1})(p^n x_1, \ldots, p^n x_k)\big)$$
$$= \dim p^n G/p^{n+1} G \geqslant k,$$
$$\chi_3(p, n, k) = \exists x_1 \ldots x_k \big(\textstyle\bigwedge\!\!\bigwedge \{p^n x_i = 0\colon 1 \leqslant i \leqslant k\} \wedge$$
$$\wedge \operatorname{indep}(\operatorname{mod} p^n)(p^{n-1} x_1 \ldots p^{n-1} x_k)\big)$$
$$= \alpha_{p,n}(G) \geqslant k,$$
$$\chi_4(n) = \exists x(nx \neq 0).$$

We used the abbreviations:

$$\mathrm{indep}(y_1 \dots y_k) = \bigwedge\!\!\bigwedge \{ \sum_{1 \leqslant i \leqslant k} r_i y_i \neq 0 \colon 0 \leqslant r_i < p, \langle r_1 \dots r_k \rangle \neq \bar{O} \},$$

$$\mathrm{indep}(\mathrm{mod}\, p^m)(y_1 \dots y_k) = \forall z \bigwedge\!\!\bigwedge \{ \sum_{1 \leqslant i \leqslant k} r_i y_i \neq p^m z \colon 0 \leqslant r_i < p, \bar{r} \neq \bar{O} \}.$$

The sentences χ_i, $1 \leqslant i \leqslant 4$, and their negations are called first-order *core sentences.*

THEOREM 1.11. *Two abelian groups are elementary equivalent iff they have the same Szmielew-invariants.*

Proof. [4] Theorem 2.6.

DEFINITION 1.12. An abelian group G is said to be *equationally* compact if for all sets Σ of equations with countably many variables and countably many parameters from G such that every finite subset of Σ has a solution in G the whole set Σ has a solution in G.

THEOREM 1.13. *An equationally compact group G is $\varkappa$-saturated iff for all Szmielew-invariants ι we have*

$$\text{if} \quad \iota(G) \text{ is infinite,} \quad \text{then} \quad \iota(G) \geqslant \varkappa.$$

Proof. [4] Theorems 1.11. and 2.7.

THEOREM 1.14. *An abelian group G is equationally compact iff for all H such that G is a pure subgroup of H, G is a direct summand of H.*

Proof. [6] Theorem 38.1 and Exercise 38.5.

LEMMA 1.15. *Let A, H be subgroups of G, H equationally compact, $A \cap H = 0$ and $A \oplus H$ a pure subgroup of G, then there is a subgroup K of G such that $G = H \oplus K$ and $A \subseteq K$.*

Proof. Apply Theorem 1.14 to G/A and $(H+A)/A$.

LEMMA 1.16. *Assume A, H are pure subgroups of G and $A \cap H = 0$. Then $A+H$ is a pure subgroup of G iff for all n $A \cap (H+nG) \subseteq nG$.*

Proof. First assume $A \oplus H$ is a pure subgroup of G. For $a \in A \cap (H+nG)$ we get $a = h+ng$ for some $h \in H$, $g \in G$. Thus $a-h \in (A \oplus H) \cap nG = n(A \oplus H) = nA \oplus nH$. Therefore $a \in nA \subseteq nG$. Now assume the second line. If $a \in A$, $h \in H$ and $a+h \in nG$ then $a \in A \cap (H+nG) \in A \cap nG = nA$ this implies $h \in nH$.

C. Profinite abelian groups. A topological group is called *profinite* iff it is a totally disconnected, compact Hausdorff group. The profinite groups are just the inverse (or projective) limits of finite groups. They may also be characterized as the groups appearing as Galois groups of field extensions endowed with the Krull topology. In the case of abelian profinite groups it is however more convenient to use Pontryagin duality.

Consider Q/Z as a topological group with the discrete topology.

DEFINITION 1.17. Let A be a topological group. The *character group* A^* of A consists of all continuous homomorphisms from A in Q/Z. A^* is turned into a topological group with the neighbourhood system at 0:

$$\{\mathrm{Ann}\, C \colon C \text{ compact open subgroup of } A\}$$

where

$$\mathrm{Ann}\, C = \{g \in A^* \colon \text{for all } c \in C \;\; g(c) = 0\}.$$

THEOREM 1.18. (*Pontrjagin duality theorem*)

(1) *If A is an abelian torsiongroup with the discrete topology then A^* is an abelian profinite group.*

(2) *If A is an abelian profinite group then A^* is a discrete abelian torsiongroup.*

(3) *If A is a discrete abelian torsiongroup* (*or a profinite abelian group*) *and with every $a \in A$ we associate the element $\varphi_a \in A^{**}$ defined by:*

$$\varphi_a(f) = f(a) \quad \text{for all } f \in A^*$$

then $a \mapsto \varphi_a$ is a topological isomorphism from A onto A^{**}.

Proof. [6] Theorem 48.1 and Exercise 48.1. It is possible to describe the structure of A^* completely in terms of certain invariants of A. For us the following will suffice:

THEOREM 1.19. *Let A be an abelian torsiongroup, A_p its p-component and B_p the p-basic subgroup of A_p. Thus*

$$B_p \cong \bigoplus_{k \geqslant 1} Z(p^k)^{(m_{p,k})}$$

and

$$A_p/B_p \cong Z(p^{\infty})^{(n_p)}.$$

Then

$$A^* \cong \prod_p \Big[\bigoplus_{k \geqslant 1} Z(p^k)^{(m'_{p,k})} \oplus Z_p^{(n'_p)} \Big]^{-}$$

where $^-$ denotes p-adic completion and

$$m'_{p,k} = \begin{cases} m_{p,k} & \text{if } m_{p,k} \text{ is finite,} \\ \text{is infinite} & \text{if } m_{p,k} \text{ is infinite,} \end{cases}$$

$$n'_p = \begin{cases} n_p & \text{if } n_p \text{ is finite and } B_p \text{ is bounded,} \\ \text{is infinite} & \text{otherwise.} \end{cases}$$

Proof. [6] Theorem 47.1.

Remark. In general n_p is not an invariant of A, but it is when B_p is bounded.

THEOREM 1.20. *The elementary theory of the class of profinite abelian groups* T_{pf} *is axiomatized by the axioms for abelian groups and* $\{\sigma_{p,k}\colon p \text{ prime}, k\in\omega\}$, *where G satisfies* $\sigma_{p,k}$ *iff* $\dim p^k G[p] \leqslant \dim p^k G/p^{k+1} G$. *Furthermore* T_{pf} *is decidable.*

Proof. [1].

§ 2. Algebraically complete groups

DEFINITION 2.1. A topological abelian group G is *locally pure* if it satisfies for all n

$$\forall X\ \exists Y\ \forall y\big(ny\in Y\to \exists x(x\in X\wedge nx = ny)\big).$$

This definition is best understood in terms of $\aleph_1$-saturated groups.

LEMMA 2.2. *An* $\aleph_1$*-saturated group* $(G, \mathscr{B})$ *is locally pure iff there is a neighbourhood basis* $\mathscr{B}_1$ *of G at* 0 *consisting exclusively of pure open subgroups. Moreover* $\mathscr{B}_1$ *may be chosen such that every* $U\in\mathscr{B}_1$ *is an intersection of countably many elements from* $\mathscr{B}$.

Proof. Easy, following the pattern of the proof of Lemma 1.10. See also [5].

DEFINITION 2.3. A topological abelian group G is called *algebraically complete* if it is locally pure, Hausdorff and satisfies for all n $\forall x\big(\forall X\,\exists y\,(x+ny\in X)\to \exists y(x = ny)\big)$.

The theory of algebraically complete topological abelian groups is denoted by ACAG. Examples of models of ACAG are discrete topological abelian groups and as we shall see later profinite abelian groups. It is also easily seen that a direct sum of algebraically complete groups with the product topology is again algebraicall complete.

The most impostant property of models of ACAG is given in the following lemma.

LEMMA 2.4. *Let* $\langle G, \mathscr{B}\rangle$ *be a* $\varkappa$*-saturated model of* ACAG, $C\subseteq G$ *with* $\operatorname{card} C < \varkappa$ *and V an open neighbourhood at* 0. *Then there is a pure open subgroup U of G,* $U\subseteq V$ *and a subgroup A with* $C\subseteq A$ *and* $G = A\oplus U$.

Proof. Let A_0 be the pure subgroup generatee by C. We have $\operatorname{card} A_0 = \max(\operatorname{card} C, \aleph_0) < \varkappa$. For any $a\in A_0$ we find by algebraic completeness $U_a\in\mathscr{B}$ such that $a\in U_a+nG$ implies $a\in nG$. By Lemmas 1.9 and 2.2 choose a pure open subgroup U which is contained in all U_a and $U\subseteq V$. Thus we have

$$A_0\cap(U+nG)\subseteq nG.$$

Furthermore we have $U = \bigcap\{U_\nu\colon \nu<\mu\}$ for certain $U_\nu\in\mathscr{B}$ and $\mu<\varkappa$. From this and $\varkappa$-saturation of $\langle G, \mathscr{B}\rangle$ we get that U itself is equationally compact (in general U is not $\varkappa$-saturated). Now the assertion of the lemma follows from the Lemmas 1.15 and 1.16.

It is our aim to describe the structure of saturated algebraically complete groups. Here are the invariants that we need for this purpose:

DEFINITION 2.5. For any Szmielew-invariant ι set:

$$\iota^*(G) = \sup\{\iota(G/U)\colon U \text{ pure open subgroup of } G\},$$
$$\iota_*(G) = \inf\{\iota(G)\colon U \text{ pure open subgroup of } G\}.$$

Though Definition 2.5 makes sense for any topological group G it will be most useful in $\aleph_1$-saturated locally pure groups.

LEMMA 2.6. *For any uncountable saturated algebraically complete group G we have for all Szmielew-invariants* ι:

$$\iota^*(G) = \iota(G).$$

Proof. We only give the proof for $\iota = \beta_p$. It will be clear how to prove the remaining cases. The inequality $\iota(G)\geqslant \iota^*(G)$ is obvious. If $C\subseteq p^n G$ is a set of representatives for an independent subset of $p^n G/p^{n+1} G$ with $\operatorname{card} C<\varkappa$ then we apply Lemma 2.4 to get a pure open subgroup U and a subgroup A such that $C\subseteq A$ and $G = A\oplus U$. This implies $\dim p^n A/p^{n+1} A\geqslant \operatorname{card} C$ and the lemma follows.

The situation for ι_* is a bit different.

LEMMA 2.7. *For any Szmielew-invariant* ι *and any* $\aleph_1$*-saturated model G of* ACAG *such that* $\iota_*(G) > 0$ *we have already* $\iota_*(G) = \infty$ *and* $\iota(G) = \infty$.

Proof. Let us again only treat the case $\iota = \beta_p$. If $\beta_{p,*}(G)$ were finite we would find an open neighbourhood U such that $\beta_p(U) = \beta_{p,*}(G)$ and $B\subseteq U$ such that $\{b+p^{n+1}U\colon b\in B\}$ is a basis for $p^n U/p^{n+1} U$. By Lemma 2.4 we find an open neighbourhood $V\subseteq U$ such that $U = A\oplus V$ for some A with $B\subseteq A$. But this implies $\beta_p(V) = 0$, contradicting $\beta_{p,*}(G) > 0$. Since we have for pure subgroups U $\iota(G)\geqslant \iota(U)$ the second assertion is clear.

LEMMA 2.8. *For any Szmielew-invariant* ι *there is a set* $S_*(\iota)$ *of* L^t*-sentences such that for any* $\aleph_1$*-saturated model G of* ACAG: *G satisfies* $S_*(\iota)$ *iff* $\iota_*(G)\geqslant 1$.

Proof. Consider the sentences:

$$\chi_5(p, n) = \forall X\exists x(x\in X\wedge p^{n+1}x = 0\wedge p^n x\neq 0),$$
$$\chi_6(p, n) = \forall X\exists x\big(x\in X\wedge \forall z(p^{n+1}z\neq p^n x)\big),$$
$$\chi_7(p, n) = \forall X\exists x\big(x\in X\wedge p^n x = 0\wedge \forall z(p^n z\neq p^{n-1}x)\big),$$
$$\chi_8(n) = \forall X\exists x(x\in X\wedge nx\neq 0).$$

Considering that in G the second order variables range over pure open subgroups it is easily seen that $S_*(\gamma_p) = \{\chi_5(p, n)\colon n\geqslant 1\}$ does the job and likewise the sentences χ_6, χ_7, χ_8 correspond to the Szmielew-invariants β_p, $\alpha_{p,n}$ and δ.

The sentences χ_1 to χ_8 and their negations are called *core sentences*!

Let A be a (non-topological) abelian group, $\varkappa$ a cardinal. $A^{\varkappa}$ denotes the (full) direct product of $\varkappa$ copies of A. $A^{\varkappa}$ is turned into a topological group by the neighbourhood basis at 0: $\{V_\alpha: \alpha < \varkappa\}$ where $V_\alpha = \{g \in A: \forall \nu < \alpha (g_\nu = 0)\}$.

DEFINITION 2.9.

$\exp(A, \varkappa) = \{g \in A^{\varkappa}: g_\alpha \text{ is eventually } 0\}$,

$\exp(A, \varkappa)$ is endowed with the topology induced from $A^{\varkappa}$.

It is easy to check that $\exp(A, \varkappa)$ is a model of ACAG.

THEOREM 2.10. *If G is a saturated model of* ACAG *of cardinality $\varkappa$ then $G \cong B \oplus \exp(A, \varkappa)$ where $\oplus$ denotes topological direct sum, B is a saturated abelian group of cardinality $\varkappa$ with the Szmielew-invariants $\iota(B) = \iota(G)$ and discrete topology, A is the saturated abelian group of cardinality $\varkappa$ with the Szmielew-invariants $\iota(A) = \iota_*(G)$.*

COROLLARY 2.11. *Two algebraically complete groups G_1, G_2 are L^t-equivalent iff for all Szmielew-invariants ι:*

$$\iota(G_1) = \iota(G_2) \quad \textit{and} \quad \iota_*(G_1) = \iota_*(G_2).$$

Proof of Corollary 2.11. Since $\iota(G)$, $\iota_*(G)$ are each described by a set of L^t-sentences one implication is obvious and in proving the reverse implication we can assume that G_1 and G_2 are saturated of the same cardinality $\varkappa^+$, for some regular cardinal $\varkappa \geqslant \aleph_0$. Here we use Theorem 1.8 and $\varkappa^+ = 2^{\varkappa}$. Now it follows from Theorem 2.10 that G_1 and G_2 are topologically isomorphic and thus a fortiori L^t-equivalent. Ways and means to eliminate the use of the continuum hypothesis are described in [4] page 120.

The remainder of § 2 will be devoted to the proof of Theorem 2.10. Let us fix an uncountable saturated algebraically complete group $\langle G, \mathscr{B}\rangle$, $\operatorname{card} G = \operatorname{card} \mathscr{B} = \varkappa$.

LEMMA 2.12. *There is a pure open subgroup H of G such that for all Szmielew-invariants ι:*

(i) $\iota(H) = \iota_*(G)$,

(ii) $\iota(G/H) = \iota^*(G)$,

(iii) *H is an intersection of $< \varkappa$ many sets in $\mathscr{B}$.*

Proof. We start with two observations:

(1) if H_1, H_2 are pure subgroups of G, $H_2 \subseteq H_1$ then $\iota(H_2) \leqslant \iota(H_1)$ and $\iota(G/H_2) \geqslant \iota(G/H_1)$.

(2) if $\iota_*(G) \geqslant \aleph_0$ (or $\iota^*(G) \geqslant \aleph_0$) then saturation of G implies that $\iota_*(G) = \varkappa$ (resp. $\iota^*(G) = \varkappa$).

By (1) and (2) it is possible to choose for each Szmielew-invariant i pure open subgroups $U^*(\iota)$, $U_*(\iota)$ such that

$$\iota(G/U^*(\iota)) = \iota^*(G), \quad \iota(U^*(\iota)) = \iota_*(G).$$

Let $H = \bigcap \{U^*(\iota) \cap U_*(\iota): \iota \text{ a Szmielew-invariant}\}$. Saturation guarantees that H is again a pure open subgroup and (i), (ii) follow by (1). Since there are only countably many Szmielew-invariants Lemma 2.2 tells us that we can find H such that (iii) is also satisfied.

The group H of Lemma 2.12 will be kept fixed for the remainder of § 2. Clause (iii) implies that H is equationally compact, thus we get from Theorem 1.14 a complementary summand B:

$$G \cong B \oplus H.$$

If we endow B with the discrete topology and H with the topology inherited from G then the algebraic isomorphism is in fact a topological isomorphism. This proves already one part of Theorem 2.10. It remains to show that H has the form indicated in Theorem 2.10.

Let $\{g_\alpha: \alpha < \varkappa\}$, $\{U_\alpha: \alpha < \varkappa\}$ be enumerations of G and $\mathscr{B}_H = \{U \in \mathscr{B}: U \subseteq H\}$ respectively.

LEMMA 2.13. *There are pure open subgroups H_α and subgroups $A_{\alpha\beta}$ of H for $\alpha < \beta < \varkappa$ such that*

(1) $H_0 = H$,

(2) *if* $\alpha < \beta < \varkappa$ *then* $H_\beta \subset H_\alpha$,

(3) $H_\alpha = \bigcap \{H_\nu: \nu < \alpha\}$ *for α a limit number,*

(4) $H_{\alpha+1} \subseteq U_\alpha$,

(5) *every H_α is the intersection of less than $\varkappa$ set in $\mathscr{B}$,*

(6) $H_\alpha = A_{\alpha\beta} \oplus H_\beta$ *for* $\alpha < \beta < \varkappa$,

(7) $A_{\alpha\gamma} = A_{\alpha\beta} \oplus A_{\beta\gamma}$ *for* $\alpha < \beta < \gamma$,

(8) if $\alpha < \beta < \varkappa$ *then* $A_{0\alpha} \subseteq A_{0\beta}$.

Notation: *$\pi_\alpha: H \to H_\alpha$ denotes the canonical homomorphism associated with the decomposition $H = A_{0\alpha} \oplus H_\alpha$.*

(9) $\pi_\alpha(g_\alpha) \in A_{\alpha,\alpha+1}$,

(10) *for all Szmielew-invariants ι:* $\iota(A_{\alpha,\alpha+1}) = \iota(H)$.

Proof. For $\alpha = 0$ there is nothing to be done.

α a limit number: Take $H_\alpha = \bigcap \{H_\nu: \nu < \alpha\}$. By saturation H_α is again a pure open subgroup of H. Set $A^* = \bigcup \{A_{0\nu}: \nu < \alpha\}$. We certainly have $A^* \cap H_\alpha = 0$. But $A^* \oplus H$ is also a pure subgroup of H, for if $a \in (A^* \oplus H_\alpha) \cap nH$ then we have $a = a_0 + h$ with $a_0 \in A_{0\nu}$ for some $\nu < \alpha$ and $h \in H_\alpha$. Since $A_{0\nu} \oplus H_\nu = H$ we get that $a_0 \in nA_0 \subseteq nA^*$ and $h \in nH_\nu$. Purity of H_α then yields $h \in nH_\alpha$ and thus $a \in nA^* \oplus nH_\alpha$. Since H_α is the intersection of less than $\varkappa$ elements from $\mathscr{B}$ saturation of $\langle G, \mathscr{B}\rangle$ implies equational compactness of H_α. Now Lemma 1.15 supplies us with a subgroup $A_{0\alpha}$ of H such that $A_{0\nu} \subseteq A_{0\alpha}$ for all $\nu < \alpha$ and $H = A_{0\alpha} \oplus H_\alpha$.

We define the remaining complements by $A_{\nu\alpha} = A_{0\alpha} \cap H_\nu$. Now one easily checks that $H_\nu = A_{\nu\alpha} \oplus H_\alpha$ and $A_{\nu\alpha} = A_{\nu\gamma} \oplus A_{\gamma\alpha}$ for $\nu < \gamma < \alpha$.

Successor step: Assume H_α, $A_{\beta\alpha}$ for $\beta < \alpha$ already constructed in accordance with (1) to (10). Let C be a pure subgroup of H_α such that

(i) $\pi_\alpha(g_\alpha) \in C$,

(ii) for any Szmielew-invariant ι and any direct summand C_0 of H_α containing C we have $\iota(C_0) \geqslant \min(\iota(H), \aleph_0)$.

Since by choice of H, $\iota(H) = \iota^*(H) = \iota(H_\alpha)$, it is possible to include in C for each Szmielew-invariant ι sufficiently many appropriately independent elements from H_α to assure (ii). Obviously C may even be taken to be countable. Using Lemma 2.4 we find a pure open subgroup $H_{\alpha+1}$ and a subgroup $A_{\alpha,\alpha+1}$ of H_α such that $H_\alpha = A_{\alpha,\alpha+1} \oplus H_{\alpha+1}$ and $C \subseteq A_{\alpha,\alpha+1}$. Moreover Lemma 2.4 tells us that we may choose $H_{\alpha+1}$ as small as we wish, in particular it is possible to have a set $V \in \mathscr{B}$ with $C \cap V = \emptyset$ and $H_\alpha \supseteq V \supseteq H_{\alpha+1}$ and $H_\alpha \supseteq H_{\alpha+1}$. This has the effect that whenever $\iota(A_{\alpha,\alpha+1}) \geqslant \aleph_0$ then we may use saturation of $\langle G, \mathscr{B}\rangle$ to get $i(A_{\alpha,\alpha+1}) = \varkappa$. So clause (10) of Lemma 2.13 is satisfied. If we define $A_{\beta,\alpha+1} = A_{\beta\alpha} \oplus A_{\alpha,\alpha+1}$ for $\beta < \alpha$ it is easily seen that (6), (7) and (8) are true. By choice of C also (9) has been taken care of. This proves Lemma 2.13.

Let A be isomorphic to A_{01}. We claim

$$H \cong \exp(A, \varkappa). \tag{2.14}$$

As direct summands of the equationally compact group G all $A_{\alpha,\alpha+1}$ are themselves equationally compact. Clause (10) of Lemma 2.13 and Theorem 1.13 imply that each $A_{\alpha,\alpha+1}$ is in fact $\varkappa$-saturated. Since $\operatorname{card} G = \varkappa$ and $A_{\alpha,\alpha+1}$ cannot be finite we also have $\operatorname{card} A_{\alpha,\alpha+1} = \varkappa$. Let $f_\alpha\colon A_{\alpha,\alpha+1} \to A$ be isomorphisms which exist by Theorem 1.11 and the fact that any two elementary equivalent saturated structures of the same cardinality are isomorphic. For $h \in H$ we define $\hat{h} \in A^\varkappa$ by $\hat{h}_\alpha = f_\alpha(\pi_\alpha(h))$.

Clause (9) of Lemma 1.13 implies that $\hat{h} \in \exp(A, \varkappa)$. It follows easily from the definitions that for all $h \in H$ and

$$\hat{h} \in V_\alpha \quad \text{iff} \quad h \in H_\alpha. \tag{2.15}$$

If $\hat{h} = 0$ then (2.15) yields $h \in H_\alpha$ for all $\alpha < \varkappa$, this implies $h = 0$ since G is Hausdorff and $\{H_\alpha\colon \alpha < \varkappa\}$ is a neighbourhood basis at 0 (clause (4) of 2.13).

So far we have that ^ is a continuous embedding from H into $\exp(A, \varkappa)$ with continuous inverse. It remains to show that ^ is surjective. To this end we define $K_\alpha = \{g \in \exp(A, \varkappa)\colon \text{for all } \beta \geqslant \alpha\ (g_\beta = 0)\}$ and we prove by induction on α

$$K_\alpha = \hat{A}_{0\alpha}. \tag{2.16}$$

This is trivial for $\alpha = 0$ and the induction is easily carried on beyond successor steps. So let α be a limit ordinal and $k \in K_\alpha$. By induction hypothesis there are elements $h_\nu \in H$ such that

$$(\hat{h}_\nu)_\mu = \begin{cases} k_\mu & \text{if } \mu < \nu, \\ 0 & \text{if } \mu \geqslant \nu. \end{cases}$$

By clause (5) of Lemma 2.13 and saturation of G we find an element $h^\circ \in H$ such that for all $\nu < \alpha$: $h^\circ - h_\nu \in H_{\nu+1}$. Let h be the projection of h° on $A_{0\alpha}$ then we have for $\mu \geqslant \alpha$ $\pi_\mu(h) = 0$ and for $\mu < \alpha$ $\pi_\mu(h) = \pi_\mu(h^\circ) = \pi_\mu(h_\mu) = k_\mu$.

This completes the proof of Theorem 2.10.

Corollary 2.11 could be used to give an alternative proof of the decidability of ACAG following the line given in [4]. Let us state another consequence of Corollary 2.11.

COROLLARY 2.17 *Every L^t-sentence is in* ACAG *equivalent to a Boolean combination of core sentences.*

§ 3. The L^t-theory of profinite abelian groups

We begin with an auxiliary lemma on abelian torsiongroups.

LEMMA 3.1. *Let G be an abelian torsiongroup, $n \geqslant 2$ and C an arbitrary finite subgroup of G. There are finite subgroups B_0, B_1 and a subgroup H of G such that*

(i) $C \subseteq B_0 \oplus B_1$, (ii) $G = B_0 \oplus H$, (iii) $B_1 \subseteq nH$.

Proof. Let $G = \bigoplus_p G_p$ be the decomposition of G into its primary summands. The group $C' = \bigoplus_p (C \cap G_p)$ is again finite and $C \subseteq C'$. From this it is obvious that it suffices to verify the lemma for all summands G_p. Thus we may assume from the start that G is a p-group and $n = p^m$. We proceed by induction on m. Assume the result true for all $m' \leqslant m$. So we have finite subgroups D_0, D_1 of G and a subgroup H such that $C \subseteq D_0 \oplus D_1$, $G = D_0 \oplus H$ and $D_1 \subseteq p^m H$. We choose a finite subgroup $D_2 \subseteq H$ with $D_1 \subseteq p^m D_2$. Now we proceed by induction on $k = \operatorname{card}\{d \in D_2\colon d \notin pH\}$.

If $k = 0$ then $D_2 \subseteq pH$. This shous $D_1 \subseteq p^{m+1} H$ and we can use the same D_0, D_1 also for the case $m+1$.

If $k > 0$ chose $d \in D_2$ with $d \notin pH$. Thus $\langle d \rangle$ is a pure subgroup of H and therefore a direct summand: $H = \langle d \rangle \oplus H_0$. For $D_2' = H_0 \cap D_2$ we have $D_2 = \langle d \rangle \oplus D_2'$.

Since $\operatorname{card}\{d \in D_2'\colon d \notin pH_0\} \leqslant \operatorname{card}\{d \in D_2'\colon d \notin pH\} < k$ we find by the hypothesis of the induction on k finite subgroups D_0^*, D_2^* and a subgroup H_0^* of H_0 such that

$$D_2' \subseteq D_0^* \oplus D_2^*,$$
$$H_0 = D_0^* \oplus H_0^*,$$
$$D_2^* \subseteq p^{m+1} H_0^*.$$

In this case take $B_0 = D_0 \oplus \langle d \rangle \oplus D_1^*$ and $B_1 = D_2^*$ to get

$$C \subseteq B_1 \oplus B_2,$$
$$G = B_0 \oplus H_0^*,$$
$$B_1 \subseteq p^{m+1} H_0^*.$$

LEMMA 3.2. *Every profinite abelian group is locally pure.*

Proof. By Theorem 1.18 we may assume without loss of generality that $G = A^*$ for some abelian discrete torsiongroup A. For every $n \in \omega$ and every finite subgroup C of A we have to produce another finite subgroup B of A such that

$$\forall y (ny \in \mathrm{Ann}\, B \to \exists x \in \mathrm{Ann}\, C (nx = ny)).$$

Choose by Lemma 3.1 finite subgroups B_0, B_1 of A such that $C \subseteq B_0 \oplus B_1$ and for some A_0: $A = B_0 \oplus B_1$ and $B_1 \subseteq nA_0$. Let B_2 be a finite subgroup of A such that already $B_1 \subseteq nB_2$ and take B to be $B = B_0 \oplus B_2$. Consider any $g \in \mathrm{Ann}\, B$. Let f be the unique element in A^* determined by:

$$f(a) = \begin{cases} 0 & \text{if } a \in B_0, \\ g(a) & \text{if } a \in A_0. \end{cases}$$

For any $c \in C$ given by $c = b_0 + nb_2$ with $b_0 \in B_0$, $b_2 \in B_2$ we have $f(c) = 0 + ng(b_2) = 0$. Thus $f \in \mathrm{Ann}\, C$. Furthermore we have for any $a = b_0 + a_0$, $b_0 \in B$, $a \in A_0$ $nf(a) = ng(a_0)$ by definition of f and $ng(a) = ng(a_0)$ since $ng(b_0) = 0$. Thus we arrive at $nf = ng$ is required.

LEMMA 3.3. *Every profinite abelian group G is algebraically complete.*

Proof. Let $G = A^*$ for some discrete abelian torsiongroup A. By Lemma 3.2 and since any profinite group is Hausdorff we need only to show that for any element $g \in G$ satisfying

(1) for all finite subgroups C of A there is h such that $nh + g \in \mathrm{Ann}\, C$

we have

(2) $$g \in nG.$$

Consider for any finite subgroup C of A the set

$$W_C = \{(h, k) \in G + G\colon\ k \in \mathrm{Ann}\, C \text{ and } g + nh = k\}.$$

Since $\mathrm{Ann}\, C$ is clopen $V_C = \{(h, k) \in G + G\colon\ k \notin \mathrm{Ann}\, C\}$ is an open subgroup of the topological product group $G + G$. Also for any finite subgroup D of A

$$U_D = \{(h, k) \in G + G\colon\ g + nh - k \notin \mathrm{Ann}\, C\}$$

is open in $G + G$. Since the complement of W_C in $G + G$ equals $V_C \cup \bigcup_D U_D$ we see that W_C is closed in $G + G$. By (1) it is non-empty. Furthermore we have for any two finite subgroups C_1, C_2 of A: $W_C \subseteq W_{C_1} \cap W_{C_2}$ where C is the finite subgroup generated by $C_1 \cup C_2$. By compactness of the topological group $G + G$ there is $(h, k) \in \bigcap \{W_C\colon C \text{ finite subgroup of } A\}$. This implies in particular $k \in \bigcap_C \mathrm{Ann}\, C$, i.e. $k = 0$ and therefore $g = -nh \in nG$.

LEMMA 3.4. *For any abelian profinite group $G = A^*$ and any Szmielew-invariant ι:*

G satisfies $S_*(\iota, k)$ for all k if $\iota(G) \geqslant \aleph_0$,
G satisfies $\neg S_*(\iota, k)$ for all k if $\iota(G) < \aleph_0$

(for the definition of $S_*(\iota, k)$ see Lemma 2.7).

Proof. It follows from the structure Theorem 2.10 that G satisfies $S_*(\iota, 1)$ iff G satisfies $S_*(\iota, k)$ for all k. Lemma 2.12 (i) implies that G satisfies $\neg S_*(\iota, 1)$ if $\iota(G) < \aleph_0$. It remains thus to show that G satisfies $S_*(\iota, 1)$ if $\iota(G) \geqslant \aleph_0$. This will follow from:

(*) for any finite subgroup C of A $\iota(\mathrm{Ann}\, C) \geqslant \aleph_0$.

It is easily checked that $\mathrm{Ann}\, C = (A/C)^*$. Now (*) follows from Theorem 1.19 if we observe e.g. that the p-basic subgroup B_p' of A/C is unbounded if the p-basic subgroup B_p of A is unbounded and that the number of summands $Z(p^\infty)$ in A/B_p equals the number of summands $Z(p^\infty)$ in $(A/C)/B_p'$ if B_p is bounded, since in this case we have $C \subseteq B_p$.

THEOREM 3.5. *Two profinite abelian groups are L^t-equivalent iff they are elementary equivalent.*

Proof. Follows from Corollary 2.11 and Lemmas 3.2–3.4.

COROLLARY 3.6. *The L^t-theory of the class of profinite abelian groups, T^t_{pf}, can be axiomatized by $T_{pf} + \mathrm{ACAG} + \{\chi_5(p, n) \to \chi_6(p, n)$ for all primes p and $n \in \omega\}$.*

Proof. By Lemma 3.3 $T_{pf} + \mathrm{ACAG} \subseteq T^t_{pf}$, while $T^t_{pf} \vdash \chi_5(p, n) \to \chi_6(p, n)$ follows from Theorem 1.18 and Lemma 3.4. To prove the converse inclusion we shall show that for any L^t-sentence φ which is true in some model G of $T_{pf} + \mathrm{ACAG} + \{\chi_5(p, n) \to \chi_6(p, n)\colon p, n\}$ there is a profinite abelian group H and φ is true in H. By Corollary 2.17 it suffices to find for every finite set of primes P and $N, K \in \omega$ a profinite abelian group H such that for all core sentences χ of the form $\chi_i(p, n, k)$, $\chi_i(p, n)$ or $\chi_i(n)$ with $p \in P$, $n \leqslant N$, $k \leqslant K$ $G \models \chi$ iff $H \models \chi$. Given G and P, N, K we will construct an abelian torsiongroup A:

$$A = \bigoplus_{p \in P} \bigoplus_{n \in N+1} Z(p^n)^{\alpha_{p,n}} \oplus Z(q^t) \oplus \bigoplus_{p \in P} Z(p^\infty)^{\gamma_p}$$

where $q \notin P$, such that $H = A^*$ is the profinite group looked for. The following equalities for arbitrary abelian groups B will frequently be used in the following

(I) $$\dim p^k B[p] = \dim p^{k+1} B[p] + \alpha_{p,k+1}(B),$$

(II) $$\dim p^k B/p^{k+1} B = \dim p^{k+1} B/p^{k+2} B + \alpha_{p,k+1}(B).$$

For a proof see [4]. We shall also use frequently Theorem 1.19 without mentioning.

Let us first assume that G is unbounded. For $p \in P$ and $0 \leqslant n \leqslant N$ we specify $\alpha_{p,n}$ as follows:

If $\alpha_{p,n,*}(G) \geqslant 1$, then $\alpha_{p,n} = \aleph_0$.
If $\alpha_{p,n,*}(G) = 0$ and $\alpha_{p,n}(G) = \infty$ then $\alpha_{p,n} = K$.
If $\alpha_{p,n}(G)$ is finite, then $\alpha_{p,n} = \alpha_{p,n}(G)$.

By Lemma 2.7 we can have $\alpha_{p,n}(G) < \aleph_0$ only if $\alpha_{p,n,*}(G) = 0$, so there is no collision in the above definition.

The following determines $\alpha_{p,N+1}$:

If $\chi_5(p, N)$ is true in G, then $\alpha_{p,N+1} = \aleph_0$.
If $\chi_5(p, N)$ is not true in G and $\dim p^M G[p]$ is infinite then $\alpha_{p,N+1} = K$.
In all other cases $\alpha_{p,N+1} = \dim p^M G[p] - \Sigma\{\alpha_{p,i}: M < i \leqslant N\}$. Here M is the least number $\leqslant N+1$ such that for all m, $M < m \leqslant N+1$ $\alpha_{p,n}(G)$ is finite.

The number t is determined such that $N < q^t$.

Finally γ_p is determined as follows:

If $\chi_6(p, N)$ is true in G then $\gamma_p = \aleph_0$.
If $\chi_6(p, N)$ is not rue in G but $\dim p^M G/p^{M+1} G$ is infinite, then $\gamma_p = K$.

In all other cases $\gamma_p = \dim p^M G/p^{M+1} G - \dim p^M G[p]$. Since G is a model of T_{pf} γ_p is always non-negative. This completes the description of A, let us now check that A^* has the desired properties.

For $\chi = \chi_3(p, n, k)$ $p \in P$, $n \leqslant N$, $k \leqslant K$ it is clear that $G \models \chi$ iff $A^* \models \chi$, and the same is easily seen to be true for $\chi_7(p, n)$. For $n < M$ we have by (I) $G \models \chi_1(p, n, K)$ and $A^* \models \chi_1(p, n, K)$. If $M = N+1$ this is already the whole story. So assume $M \leqslant N$. If $\dim p^M G[p]$ is infinite, then by choice of M and (I) $\dim p^n G[p]$ is infinite for all n such that $M \leqslant n \leqslant N$. We defined in this case $\alpha_{p,N+1} \geqslant K$. Thus for all n, $M \leqslant n \leqslant N$ $\dim p^n A^*[p] \geqslant K$. If $\dim p^M G[p]$ is finite then

$$\begin{aligned}\dim p^M A^*[p] &= \dim p^{N+2} A^*[p] + \Sigma\{\alpha_{p,i}(A^*): M < i \leqslant N+2\}\\ &= \Sigma\{\alpha_{p,i}: M < i \leqslant N+1\}\\ &= \dim p^M G[p], \quad \text{by definition of } \alpha_{p,N+1}.\end{aligned}$$

Since $\dim p^M G[p]$ is finite (I) implies that in this case we have already $\dim p^n A^*[p] = \dim p^n G[p]$ for all n, $M \leqslant n \leqslant N$. Let us now consider χ_2. For $n < M$ we have as above $G \models \chi_2(p, n, K)$ and $A^* \models \chi_2(p, n, K)$. Next assume $M \leqslant N$ and $\dim p^M G/p^{M+1} G$ is infinite, then for all n, $M \leqslant n \leqslant N$ $\dim p^n G/p^{n+1} G$ is infinite and $\gamma_p \geqslant K$ implies that for all n, $M \leqslant n \leqslant N$ $\chi_2(p, n, K)$ is true in A^*. If $\dim p^M G/p^{M+1} G$ is finite then

$$\begin{aligned}\dim p^M A/p^{M+1} A &= \Sigma\{\alpha_{p,i}(A^*): M < i \leqslant N+2\} + \dim p^{N+2} A^*/p^{N+3} A^*\\ &= \Sigma\{\alpha_{p,i}: M < i \leqslant N+1\} + \gamma_p\\ &= \dim p^M G[p] + \gamma_p\\ &= \dim p^M G/p^{M+1} G.\end{aligned}$$

Since $\dim p^M G/p^{M+1} G$ is finite, we get from (II) that for all n, $M \leqslant n \leqslant N$ $\dim(p^n A^*/p^{n+1} A^*) = \dim p^n G/p^{n+1} G$.

The direct factor $\boldsymbol{Z}(q^t)$ assures that A^* satisfies $\chi_4(N)$. Considering χ_5 we remark first that

$$\begin{aligned}A^* \models \chi_5(p, n) \quad &\text{iff} \quad \Sigma\{\alpha_{p,i}(A^*): n < i \leqslant N+1\} \geqslant \aleph_0\\ &\text{iff} \quad \Sigma\{\alpha_{p,i}(A): n < i \leqslant N+1\} \geqslant \aleph_0.\end{aligned}$$

If $G \models \chi_5(p, N)$ then we have by (I) $G \models \chi_5(p, n)$ for all $n \leqslant N$. In this case we defined $\alpha_{p,N+1} = \aleph_0$ and we get for all $n \leqslant N$ that $\chi_5(p, n)$ is true in A^*. If $G \models \neg\chi_5(p, N)$ and $G \models \chi_5(p, n)$ for some $n < N$ then (I) implies that there is some i, $n < i \leqslant N$ such that $\alpha_{p,i,*}(G) \neq 0$ and thus we have also $A^* \models \chi_5(p, n)$. If $G \models \neg\chi_5(p, n)$ then we have for all i, $n < i \leqslant N$ $\alpha_{p,i} < \aleph_0$ and since this implies $G \models \chi_5(p, N)$ also $\alpha_{p,N+1} < \aleph_0$. Thus $A^* \models \neg\chi_5(p, n)$. Considering χ_6 we remark first that

$$A^* \models \chi_6(p, n) \text{ iff for some } i,\ n < i \leqslant N+1\ \alpha_{p,i} = \aleph_0 \text{ or } \gamma_p = \aleph_0.$$

If $G \models \chi_6(p, N)$ then we have by (II) $G \models \chi_6(p, n)$ for all $n \leqslant N$. In this case we defined $\gamma_p = \aleph_0$. Thus $\chi_6(p, n)$ is true in A^* for all $n \leqslant N$. If $G \models \neg\chi_6(p, N)$ and $G \models \chi_6(p, n)$ for some $n < N$, then (II) implies $\alpha_{p,i,*}(G) \neq 0$ for some i, $n < i \leqslant N$. But as we have seen above this implies $\alpha_{p,i,*}(A^*) \neq 0$ and thus $A \models \chi_6(p, n)$. If $G \models \neg\chi_6(p, n)$, then we have for all i, $n < i \leqslant N$ $\alpha_{p,i} < \aleph_0$ and since G is a model of $\chi_5(p, n) \to \chi_6(p, n)$ we have also $\alpha_{p,N+1} < \aleph_0$. Furthermore $G \models \neg\chi_6(p, N)$ and therefore $\gamma_p < \aleph_0$. Alltogether we find $A^* \models \neg\chi_6(p, n)$.

Now consider the case that G is bounded. We may assume that for any prime p and $n > 0$ such that $\alpha_{p,n}(G) \neq 0$ we have $p \in P$ and $n \leqslant N$. In this case χ_1, χ_2 and χ_5, χ_6 are completely determined by χ_3 and χ_7 respectively using (I) and (II). So we construct

$$A = \bigoplus_{p \in P} \bigoplus_{n \leqslant N} \boldsymbol{Z}(p^n)^{\alpha_{p,n}}$$

where $\alpha_{p,n}$, $p \in P$, $1 \leqslant n \leqslant N$ are defined as above. The validity of $\chi_4(n)$ and $\chi_8(n)$ is obviously not changed when passing from G to A^*.

Corollary 3.7. *T_{pf}^t is decidable.*

Proof. Using the fact that a sentence which is consitent with T_{pf}^t has already a model A^* where $A = \bigoplus_{p\in P}\bigoplus_{n\leq N} Z(p^n)^{\alpha_{p,n}} \oplus \bigoplus_{p\in P} Z(p^\omega)^{\gamma_p}$ is determined by the finite tupel of numbers $\alpha_{p,n}$ and γ_p, it is possible to enumerate recursively all L^t-sentences which are consitent with T^t_{pf}. This implies decidability.

References

[1] S. A. Basarab, *On the elementary theories of abelian profinite groups and abelian torsion groups*, Revue Romaine de Math. Pure 22 (1977), pp. 299–311.

[2] G. Cherlin and P. H. Schmitt, *Undecidable L^t-theories of topological abelian groups*, J. Symb. Logic 46 (1981), pp. 761–772.

[3] — — *A decidable L^t-theory of topological abelian groups*, to appear in Ann. Math. Logic.

[4] P. Eklof and E. R. Fischer, *The elementary theory of abelian groups*, Ann. Math. Logic 4 (1972), pp. 115–171.

[5] J. Flum and M. Ziegler, *Topological model theory*, Springer Lecture Notes 769.

[6] L. Fuchs, *Infinite abelian groups*, Vol. I, Pergamon Press 1970.

Accepté par la Rédaction le 3. 8. 1981

On the span of weakly-chainable continua

by

Lex G. Oversteegen (Birmingham, Ala.)
and **E. D. Tymchatyn*** (Saskatoon, Sas.)

Abstract. A continuum is weakly chainable provided it is the continuous image of the pseudo-arc. It is an open problem to classify weakly chainable atriodic tree-like continua. In particular, the following problem, due to Mohler, is open: Suppose X is a weakly-chainable atriodic tree-like continuum, is X chainable? In this paper we will give a necessary condition for weak-chainability of certain continua by proving the following theorem: Suppose X is a weakly-chainable (atriodic) tree-like continuum such that every proper subcontinuum is chainable, then the span of X is zero. This answers a question of Ingram. We will also investigate some related problems.

1. Introduction and preliminaries. By a *mapping* we mean a continuous function and by a continuum a compact, connected metric space. A *tree* is a finite, connected and simply connected graph. A continuum is *tree-like* (*arc-like*) if it is an inverse limit of trees (arcs, respectively). A continuum X is *atriodic*, provided for every pair Y_1, Y_2 ($Y_2 \subset Y_1$) of subcontinua of X, $Y_1 \setminus Y_2$ has at most two components.

Let (X, d) be a connected metric space. For $i = 1, 2$ let π_i: $X\times X \to X$ be the ith coordinate projection. We define the *surjective span* $\sigma^*(X)$ (respectively the *surjective semi-span* $\sigma_0^*(X)$) (see [6], [7]), of X to be the least upper bound of the set of real numbers $\alpha \geqslant 0$ with the following property: there exists a connected set $C_\alpha \subset X\times X$ such that $d(x, y) \geqslant \alpha$ for $(x, y)\in C_\alpha$ and $\pi_1(C_\alpha) = X = \pi_2(C_\alpha)$ [resp. $\pi_1(C_\alpha) = X$]. The *span* $\sigma(X)$ [resp. *semi-span* $\sigma_0(X)$] of X is defined by

$$\sigma(X) = \sup\{\sigma^*(A)|\ A\subset X, A\neq \emptyset \text{ connected}\}$$

$$(\text{resp. } \sigma_0(X) = \sup\{\sigma_0^*(A)|\ A\subset X,\ A\neq\emptyset \text{ connected}\}.$$

It is known that the (semi-) span of a chainable continuum is zero. It is an open question of Lelek whether a continuum of span zero is chainable. It follows from [8] that such a continuum is atriodic tree-like.

* This research was supported in part by NSERC grant no. A5616 and by a University of Saskatchewan research grant.