A boundary set for the Hilbert cube containing no arcs

by

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Abstract. There is a σ-Z-set $B \subset Q$ so that $Q - B \approx l_2$ but B contains no arcs.

0. Introduction. A *boundary set* for the Hilbert cube Q is a σ-Z-set $B \subset Q$ for which $Q - B \approx l_2$, the separable Hilbert space. This concept is due to Curtis [4]. Well-known examples of boundary sets are Anderson's [1] capsets and fd-capsets. In this paper we present an example of a boundary set for Q containing no arcs. This answers a question of Curtis [4]. Our result implies that in every Q-manifold M there is a σ-compact σ-Z-set B such that $M - B$ is an l_2-manifold but B contains no arcs.

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1. Triple-convex subspaces of Q. As usual, $Q = \prod_1^\infty [-1, 1]_i$. We use the metric $d(x, y) = \sup\{2^{-n}|x_n - y_n|:\ n \in N\}$ on Q. Define a function μ: $Q^3 \to Q$ by

$$\mu(x, y, z)_n = \text{the middle one of } x_n, y_n \text{ and } z_n.$$

Clearly, μ is continuous and $\mu(x, x, y) = \mu(x, y, x) = \mu(y, x, x) = x$ for all $x, y \in Q$. The function μ is called the *standard mixer* on Q, [8], and a subset $X \subset Q$ is called *triple-convex* provided that $\mu(X^3) = X$, [9]. Notice that the intersection of an arbitrary family of triple-convex subspaces of Q is again triple-convex. Therefore for each $X \subset Q$, the intersection of all triple-convex subsets of Q containing X is the smallest triple-convex subset $\hat{X}$ of Q that contains X. The "triple-convex closure" $\hat{X}$ of X can also be found in a more constructive way. Inductively, define subsets $X_n \subset Q$ by

$$X_1 = X, \qquad X_{n+1} = \mu(X_n^3).$$

Notice that $X_1 \subset X_2 \subset \ldots$ and that clearly $\hat{X} = \bigcup_1^\infty X_n$. The closure of $\hat{X}$ in Q will be denoted by $\tilde{X}$. Observe that by continuity of μ, $\tilde{X}$ is triple-convex and that $\tilde{X}$ is the smallest triple-convex *closed* subset of Q containing X.

The proof of the following lemma is implicit in the proof of [8, Theorem 1.3] and a sketch of the proof will only be included for completeness sake.

1.1. LEMMA. *Let $X \subset Q$ and let $f\colon S^n \to X$ $(n \geq 1)$ be continuous. Then f can be extended to a map $\bar{f}\colon B^{n+1} \to X_2 \subset \hat{X}$.*

Proof. We use the standard representations

$$S^n = \{(x_0, \ldots, x_n) \in R^{n+1}\colon \sum_{i=0}^{n} x_i^2 = 1\},$$

$$B^{n+1} = \{(x_0, \ldots, x_n) \in R^{n+1}\colon \sum_{i=0}^{n} x_i^2 \leq 1\}.$$

Let $u \in B^{n+1}$ be defined by

$$u_0 = 1 \quad \text{and} \quad u_i = 0 \quad \text{for} \quad 1 \leq i \leq n.$$

For each $v \in B^{n+1}$ the equation

$$\sum_{i=0}^{n-1} v_i^2 + y^2 = 1$$

has exactly two solutions $y = g_1(v) \geq 0$ and $y = g_2(v) \leq 0$ each depending continuously on v. For each $v \in B^{n+1} - \{u\}$ the line through u and v meets $S^n - \{u\}$ in exactly one point $g_3(v)$ depending continuously on v. We put $g_3(u) = u$ for convenience. This leads us to a function

$$g = (g_1, g_2, g_3)\colon B^{n+1} \to (S^n)^3$$

which is continuous in all points $v \neq u$. Define $\bar{f}\colon B^{n+1} \to X_2$ as the composition

$$B^{n+1} \xrightarrow{g} (S^n)^3 \to X^3 \xrightarrow{\mu} X_2,$$

where the map in the middle is (f, f, f). Then $\bar{f}$ extends f since for each $v \in S^n$, two points out of $g_1(v)$, $g_2(v)$ and $g_3(v)$ equal v. The easy check that $\bar{f}$ is continuous is left to the reader (for details see the proof of [8, Theorem 1.3]). ■

We claim that any closed and connected triple-convex subspace of Q is an Absolute Retract. This is known. Since any closed triple-convex subspace of Q has a binary normal subbase (this will not be defined here), [9], and since any continuum with a binary normal subbase is an AR, [5], our claim follows. For the readers convenience we will give another proof of this fact using standard apparatus only. So, let $X \subset Q$ be a triple-convex continuum. First, the connectedness of X implies that X is locally connected, [8, Lemma 1.1]. Second, if $Y = \prod_1^\infty H_i$, where $H_i \subset [-1, 1]$ is an interval for all $i \in N$, then clearly $X \cap Y$ is triple-convex, hence each map $f\colon S^n \to X \cap Y (n \geq 1)$ is null-homotopic (Lemma 1.1). Now let $\mathcal{U}$ be an open cover of X. Let $\mathcal{V}$ be an open star-refinement of $\mathcal{U}$ consisting of open subsets of X which are the intersection of some basic open (hence triple-convex) subset of Q with X. Let $\mathcal{H}$ be an open refinement of $\mathcal{V}$ consisting of connected open subsets only. Such a refinement exists since X is locally connected. Let P be a polyhedron and let P_0 be a subpolyhedron of P containing all the vertices of P, and let $f\colon P_0 \to X$ be continuous so that the partial image of every simplex of P under f is contained in some $H \in \mathcal{H}$. Using the fact that each $H \in \mathcal{H}$ is path-connected and using the above remark on the possibility to extend maps, by a straightforward partial realization argument we can extend f to a map $\bar{f}\colon P \to X$ so that for every simplex σ in P there is a $U \in \mathcal{U}$ containing $\bar{f}(\sigma)$. Consequently, X is an ANR, [2, Theorem 8.1], and since by Lemma 1.1, X is C^∞, X is even an AR.

1.2. LEMMA. *If X is a continuum (resp. Peano continuum) then so is X_i, for all $i \in N$.*

Proof. For $i \geq 2$, X_i is a continuous image of X_{i-1}. ■

Observe that this lemma implies that $\tilde{X} \in \mathrm{AR}$ if $X \subset Q$ is a continuum. A subcube of Q is a product $\prod_1^\infty H_n$, where, for each $n \in N$, $H_n \subset [-1, 1]$ is an interval (not necessarily closed).

Let $S \subset Q$ be a compact subcube. It is easily seen that the function $r\colon Q \to S$ defined by

$$r(x)_n = \text{the middle one of } x_n,\ \min \pi_n(S) \text{ and } \max \pi_n(S),$$

is a retraction of Q onto S. We will call r the *canonical retraction* of Q onto S. This type of retraction was studied in van Mill and van de Vel [8].

1.3. LEMMA. *If S_1 and S_2 are intersecting compact subcubes in Q and $r_i\colon Q \to S_i$ denote the canonical retractions, then the formula*

$$r(x) = \mu(r_1 \circ r_2(x), r_2 \circ r_1(x), x) \quad \text{for } x \in Q$$

defines the canonical retraction $Q \to S_1 \cap S_2$.

Proof. Left to the reader. ■

If $X \subset Q$, let $I(X)$ denote the smallest closed subcube of Q containing X, i.e.

$$I(X) = \prod_1^\infty [\inf \pi_n(X), \sup \pi_n(X)].$$

1.4. LEMMA. *For any set X in Q we have $\tilde{X} \subset I(X)$ and hence, for all n,*

$$\operatorname{diam} X \leq \operatorname{diam} X_n \leq \operatorname{diam} \tilde{X} \leq \operatorname{diam} X.$$

Proof. This is clear since each subcube is triple-convex. ■

1.5. LEMMA. *If $X \subset Q$ and $x \in X_n$, then there is a set $F \subset X$ with $x \in I(F)$ and $|F| \leq 3^{n-1}$.*

Proof. If $n = 1$ then there is nothing to prove. So assume the statement to be true for n and take $x \in X_{n+1}$ arbitrarily. There are $p, q, r \in X_n$ with $\mu(p, q, r) = x$. Find, by induction hypothesis, sets $F, G, H \subset X$ with $|F|, |G|, |H| \leq 3^{n-1}$

and $p \in I(F)$, $q \in I(G)$ and $r \in I(H)$. Then

$$\{p, q, r\} \subset I(F) \cup I(G) \cup I(H) \subset I(F \cup G \cup H),$$

and consequently, $x = \mu(p, q, r) \in I(F \cup G \cup H)$. Clearly $|F \cup G \cup H| \leqslant 3^n$. ∎

1.6. LEMMA. *Let F be a finite set in X and $r\colon Q \to I(F)$ be the canonical retraction. Then $r(\hat{X}) \subset \hat{X}$ and hence, $r(\tilde{X}) \subset \tilde{X}$. If F in addition is an ε-net in X then $d(r(x), x) < \varepsilon$ for all $x \in I(X)$.*

Proof. Suppose that $|F| = n+1$ and that the first statement is true for sets of cardinality n. Pick $y \in F$ and let $r_0\colon Q \to I(F-\{y\})$ be the canonical retraction. It is trivial to verify that $r(x) = \mu(x, r_0(x), y)$ for $x \in Q$; hence if $x \in \hat{X}$ then $r(x) \in \hat{X}$ since $r_0(x), y \in \hat{X}$.

The proof of the remaining part is left to the reader. ∎

In verifying that certain subsets of Q are boundary sets, we will make use of the following result due to Curtis [4], the proof of which is based on Toruńczyk's characterization of l_2.

1.7. THEOREM. *Let B be a σ-Z-set in a topological copy A of Q. Then B is a boundary set in A iff*

(C1) *for each $\varepsilon > 0$ and each map $f\colon I^n \to A$, where $n \in N$, there is a compactum $K \subset B$ such that for every neighborhood $N(K)$ of K in A there is a map $g\colon I^n \to N(K)$ with $d(g,f) < \varepsilon$,*

(C2) *for every $x \in B$ and for every neighborhood U of x there is a neighborhood V of x such that for each compactum $K \subset V \cap B$ there is a compactum $K' \subset U \cap B$ such that for every neighborhood $N(K')$ in A of K' there is a neighborhood $N(K)$ of K in A such that every map $f\colon S^n \to N(K)$ $(n \geqslant 0)$ is null-homotopic in $N(K')$.*

Notice that our example shows that condition (C1) in the above theorem cannot be replaced by the more natural condition: for every $\varepsilon > 0$ and for every map $f\colon I^n \to A$ there is a map $g\colon I^n \to B$ with $d(g,f) < \varepsilon$.

2. Verifying condition (C1). In this section we will show that pairs of the form $(\tilde{X}, \hat{X})$ always satisfy condition (C1) of Theorem 1.7.

2.1. THEOREM. *Let X be a continuum in Q. Then the pair $(A, B) = (\tilde{X}, \hat{X})$ satisfies condition* (C1).

Proof. Fix $\varepsilon > 0$ and a map $f\colon I^n \to X$ and let $\delta = \varepsilon \cdot 2^{-n-2}$. Take a continuum K in $\tilde{X}$ which is a δ-net in $\tilde{X}$ (e.g., let $K = X_k$ for a large k; observe that by Lemma 1.2 X_k is a continuum). The proof will be concluded once we show that

(∗) for every neighborhood U of K_{n+1} in $\tilde{X}$ there is a map $g\colon I^n \to U$ with $d(g,f) < \varepsilon$.

To this end, fix U and using n-times the continuity of μ, take a neighborhood V of K with $V_{n+1} \subset U$. Since $\tilde{X} \in \mathrm{AR}$, see section 1, there is a Peano continuum Y with $K \subset Y \subset V$. Let $\mathscr{T}$ be a triangulation of I^n such that $\operatorname{diam} f(\sigma) < \delta$ for every $\sigma \in \mathscr{T}$ and let $g_0\colon \mathscr{T}^0 \to K$ be a map such that $d(g(x), f(x)) < \delta$ for $x \in \mathscr{T}^0$. By the triangle inequality, (∗) follows from the case $i = n$ of the following claim ($\mathscr{T}^i$ denotes the ith skeleton of $\mathscr{T}$):

$(*)_i$ there is a map $g_i\colon \mathscr{T}^i \to Y_{i+1}$ extending g_0 and such that $\operatorname{diam} g_i(\sigma) < 3 \cdot 2^i \cdot \delta$, for all $\sigma \in \mathscr{T}^i$.

To prove $(*)_i$, first consider the case $i = 1$. Given $\sigma \in \mathscr{T}^1$ let $\{a, b\} = g_0(\partial\sigma)$. Then $L = \mu(\{a\} \times \{b\} \times Y)$ is a Peano continuum in Y_2 containing $\{a, b\}$; moreover $L \subset I(\{a, b\})$ and therefore $\operatorname{diam} L \leqslant d(a, b) < 3\delta$. Take a map $\sigma \to L$ extending $g_0|\{a, b\}$ and proceed in this way with all $\sigma \in \mathscr{T}^1$ to get the desired g_1.

Now suppose that $i \geqslant 1$ and that $g_i\colon \mathscr{T}^i \to Y_{i+1}$ as described in $(*)_i$ is known. Given $\sigma \in \mathscr{T}^{i+1}$ the map $g_i|\partial\sigma$ extends, by Lemma 1.1, to a map $\sigma \to L = \mu\big((g_i(\partial\sigma))^3\big)$. Then $L \subset Y_{i+2}$ and $\operatorname{diam} L = \operatorname{diam} g_i(\partial\sigma) < 3 \cdot 2^{i+2} \cdot \delta$. Thus the collection of so obtained maps $\sigma \to Y_{i+2}$ defines the desired extension g_{i+1} of g_i. ∎

3. Verifying condition (C2). In this section we will show that pairs of the form $(\tilde{X}, \hat{X})$ satisfy condition (C2) of Theorem 1.7.

3.1. LEMMA. *Let $X \subset Q$ be a continuum and let $Q' \subset Q$ be a subcube. If $K \subset \hat{X} \cap Q'$ is compact, then there is a continuum $K' \subset \hat{X} \cap Q'$ containing K.*

Proof. By a result of Curtis [3, Lemma 1.3] it suffices to show that $\hat{X} \cap Q'$ is continuum-connected (each pair of its points is contained in a continuum) and locally continuum-connected. Since sets of the form $\hat{X} \cap Q_1$, where Q_1 is a subcube contained in Q', form a neighborhood basis of the points of $\hat{X} \cap Q'$, it thus suffices to prove the assertion in the case where K is a 2-point set, say $K = \{x, y\}$. Then, however, $K \subset X_n$ for some n and, since X_n is a continuum (Lemma 1.2), so is $L = \mu(\{x\} \times \{y\} \times X_n)$. It is clear that $\{x, y\} \subset L \subset \hat{X} \cap Q'$. ∎

3.2. THEOREM. *Let X be a continuum in Q. Then the pair $(A, B) = (\tilde{X}, \hat{X})$ satisfies condition* (C2).

Proof. Let $x \in Q$ and let U be a neighborhood of x. Let $V \subset U$ be a subcube neighborhood of x. Now choose any compactum $K \subset V \cap \hat{X}$. We may assume that $K \neq \emptyset$. By Lemma 3.1, there is a continuum $S \subset V \cap \hat{X}$ containing K. Define $K' = \mu(S^3)$. Notice that, since V is a subcube and since $\hat{X}$ is triple-convex,

$$K \subset K' \subset V \cap \hat{X} \subset U \cap \hat{X}.$$

We claim that for every neighborhood $N(K')$ of K' in Q there is a neighborhood $N(K)$ of K in Q such that every map $f\colon S^n \to N(K) \cap \tilde{X}$ $(n \geqslant 0)$ is null-homotopic in $N(K') \cap \tilde{X}$. To this end, let $N(K')$ be any neighborhood of K'. By continuity of μ there is a neighborhood E of S so that $\mu(E^3) \subset N(K')$. Since $\tilde{X} \in \mathrm{AR}$ there is a closed neighborhood $N(K)$ of S in Q such that $N(K) \cap \tilde{X}$ is a Peano continuum while moreover $N(K) \subset E \cap N(K')$. Since $K \subset S$, $N(K)$ is a neighborhood of K. If $\{p, q\}$ is a pair of points in $N(K) \cap \tilde{X}$, then the local connectivity of $N(K) \cap \tilde{X}$ implies that there is a path in $N(K) \cap \tilde{X} \subset N(K') \cap \tilde{X}$ connecting p and q. Now let $f\colon S^n \to N(K) \cap \tilde{X}$ $(n \geqslant 1)$ be any map. By Lemma 1.1, f can be extended to a map $\bar{f}\colon B^{n+1} \to N(K)_2 \cap \tilde{X}$. Since

$$N(K)_2 \subset E_2 = \mu(E^3) \subset N(K'),$$

this implies that $\bar{f}(B^{n+1}) \subset N(K') \cap \tilde{X}$. ■

4. Recognizing freely embedded cubes and Hilbert spaces. A subset $X \subset Q$ is called *free* provided that for any two disjoint finite subsets $F, G \subset X$ there is an $n \in N$ with $\pi_n(F) = -1$ and $\pi_n(G) = 1$ (π_n denotes the projection onto the nth coordinate).

4.1. Lemma. *Let X be a space. Then there is an embedding $f\colon X \to Q$ so that $f(X)$ is free.*

Proof. Let X' be a compactification of X. By van Mill [6, Lemma 1.1] there is an embedding $g\colon X' \to Q$ so that for any two disjoint closed subsets $A, B \subset X'$ there is an $n \in N$ with $\pi_n g(A) = -1$ and $\pi_n g(B) = 1$. It is clear that $f = g|X$ is as required. ■

4.2. Lemma. *Let $\mathscr{A}$ be a finite collection of subsets of $X \subset Q$ so that any two elements of $\mathscr{A}$ meet. If $r\colon Q \to \bigcap_{A \in \mathscr{A}} I(A)$ is the canonical retraction, then $r(\hat{X}) \subset \hat{X}$, and hence $r(\tilde{X}) \subset \tilde{X}$, while moreover $\hat{X} \cap \bigcap_{A \in \mathscr{A}} I(A) \supset r(\hat{X}) \neq \emptyset$.*

Proof. We will induct on the cardinality of $\mathscr{A}$. If $|\mathscr{A}| = 1$, then Lemma 1.6 can be applied. So suppose that the statement is true for collections of sets of cardinality n, and let $\mathscr{A} = \{A_1, \ldots, A_{n+1}\}$. Let $t\colon Q \to \bigcap_{i \leqslant n} I(A_i)$ and $s\colon Q \to I(A_{n+1})$ be the canonical retractions. By Lemma 1.3 for all $x \in Q$,

$$r(x) = \mu(s \circ t(x), t \circ s(x), x)$$

defines the canonical retraction from Q onto $\bigcap_{i \leqslant n+1} I(A_i)$ provided that

$$(*) \qquad \bigcap_{i \leqslant n+1} I(A_i) \neq \emptyset.$$

Suppose for a moment that $(*)$ is true. If $x \in \hat{X}$, then, by induction hypothesis both $s \circ t(x)$ and $t \circ s(x)$ belong to $\hat{X}$. Consequently, $r(x) \in \hat{X}$.

So the only remaining thing to verify is $(*)$. If $n = 1$ then $(*)$ is trivially true since any two elements of $\mathscr{A}$ meet. Therefore assume that n is at least 2. By induction hypothesis there exist points

$$x \in \bigcap_{\substack{i \leqslant n+1 \\ i \neq 1}} I(A_i) \cap \hat{X},$$

$$y \in \bigcap_{\substack{i \leqslant n+1 \\ i \neq 2}} I(A_i) \cap \hat{X}, \quad \text{and}$$

$$z \in \bigcap_{\substack{i \leqslant n+1 \\ i \neq 3}} I(A_i) \cap \hat{X}.$$

Clearly $\mu(x, y, z) \in \bigcap_{i \leqslant n+1} I(A_i) \cap \hat{X}$. ■

We now come to the main result in this paper.

4.3. Theorem. *If X is a free continuum in Q then* a) $\tilde{X} \approx Q$, *and* b) $\hat{X}$ *is a boundary set in* $\tilde{X}$.

Proof. As noted before, the connectedness of X implies that $\tilde{X}$ is an AR. We will show that the identity map on $\tilde{X}$ can be approximated by maps having disjoint images. Applying Toruńczyk [10] then yields $\tilde{X} \approx Q$. To this end, let $\varepsilon > 0$ and take disjoint finite ε-nets $F, G \subset X$. Since X is free, $I(F) \cap I(G) = \emptyset$. The desired result now directly follows from Lemma 1.6. This proves a).

For b), observe that by Curtis' result Theorem 1.7 and by the results in sections 2 and 3, it suffices to show that, for each n, X_n is a Z-set in X (i.e., given $\varepsilon > 0$, there is a map $f\colon \tilde{X} \to \tilde{X} - X_n$ with $d(f, \mathrm{id}) < \varepsilon$). To this end, fix n and ε and let $\mathscr{A}$ be a family of $2 \cdot 3^{n-1} + 1$ finite disjoint ε-nets in X. Put

$$\mathscr{B} = \{\bigcup \mathscr{E}\colon \mathscr{E} \subset \mathscr{A} \text{ and } |\mathscr{E}| = 3^{n-1} + 1\}.$$

Clearly any two elements of $\mathscr{B}$ meet and each $B \in \mathscr{B}$ is an ε-net. Let $r\colon Q \to \bigcap_{B \in \mathscr{B}} I(B)$ be the canonical retraction. It is easy to see that if $x \in I(X)$ then $d(x, r(x)) < \varepsilon$. We therefore only have to check that

$$\bigcap_{B \in \mathscr{B}} I(B) \cap X_n = \emptyset,$$

for, by Lemma 4.2, $r(\tilde{X}) \subset \tilde{X}$. Take $x \in X_n$. By Lemma 1.5 there exists a set $F \subset X$ of cardinality 3^{n-1} such that $x \in I(F)$. Since $\mathscr{A}$ is a disjoint family, at most 3^{n-1} elements of $\mathscr{A}$ can meet F. Consequently, there is a $B \in \mathscr{B}$ with $F \cap B = \emptyset$. Since X is free, $I(F) \cap I(B) = \emptyset$. We conclude that $x \notin \bigcap_{B \in \mathscr{B}} I(B)$. ■

4.4. Remark. In view of the above theorem we only need to find a free continuum in Q so that $\hat{X}$ contains no arcs. It turns out, see section 5, that if X is any free continuum containing no arcs, then $\hat{X}$ contains no arcs. This gives us a rich supply of boundary sets containing no arcs. The proof of this fact is, though elementary, surprisingly complicated.

5. $\hat{X}$ contains no arcs. In this section we will show that $\hat{X}$ contains no arcs, provided that $X \subset Q$ is a free continuum which contains no arcs. For the remaining part of this section, let X denote a fixed free continuum in Q. We will often use without explicit reference the fact that for any two disjoint finite subsets $F, G \subset X$ it is true that $I(F) \cap I(G) = \emptyset$. If $A \subset X$ then, for convenience, put $h(A) = I(A) \cap \hat{X}$.

5.1. LEMMA. *If $x \in \hat{X}$ then there is a finite collection of finite subsets $\mathscr{A}$ of X with* $\bigcap_{A \in \mathscr{A}} h(A) = \{x\}$.

Proof. Clearly the statement is true for points in X_1. Suppose that the statement is true for points in X_n and take $x \in X_{n+1}$ arbitrarily. There are $p_1, p_2, p_3 \in X_n$ with $\mu(p_1, p_2, p_3) = x$ and by induction hypothesis, there are families $\mathscr{A}_i$ of finitely many finite subsets of X with $\bigcap_{A \in \mathscr{A}_i} h(A) = \{p\}$ $(1 \leqslant i \leqslant 3)$. Put

$$\mathscr{A} = \{F \cup G\colon i, j \leqslant 3, i \neq j, F \in \mathscr{A}_i \text{ and } G \in \mathscr{A}_j\}.$$

We claim that $\mathscr{A}$ is as required. It is clear that $x \in \bigcap_{A \in \mathscr{A}} h(A)$. Let us assume there is a point $y \in \bigcap_{A \in \mathscr{A}} h(A)$ distinct from x. Choose $n \in N$ with $x_n \neq y_n$ and, without loss of generality assume that $x_n < y_n$. Take $s \in (x_n, y_n)$. Without loss of generality we may assume that $\pi_n(p_1) \leqslant s$ and $\pi_n(p_2) \leqslant s$. Take $t \in (s, y_n)$. If $B \cap \pi_n^{-1}[t, 1] \neq \emptyset$ for every $B \in \mathscr{A}_1$, then, by Lemma 4.2, $\bigcap_{B \in \mathscr{A}_1} h(B) \cap \pi_n^{-1}[t, 1] \neq \emptyset$, in which case $\pi_n(p_1) \geqslant t$. So there exists an $F \in \mathscr{A}_1$ which misses $\pi_n^{-1}[t, 1]$. Similarly, there is a $G \in \mathscr{A}_2$ which misses $\pi_n^{-1}[t, 1]$. Then $F \cup G \in \mathscr{A}$ and since $h(F \cup G) \subset \pi_n^{-1}[-1, t]$, this implies that $y_n \leqslant t$, a contradiction. ■

If $x \in \hat{X}$ then a finite subset $F \subset X$ is called a *center* for x provided that there exist $A_1, \ldots, A_n \subset F$ with $\bigcap_{i \leqslant n} h(A_i) = \{x\}$.

5.2. LEMMA. *If F is a center for $x \in \hat{X}$ and if $x \in h(A)$ for certain finite $A \subset X$, then $x \in h(A \cap F)$.*

Proof. Choose $A_1, \ldots, A_n \subset F$ with $\bigcap_{i \leqslant n} h(A_i) = \{x\}$. If $x \notin h(A \cap F)$ then, by Lemma 4.2, there is an $i \leqslant n$ with $A_i \cap (A \cap F) \in \emptyset$. Then $A_i \cap A = \emptyset$ and since $x \in h(A_i) \cap h(A)$, this is a contradiction. ■

5.3. COROLLARY. *If F and G are centers for $x \in \hat{X}$, then so is $F \cap G$.*

Proof. Choose finitely many $A_1, \ldots, A_n \subset F$ with $\{x\} = \bigcap_{i \leqslant n} h(A_i)$. By Lemma 5.2, $\{x\} = \bigcap_{i \leqslant n} h(A_i \cap G)$. Since $A_i \cap G \subset F \cap G$ for all $i \leqslant n$, $F \cap G$ is a center for x. ■

If $x \in \hat{X}$ then, by 5.1, x has a center. By 5.3,

$$(*) \qquad F(x) = \bigcap \{F \subset X\colon F \text{ is a center for } x\}$$

is the smallest center for x. Put $X(m) = \{x \in X\colon |F(x)| \leqslant m\}$. The hyperspace of nonempty closed subsets of X, with topology generated by the Hausdorff distance, will be denoted by 2^X.

5.4. LEMMA. *Let x_n, $n \geqslant 1$, be points in $X(m)$ such that $x = \lim_{n\to\infty} x_n \in Q$ and $G = \lim_{n\to\infty} F(x_n) \in 2^X$ exist. Then $x \in X(m)$ and G is a center for x.*

Proof. With k large enough there are sets $A_n^1, \ldots, A_n^k \subset F(x_n)$ with $\{x_n\} = \bigcap_{i \leqslant k} h(A_n^i)$ for each n. We may assume that $\lim_{n\to\infty} A_n^i$ exists (in the Hausdorff metric) for all $i \leqslant k$ and is equal to, say A_i. Since $\bigcup_{i=1}^k A_n^i \subset F(x_n)$ and since $|F(x_n)| \leqslant m$ for all $n \in N$, $|\bigcup_{i=1}^k A_i| \leqslant m$. It is clear that the family $\{A_i\colon 1 \leqslant i \leqslant k\}$ has the property that any two of its elements meet. By Lemma 4.2 there is a point $y \in \bigcap_{i=1}^k h(A_i)$. We will show that $y = x$, which will conclude the proof. If $x \neq y$, then we can find an index $t \in N$ such that, say, $\pi_t(x) < \pi_t(y)$. Take a point $s \in (\pi_t(x), \pi_t(y))$. We may assume that $\pi_t(x_n) < s$ for all $n \in N$. Fix $n \in N$. If $A_n^i \cap \pi_t^{-1}[s, 1] \neq \emptyset$ for all $i \leqslant k$ then, by Lemma 4.2, $\bigcap_{i=1}^k h(A_n^i) \cap \pi_t^{-1}[s, 1] \neq \emptyset$ or equivalently, $\pi_t(x_n) \geqslant s$, which is not the case. Therefore, for all $n \in N$ there us an index $i(n) \leqslant k$ with

$$A_n^{i(n)} \cap \pi_t^{-1}[s, 1] = \emptyset.$$

There is a $k_0 \leqslant k$ so that the set $\{n\colon i(n) = k_0\}$ is infinite. This implies that $A_{k_0} \subset \pi_t^{-1}[-1, s]$ and consequently, $\pi_t(y) \leqslant s$, which obviously is a contradiction. ■

5.5. COROLLARY. *For each m, the set $X(m)$ is compact and the function $F\colon \hat{X} \to 2^X$ defined by $(*)$ is finite-to-one and continuous on $X(m) - X(m-1)$.*

Proof. By the definition of a center, each $y \in F^{-1}F(x)$ is determined by a family of subsets of $F(x)$; since $F(x)$ is finite, so is $F^{-1}F(x)$. The compactness of $X(m)$ follows from 5.4. To prove the continuity of F on $X(m) - X(m-1)$, fix a sequence $(x_n)_{n=0}^{\infty}$ in $X(m) - X(m-1)$ with $\lim_{n\to\infty} x_n = x_0$. By 5.4, whenever G is a cluster point of the sequence $(F(x_n))_{n=1}^{\infty}$ in 2^X then $|G| \leqslant m$ and G is a center for x_0. Since $F(x_0)$ is contained in any center for x_0 and $|F(x_0)| = m$, by the assumption on x_0, it follows that $F(x_0) = G$ and $(F(x_n))_{n=1}^{\infty}$ converges to $F(x_0)$. ■

We now come to the main result in this section.

5.6. THEOREM. *If $\hat{X}$ contains an arc, then X contains an arc.*

Proof. If $\hat{X}$ contains an arc then, by the Baire category theorem, either $X(1) = X$ contains an arc or, for some $m \geqslant 2$, $X(m) - X(m-1)$ contains an arc (Corollary 5.5). In the latter case, since an arc is infinite, it follows from Corollary 5.5 that the space

$$H_m(X) = \{A \in 2^X\colon |A| = m\}$$

contains an arc. Since $H_m(X)$ is locally homeomorphic to X^m, this implies that X contains an arc. ∎

5.7. Remark. Let $X \subset Q$ be a free pseudo-arc. Then $\hat{X}$ is a boundary set for $\tilde{X} \approx Q$ containing no arcs. It might be interesting to point out that $\hat{X}$ is countable dimensional, i.e. a union of countable many zero-dimensional subsets. It is also easy to construct a boundary set containing no arcs which is strongly infinite dimensional. Let $X \subset Q$ be a free strongly infinite dimensional continuum containing no arcs. Then $\hat{X} \subset \tilde{X}$ is as required. We do not have an example of a boundary set $B \subset Q$ so that either $\dim A = 0$ or $\dim A = \infty$ for all $A \subset B$. If there is a continuum X with the property that for any $n \in N$ and $A \subset X^n$ either $\dim A = 0$ or $\dim A = \infty$ then it is possible to construct a "hereditary infinite dimensional" boundary set. It is unknown whether such a continuum exists. Notice however that there is a continuum with no n-dimensional ($n \geqslant 1$) subsets, [11].

Let M be a Q-manifold. Using the fact that $M \times [0, 1)$ embeds in Q as an open subset, it is easy to show that M contains a σ-compact σ-Z-set B such that B contains no arcs and $M-B$ is an l_2-manifold.

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Zero-dimensional countable dense unions of Z-sets in the Hilbert cube

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Abstract. We show that every σ-compact, nowhere locally compact, zero-dimensional metric space can be imbedded in the Hilbert cube as a countable dense union of Z-sets, and that there are exactly three such spaces for which all such imbeddings are topologically equivalent.

§ 0. Introduction. It is well know that the Hilbert cube I^∞ is countable dense homogeneous: for any two countable dense subsets D and E, there exists a homeomorphism $h\colon I^\infty \to I^\infty$ with $h(D) = E$. Thus, all dense imbeddings of the space Q of rationals into I^∞ are topologically equivalent. It seems natural to ask which other σ-compact, 0-dimensional metric spaces share this property. It is easily shown that such a space X admits a dense imbedding into I^∞ if and only if it is nowhere locally compact. Furthermore, to obtain positive results in the general case when X is uncountable, we consider only imbeddings as countable unions of Z-sets (see § 1). Thus, the question we ask is: which σ-compact, nowhere locally compact, 0-dimensional metric spaces X have the property that all imbeddings of X into the Hilbert cube as countable dense unions of Z-sets are topologically equivalent? In this note we show that there are exactly three such spaces: the space of rationals, the product of the rationals and the Cantor set, and the space which is the union of a copy of the rationals and a nowhere dense Cantor set.

Actually, the question of equivalence of imbeddings $f_1\colon X \to I^\infty$ and $f_2\colon X \to I^\infty$ of a 0-dimensional space X reduces to the question of whether the complements $I^\infty \backslash f_1(X)$ and $I^\infty \backslash f_2(X)$ are homeomorphic (see § 4). This rather curious result is of course strictly limited to the 0-dimensional case (compare for instance with Chapman's complement theorem for Z-sets in I^∞ [3], or with the fact that the complements of both capsets and fd-capsets in I^∞ are homeomorphic to l^2 [1]).

§ 1. Preliminaires. All spaces considered are separable metric. We shall frequently use the following classical characterizations for certain 0-dimensional spaces (for techniques of proof and references, see [6]):

1.1. Lemma. *$X \approx Q$, the space of rationals, if and only if X is countable and has no isolated points.*