Continua whose local homeomorphisms are homeomorphisms

by

Akira Tominaga (Hiroshima)

Abstract. Let $f: X \to Y$ be a local homeomorphism between continua. Then an answer is given to the question: Under what conditions for $X$, is $f$ a homeomorphism?

1. Introduction. Let $X, Y$ be continua and $f: X \to Y$ a local homeomorphism of $X$ onto $Y$. Then we give sufficient conditions for $X$ that $f$ is a homeomorphism. If $X$ is a chainable continuum or more generally a tree-like continuum, then $f$ is a homeomorphism ([5, p. 261], [3, p. 67], [2, p. 317], [4, p. 50]).

Our results are the following:

Theorem 1. Let $X, Y$ be continua and $f: X \to Y$ a local homeomorphism of $X$ onto $Y$. If $X$ is the limit of an inverse sequence, with bonding maps onto, of simply connected Peano continua and $X$ has the fixed point property for homeomorphisms, then $f$ is a homeomorphism.

Theorem 2. Let $X, Y$ be continua and $f: X \to Y$ a local homeomorphism of $X$ onto $Y$. If $X$ is the intersection of a monotone decreasing sequence of simply connected Peano continua and $X$ has the fixed point property for homeomorphisms, then $f$ is a homeomorphism.

Corollary. Every local homeomorphism of a compact metric AR onto a space is a homeomorphism.

2. Definitions and notation. A Peano continuum is a locally connected, connected, compact metrizable space. A space $X$ is simply connected if it is arcwise connected and each closed path in $X$ is homotopic to zero. A map means a continuous function. A local homeomorphism $f: X \to Y$ between topological spaces is a map having the following property: For each point $x$ of $X$ there exists an open neighborhood $U$ of $x$ such that $f(U)$ is open in $Y$ and $f$ restricted to $U$, $f|U$, is a homeomorphism of $U$ onto $f(U)$.

Let $(M, d)$ be a metric space and $\delta$ a positive number. For points $a, b$ of $M$, a $\delta$-chain from $a$ to $b$ is a finite sequence $x = \{a = x_1, x_2, \ldots, x_l = b\}$ of...
points such that \( d(x_i, x_{i+1}) < \delta \) \((1 \leq i < \infty)\). If \( a = b \), then \( a \) is a \( \delta \)-loop based at \( a \). Let \( x = [a_1, \ldots, a_\ell] \), \( y = [b_1, \ldots, b_\ell] \) be \( \delta \)-chains. If \( [a_1, \ldots, a_\ell, b_1, \ldots, b_\ell] \) is also a \( \delta \)-chain, then we denote it by \( a \equiv b \). Moreover \( [a_1, a_2, \ldots, a_\ell] \) is denoted by \( a^\ell \). For \( \delta > 0 \), a finite set of points \( \{x_i \in X : 1 \leq i \leq l, 1 \leq j \leq m \} \) is a \( \delta \)-net provided that the diameter of \( [x_i, x_{i+1}, x_{i+2}, \ldots, x_{i+j}] \) \((1 \leq i \leq l, 1 \leq j < m)\) is less than \( \delta \). Let \( \alpha = [a_1, a_2, \ldots, a_k] \) be a \( \delta \)-loop. If there exists a \( \delta \)-net \( \{X_{ij} : 1 \leq i \leq j \leq k \} \) such that \( X_{ij} = a_i \) and \( X_{ij} = a_j \) \((1 \leq i \leq j \leq k)\), then \( \alpha \) is a \( \delta \)-homotopic to zero in \( M \), and we denote \( \alpha \equiv 0 (\delta) \).

Let \( X, Y \) be continua, and \( f : X \to Y \) be a local homeomorphism of \( X \) onto \( Y \). Hereafter we shall exclusively use the symbols \( f, \psi, e, \mu, \gamma \), and \( \lambda \) as follows: For each \( V \in \mathcal{V} \) there exists a finite collection \( \{E_1, \ldots, E_k\} \), denoted by \( \mathcal{E}(V) \), of mutually exclusive open sets of \( K \), such that \( f^{-1}(V) = \bigcup_{i=1}^k E_i \) and \( f|E_i : E_i \to \mathcal{V} \) \((1 \leq i \leq k)\) is a homeomorphism of \( E_i \) onto \( V \).

\( \Psi \) is the covering \( \bigcup_{i=1}^k \mathcal{V} \) of \( X \).

\( \psi \) is the Lebesgue number of \( \Psi \).

\( \mu \) is the mesh of \( \Psi \).

\( \psi \) is the Lebesgue number of \( \Psi \).

\( \lambda \) is a positive number such that if \( A \) is a subset of \( X \) with diameter \( < \lambda \), then \( f(A) \) is a \( \delta \)-homeomorphism.

Let \( \beta = [b_1, \ldots, b_l] \) be a chain in \( Y \) such that \( [b_1, b_{l+1}] \) is contained in an element \( V \) of \( \mathcal{V} \). If a chain \( x = [a_1, \ldots, a_k] \) in \( X \) satisfies the condition that \( [a_1, a_{k+1}] \) is contained in an element \( \mathcal{V} \) of \( \mathcal{V} \), then we say that \( \alpha \) covers \( \beta \), or that \( \alpha \) is a lifting of \( \beta \). A homeomorphism \( g \) of \( X \) onto itself is said to be an automorphism of \( X \) with respect to \( f \) provided that \( f\circ g = f \).

By the method of lifting similar to that in the theory of covering spaces, we have (2.1) and (2.2).

(2.1) \( \mathcal{V} \) is an \( e \)-chain in \( X \) with initial point \( y \), and let \( x \) be a point of \( X \) with \( f(x) = y \). If \( e < \lambda/2 \), then there exists a unique \( e \)-chain \( X \) in \( X \) with initial point \( y \) covering \( \beta \).

(2.2) \( \mathcal{V} \) is a chain in \( Y \) from \( y \) to \( y' \), such that \( \beta \beta^{-1} \equiv 0 \) \((a) \) in \( Y \). Let \( x \) be a point of \( X \) with \( f(x) = y \), and let \( \alpha, \alpha' \) be liftings of \( \beta, \beta' \) with initial point \( x \), respectively. If \( e < \lambda/2 \), then \( \alpha, \alpha' \) have the same initial point. Whence a lifting of a loop \( \mathcal{V} \) is homotopic to zero is also a loop.

Obviously (2.3) \( \mathcal{V} \) is a \( \mu \)-chain in \( Y \) with \( \gamma \) \( \beta \), then \( \gamma \beta^{-1} \equiv 0 \) \((a) \).

3. Proof of Theorem 1. We first prove that if \( a, b \in X \) with \( f(a) = f(b) \), then there exists an automorphism \( g \) of \( X \) with respect to \( f \) such that \( g(a) = b \). Next we show that if \( a \neq b \), then \( g \) has no fixed point.

Let \( X \) be the limit of an inverse sequence, with bonding maps onto, of simply connected Peano continua \( X_t \). Let \( s_t \) be the \( t \)-th projection of \( X_t \) onto \( X_{t-1} \), and \( d_t \) a metric on \( X_t \) bounded by number 1. Then a metric \( d \) on \( X \) is given by

\[
d(x, y) = \sum_{t=1}^{\infty} 2^{-t} d_t(s_t(x), s_t(y)).
\]

We may assume that

\[
(1) < \lambda/2.
\]

Let \( \alpha \) be a positive number such that

(2) \( \alpha > 0 \).

(3) \( \alpha \) is a subset of \( X \) with \( \text{diam}(\alpha) < \tau_0 \), then \( \text{diam}(f(\alpha)) < \tau_0 \).

By (1) there exists a positive integer \( n \) and \( \gamma > 0 \) such that

(4) \( \alpha \) is a subset of \( X \) with \( \text{diam}(K(\alpha)) < \gamma \), then \( \text{diam}(\alpha(\gamma)^{-1}(K)) < \tau_0 \).

Choose \( \delta > 0 \) such that

(5) \( f(x, y) < \delta \), then \( x, y \) can be joined by an arc in \( X \) with diameter \( \gamma \).

Moreover choose \( \tau > 0 \) such that

(6) \( \tau < \min(\tau_0, \delta/2) \)

and

(7) \( \alpha \) is a \( \tau \)-chain in \( X \), then every lifting of \( f(\alpha) \) is a \( \tau/2 \)-chain.

(a) \( \mathcal{V} \) is a point of \( X \) and \( x \) a point of \( X \) with \( f(x) = y \). If \( e < \lambda/2 \), then there exists a unique \( e \)-chain \( X \) in \( X \) with initial point \( y \) covering \( \beta \).

(b) \( \mathcal{V} \) is a chain in \( Y \) from \( y \) to \( y' \), such that \( \beta \beta^{-1} \equiv 0 \) \((a) \) in \( Y \). Let \( x \) be a point of \( X \) with \( f(x) = y \), and let \( \alpha, \alpha' \) be liftings of \( \beta, \beta' \) with initial point \( x \), respectively. If \( e < \lambda/2 \), then \( \alpha, \alpha' \) have the same initial point. Whence a lifting of a loop \( \mathcal{V} \) is homotopic to zero is also a loop.

Obviously (2.3) \( \mathcal{V} \) is a \( \mu \)-chain in \( Y \) with \( \gamma \) \( \beta \), then \( \gamma \beta^{-1} \equiv 0 \) \((a) \).

3. Proof of Theorem 1. We first prove that if \( a, b \in X \) with \( f(a) = f(b) \), then there exists an automorphism \( g \) of \( X \) with respect to \( f \) such that \( g(a) = b \). Next we show that if \( a \neq b \), then \( g \) has no fixed point.
that $x_q = x_{1j} = x_j = a$ (1 ≤ i ≤ p, 1 ≤ j ≤ q) and $x_{q+1} = x_q (1 ≤ k ≤ l + m)$. Then by (8) and (4) ($x_q$) is a $t_q$-net in $X$ and by (3) ($f(x_q)$) is an $n$-net in $Y$. Therefore if $β = (f(a) = f(x_{1q}), f(x_{2q}), ..., f(x_{q-1}) = f(x_{q}))$ and $β'$ = $(f(a) = f(x_q), f(x_{q+1}), ..., f(x_{q+q}) = f(x_q))$, then $β'^{-1}$ is 0 (0). Hence by (2) and (2.2) the liftings of $β$, $β'$ with initial point $b$ have the same terminal point.

On the other hand, since $a$, $a'$ are $τ$-chains, by (6) and (3) $f(a), f(a')$ are $τ$-chains with $f(a) ⊂ β$, $f(a') ⊂ β'$. Since $β$, $β'$ are $τ$-chains, by (2.3) we have $f(a)β'^{-1}$ 0 (a) and $f(a')β'^{-1}$ 0 (a). Thus the liftings of $f(a)$ and $f(a')$ with initial point $b$ have the same terminal point, $g(x)$ (cf. (2.2)).

(c) The map $g$ is a local homeomorphism. Let $U$ be any neighborhood of $g(x)$. Then there exist $V' ⊂ V$, $E_1, E_2 ∈ E(V)$ such that $f(x) ⊂ V'$ and $g(x) ⊂ E_1$, $g(x) ⊂ E_2$. Let $V'$ be an open set such that $f(x) ⊂ V'$, $diam_V β < τ$ and $E_1 ⊂ U$, where $E_1 = (f|E_1)^{-1}(V')$. Then $E_1 = (f|E_1)^{-1}(V)$, $E_2 = (f|E_2)^{-1}(V')$. If $U$ is any point of $E_1$, and $a$ is a $τ$-chain from $a$ to $x$, then $a U$ is a $τ$-chain from $a$ in $U$ and $g(a) U$ is an $τ$-chain in $Y$. Lifting the $τ$-chain to a chain with initial point $b$, we see that $g(a) ⊂ E_2 ⊂ U$ and hence $g(E_2 ⊂ U$. Thus $g$ is continuous. Clearly $g(E_1) = E_1$, and $g$ is a local homeomorphism.

(d) The map $g$ is a homeomorphism. For suppose that there exist distinct points $a, a'$ with $g(a) = g(a')$. Suppose $a$ is a $τ$-chain from $a$ to $x$ and $a'$ is the chain with initial point $b$, covering $f(a)$. Then $a'^{-1}$, $a'^{-1}$ cover $f(a')$ and have the common initial point $b$. Then by (2.1), we have $a' = b$, contrary to $a ≠ b$.

(f) Suppose that there exist distinct points $a, b ∈ X$ with $f(a) = f(b)$. Then by (a) (e) there exists an automorphism $g$ of $X$ without fixed point, which contradicts to our assumption that $X$ has the fixed point property for homeomorphisms. Thus $f = g$ is a homeomorphism.

4. Proof of Theorem 2. Let $a, b$ be points of $X$ with $f(a) = f(b)$. We first show the existence of an automorphism $g$ of $X$ with respect to $f$ such that $g(a) = b$.

We may assume that $μ ≤ i/2$. Let $τ_{ab}$ be a positive number such that if $A$ is a subset of $X$ with $diam(A) < τ_{ab}$, then $diam(f(A)) < τ$. Then we can find a positive integer $n$ such that $X_n$ is contained in a $τ_{ab}$-neighborhood of $X$. Choose $δ > 0$ such that if $z, x' ∈ X_n$ and $d(z, x') < δ$, then $x'$ can be joined by an arc in $X_n$ with diameter $< τ_{ab}/2$. There exists $τ > 0$ such that $τ ≤ τ_{ab}/2$ and such that if $a$ is a $τ$-chain in $X$, then each lifting of $f(a)$ is a $τ$-chain.

(a) The definition of $g: X → X$ is the same as (a) in the preceding section.

(b) The map $g$ is well defined. For let $α = (a = x_{11}, x_{21}, ..., x_{1q} = x_{1q+1})$ be a $τ$-chain in $X$, and let $x_k$ be an arc in $X_n$, from $x_q$ to $x_{q+1}$, whose diameter $< τ/2$. If $ψ: I → X_q$ is a parametrization of the loop $a_1 ⊂ a_2 ⊂ ... ⊂ a_{q+1}$, then there exists a map $F: I × I → X_q$ such that

$$F(s, 0) = ψ(s),$$
$$F(s, 1) = F(0, t) = F(1, t) = a (0 ≤ s, t ≤ 1).$$

We can find numbers $0 = s_1 < s_2 < ... < s_p = 1$ and $0 = t_1 < t_2 < ... < t_q = 1$ such that $diam [z_{01}, z_{11}, z_{11+1}, z_{11+1}, z_{11+1+1}] < τ_{ab}/2$, and such that there exists a subsequence of $x_i, 0 = s_{11} < ... < s_{1p} < s_{1p+1} = 1$, with $x_{0k+1} = x_q$. Choose a point $x_{1k}$ of $X$ so that $d(x_{1k}, x_{1k}) < τ_{ab}/2$. Then $x_{1k} = x_{1k} = a (1 ≤ i ≤ p, 1 ≤ j ≤ q)$ and $x_{0k+1} = x_q (1 ≤ k ≤ l + m)$. Then $x_{0k}$ is a $τ$-net in $X$ and $f(x_{0k})$ is an $n$-net in $Y$. Therefore if we put $β = (f(a) = f(x_{1q}), f(x_{1q}), ..., f(x_{q+q})) = f(x_{q})$ and $β' = (f(a) = f(x_q), f(x_{q+1}), ..., f(x_{q+q})) = f(x_q)$, then $β'^{-1}$ is 0 (0). Hence the liftings of $β$, $β'$ with initial point $b$ have the same terminal point (cf. (2.2)). On the other hand, since $f(a) ⊂ β, f(a') ⊂ β'$, by (2.3) and (2.2) the liftings of $f(a), f(a')$ with initial point $b$ have the same terminal point, $g(x)$.

By the same procedure as (c) (d) in Section 3, we can complete the proof.

Addendum. The following Propositions 1 and 2 correspond to Theorems 1 and 2, respectively.

PROPOSITION 1. Let $X, Y$ be continua, and $f: X → Y$ a local homeomorphism of $X$ onto $Y$. If $f$ is the limit of an inverse sequence, with bonding maps onto, of simply connected Peano continua, then $f$ is a homeomorphism.

PROPOSITION 2. Let $X, Y$ be continua, and $f: X → Y$ a local homeomorphism of $X$ onto $Y$. If $f$ is the intersection of a monotone decreasing sequence of simply connected Peano continua, then $f$ is a homeomorphism.

I am much indebted to Professor Y. Kodama who indicated to me that Propositions 1 and 2 above are consequences of the Fox’s overlay theorem [1, (5.2), p. 60]. Also after submitting the manuscript, I have known Lau’s theorem (Bull. Acad. Polon. Sci. Sér. Sci. Math. 27 (1979), p. 382) closely related to this paper.
A sum theorem for $A$-weakly infinite-dimensional spaces

by

L. Polkowski (Warszawa)

Abstract. In this note we shall establish a hereditarily closure-preserving sum theorem for $A$-weakly infinite-dimensional spaces. The applications of this theorem to the closed mappings defined on $A$-weakly infinite-dimensional spaces are given in [5].

Our terminology and notation follow [2]. Let us recall that a normal space $X$ is said to be $A$-weakly infinite-dimensional (abbrev. $A$-w.i.d.) if for every sequence $(A_1, B_1), (A_2, B_2), \ldots$ of pairs of disjoint closed subsets of $X$ there exists a sequence $L_1, L_2, \ldots$ of closed subsets of $X$ such that, for each positive integer $i$, the set $L_i$ is a partition between $A_i$ and $B_i$ in $X$ (meaning that there exist disjoint open subsets $U_i, V_i$ of $X$ such that $A_i \subseteq U_i, B_i \subseteq V_i$ and $X \setminus L_i = U_i \cup V_i$), and $\cap_{i=1}^{\infty} L_i = \emptyset$. It is manifest that every closed subspace of an $A$-w.i.d. space is $A$-w.i.d.

We begin with the following obvious lemma (cf. the proof of Lemma 1.2.9 in [2]).

Lemma 1. Let $F$ be a closed subspace of a hereditarily normal space $X$ and $A, B$ a pair of disjoint closed subsets of $X$. For every partition $L$ between $A \cap F$ and $B \cap F$ in $F$ with $F \setminus L = G \cup H$, where disjoint open subsets $G, H$ of $F$ are such that $A \cap F \subseteq G$ and $B \cap F \subseteq H$, there exists a partition $L$ between $A$ and $B$ in $X$ with $X \setminus L = M \cup N$, where disjoint open subsets $M, N$ of $X$ are such that $A \subseteq M$, $B \subseteq N$, $M \cap F = G$ and $N \cap F = H$.

The next lemma deals with countable families of partitions.

Lemma 2. Let $F$ be a closed subspace of a hereditarily normal $A$-w.i.d. space $X$ and $(A_1, B_1), (A_2, B_2), \ldots$ a sequence of pairs of disjoint closed subsets of $X$. For every sequence $L_1, L_2, \ldots$, where $L_i$ is a partition between $A_i \cap F$ and $B_i \cap F$ in $F$ for $i = 1, 2, \ldots$, such that $\cap_{i=1}^{\infty} L_i = \emptyset$, there exists a sequence