Characterization of a certain subset of the Cantor set

by

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Abstract. The Cantor set C contains a subset S which is topologically characterized by the following properties: (1) S is a zero-dimensional separable metric space which is the union of a (topologically) complete subspace and a σ-compact subspace, (2) S is nowhere σ-compact and, (3) if $U \subset S$ is open and nonempty and if $D \subset S$ is countable, then $U-D$ is not complete. This implies that if aX and bX are zero-dimensional compactifications of $X = P \times Q$, i.e. the product of the irrationals and the rationals, then $(aX-X)\times C \approx (bX-X)\times C$.

0. Introduction. All topological spaces under discussion are separable metric. It is well-known that there are good usable topological characterizations of the space of rational numbers Q, the space of irrational numbers P, the Cantor set C, and the product $Q \times C$ (see, respectively, [11], [1], [5] and [1]). Recently, the author has obtained a characterization of $Q \times P$ [9]. This characterization is the following: $Q \times P$ is the unique zero-dimensional space which is the union of countably many closed and (topologically) complete subspaces and which in addition is nowhere complete and nowhere σ-compact.

In this paper we study complements of dense copies of $Q \times P$ in C. It turns out that there are only very few of these spaces. We show that if $a(Q \times P)$ and $b(Q \times P)$ are zero-dimensional compactifications of $Q \times P$ then the spaces $\big(a(Q\times P)-(Q\times P)\big)\times C$ and $\big(b(Q\times P)-(Q\times P)\big)\times C$ are homeomorphic. This is a consequence of the fact that there is only one zero-dimensional space S with the following properties:

(1) *S is the union of a complete subspace and a σ-compact subspace;*
(2) *S is nowhere σ-compact;*
(3) *if $U \subset S$ is open and nonempty and if $D \subset S$ is countable, then $U-D$ is not complete.*

From this characterization it follows that any nonempty clopen subspace of S is homeomorphic to S. This implies that S is homogeneous, in fact, any homeomorphism between closed and nowhere dense subsets of S can be extended to an autohomeomorphism of S, [9].

1. Extending small homeomorphisms. It is an interesting result due to Ryll-Nardzewski, see [8], that any homeomorphism between nowhere dense closed subsets of C extends to a homeomorphism of C. The aim of this section is to prove an estimated version of this result.

1.1. LEMMA. *Let $A, B \subset C$ be closed and nowhere dense and let $\varepsilon > 0$. Then there is a homeomorphism $\varphi\colon C \to C$ such that $\varphi(B) \cap A = \emptyset$ and $d(\varphi, \mathrm{id}) < \varepsilon$.*

Proof. Let $C_1, C_2, \ldots, C_n \subset C$ be nonempty and clopen such that

$$(1) \qquad C_i \cap C_j = \emptyset \quad \text{if} \quad i \neq j;$$

$$(2) \qquad \operatorname{diam}(C_i) < \varepsilon;$$

$$(3) \qquad A \cup B \subset \bigcup_{i=1}^{n} C_i.$$

For each $i \leqslant n$ let $E_i \subset C_i$ be clopen and nonempty such that $E_i \cap (A \cup B) = \emptyset$ and let $\varphi_i\colon E_i \to C_i - E_i$ be any homeomorphism. Define $\varphi\colon C \to C$ by

$$\varphi(x) = \begin{cases} x & \text{if} \quad x \notin \bigcup_{i=1}^{n} C_i, \\ \varphi_i(x) & \text{if} \quad x \in E_i, \\ \varphi_i^{-1}(x) & \text{if} \quad x \in C_i - E_i. \end{cases}$$

It is clear that φ is as required. ■

We now come to the main result in this section

1.2. THEOREM. *Let $A, B \subset C$ be closed and nowhere dense and let $\varphi\colon A \to B$ be a homeomorphism such that $d(\varphi, \mathrm{id}) < \varepsilon$. Then φ can be extended to a homeomorphism $\bar{\varphi}\colon C \to C$ such that $d(\bar{\varphi}, \mathrm{id}) < \varepsilon$.*

Proof. Let $\delta = \frac{1}{4}(\varepsilon - d(\varphi, \mathrm{id}))$. By Lemma 1.1 there is a homeomorphism $\xi\colon C \to C$ such that $A \cap \xi(B) = \emptyset$ and $d(\xi, \mathrm{id}) < \delta$. Define $\eta\colon A \to \xi(B)$ by

$$\eta(x) = \xi(\varphi(x))$$

and by Ryll-Nardzewski's result previously cited we may extend η to a homeomorphism $\bar{\eta}\colon C \to C$. Let V be a clopen neighborhood of A such that

$$(1) \qquad V \cap \bar{\eta}(V) = \emptyset;$$

$$(2) \qquad \text{if} \quad x \in V, \quad \text{then} \quad d(x, \bar{\eta}(x)) < 3\delta + d(\varphi, \mathrm{id}).$$

It is clear that such neighborhood exists since $A \cap \xi(B) = \emptyset$ and $d(\bar{\eta}(x), x) < d(\varphi(x), x) + \delta$ for all $x \in A$. Define $\varrho\colon C \to C$ by

$$\varrho(x) = \begin{cases} x & \text{if} \quad x \notin V \cup \bar{\eta}(V), \\ \bar{\eta}(x) & \text{if} \quad x \in V, \\ \bar{\eta}^{-1}(x) & \text{if} \quad x \in \bar{\eta}(V). \end{cases}$$

Notice that $\varrho|A = \eta$ and that $d(\varrho, \mathrm{id}) < 3\delta + d(\varphi, \mathrm{id})$. Now define $\bar{\varphi}\colon C \to C$ by $\bar{\varphi} = \xi^{-1} \circ \varrho$. Clearly, $\bar{\varphi}|A = \varphi$ and

$$d(\bar{\varphi}, \mathrm{id}) < 4\delta + d(\varphi, \mathrm{id}) = \varepsilon. \ ■$$

1.3. Remark. Notice that the estimated homeomorphism extension Theorem 1.2 is true for any metric d on C. There is also an estimated homeomorphism extension theorem for Z-sets in the Hilbert cube, but this theorem is valid only with respect to one of the standard convex metrics on the Hilbert cube, [4].

1.4. Remark. Theorem 1.2 is not stated in full generality. Using Theorem 3.1 of [9] it can be shown that Theorem 1.2 is true with C replaced by any strongly homogeneous zero-dimensional space (a space is called *strongly homogeneous* if all nonempty clopen subspaces are homeomorphic).

1.5. Remark. The reason that we are interested in an estimated homeomorphism extension theorem is that in several constructions in this paper, we obtain a homeomorphism with certain properties as a limit of inductively constructed homeomorphisms. We can only ensure convergence to a homeomorphism if at each stage the next homeomorphism can be chosen arbitrarily close to the identity. The following result, due to Anderson [2], explains this.

1.6. THEOREM (Inductive Convergence Criterion). *Let X be a compact space and let $\{h_i\}_{i=1}^{\infty}$ be a sequence of homeomorphisms of X. Then $\lim_{i\to\infty} h_i \circ \ldots \circ h_1$ exists and is a homeomorphism provided that for any i:*

$$(1) \qquad d(h_{i+1}, \mathrm{id}) < 2^{-i},$$

and

$$(2) \qquad d(h_{i+1}, \mathrm{id}) < 3^{-i} \inf\{d(h_i \circ \ldots \circ h_1(x), h_i \circ \ldots \circ h_1(y))\colon d(x, y) \geqslant 1/i\}.$$

2. A capsed for C. Let $K \subset C$ be a countable union of nowhere dense closed subsets of C. We say that K is a *capset* (abbreviation for set with the *compact absorption property*) for C if $K = \bigcup_{i=1}^{\infty} K_i$, for some tower $K_1 \subset K_2 \subset \ldots$ of compacta with the following property:

> *for each $\varepsilon > 0$, each integer m, and each nowhere dense closed $F \subset C$, there exists a homeomorphism $h\colon C \to C$ with $d(h, \mathrm{id}) < \varepsilon$, $h|K_m = \mathrm{id}$, and $h(F) \subset K_n$, for some $n \geqslant m$.*

The idea of a capset for a given topological class of compacta is due independently to Anderson [3] and Bessaga & Pełczyński [6]. The following lemma follows directly from [6, 2.1].

2.1. LEMMA. *Let K and L be capsets for C and let $\varepsilon > 0$ be given. Then there is a homeomorphism φ: $C \to C$ such that $\varphi(K) = L$ and $d(\varphi, \mathrm{id}) < \varepsilon$.*

It is not entirely clear that capsets in C exist. However, by using precisely the same technique as in Curtis & van Mill [7, Theorem 2.1] and by using Theorem 1.2 instead of the estimated homeomorphism extension theorem in the Hilbert cube, the following result can be shown:

2.2. LEMMA. *Let $K \subset C$ be a dense copy of $Q \times C$. Then K is a capset for C.*

Consequently, Lemmas 2.1 and 2.2 imply the following

2.3. THEOREM. *Let $K, L \subset C$ be dense copies of $Q \times C$ and let $\varepsilon > 0$ be given. Then there is a homeomorphism φ: $C \to C$ such that $\varphi(K) = L$ and $d(\varphi, \mathrm{id}) < \varepsilon$.*

3. Closed subsets homeomorphic to $Q \times C$. In [1], Alexandroff and Urysohn showed that all zero-dimensional, σ-compact, nowhere locally compact and nowhere countable spaces are homeomorphic to $Q \times C$ (see also [9]). In the remaining part of this paper we will make use of this characterization.

Let X be a space. Define

$$\varrho X = \{x \in X\colon x \text{ has a countable neighborhood}\}$$

and

$$\sigma X = \{x \in X\colon x \text{ has a compact open neighborhood}\}$$

respectively. Observe that both ϱX and σX are open subsets of X and that ϱX is always countable.

3.1. THEOREM. *Let X be a σ-compact zero-dimensional space. If no closed subset of X is homeomorphic to $Q \times C$ then X can be written as $A \cup B$ where A is countable and B is topologically complete.*

Proof. By transfinite induction, for each ordinal $\alpha < \omega_1$ define subsets $S_\alpha, T_\alpha \subset X$ in the following way:

$$S_0 = \varrho X \quad \text{and} \quad T_0 = \sigma(X - S_0),$$

$$S_\alpha = \varrho\big(X - \bigcup_{\beta<\alpha} (S_\beta \cup T_\beta)\big) \quad \text{and} \quad T_\alpha = \sigma\big(X - (S_\alpha \cup \bigcup_{\beta<\alpha} (S_\beta \cup T_\beta))\big).$$

By induction it is easy to show that $\bigcup_{\beta<\alpha} (S_\beta \cup T_\beta)$ is open in X for each $\alpha < \omega_1$. Let $\beta < \omega_1$ be the first ordinal for which $S_\beta = \emptyset$ and $T_\beta = \emptyset$ (it is clear that such $\beta < \omega_1$ exists since X is separable metric).

Put $Y = X - \bigcup_{\alpha<\beta} (S_\alpha \cup T_\alpha)$. We claim that $Y = \emptyset$. For suppose that this is not true. Since $S_\beta = T_\beta = \emptyset$ and since Y is closed in X, the space Y is σ-compact, nowhere locally compact and nowhere countable. Hence $Y \approx Q \times C$, which contradicts our assumptions on X.

Put $A = \bigcup_{\alpha<\beta} S_\alpha$ and $B = \bigcup_{\alpha<\beta} T_\alpha$. We claim that A and B are as required. It is clear that A is countable, so it suffices to show that B is (topologically) complete. By induction we will show that $\bigcup_{\alpha\leqslant\varkappa} T_\alpha$ is complete for all $\varkappa \leqslant \beta$. Since T_0 is locally compact, T_0 is complete. Assume that $\bigcup_{\alpha\leqslant\mu} T_\alpha$ is complete for all $\mu < \varkappa \leqslant \beta$. Since $\bigcup_{\alpha\leqslant\mu} T_\alpha$ is open in $\bigcup_{\alpha\leqslant\varkappa} T_\alpha$ for all $\mu < \varkappa$ and since a locally complete space is complete,

$$V = \bigcup_{\alpha<\varkappa} T_\alpha$$

is an open complete subspace of $\bigcup_{\alpha\leqslant\varkappa} T_\alpha$. Let Z be a (metric) compactification of $\bigcup_{\alpha\leqslant\varkappa} T_\alpha$. Then $Z - V$ is σ-compact, since V is complete. Since $T_\varkappa$ is closed in $\bigcup_{\alpha\leqslant\varkappa} T_\alpha$ and since $T_\varkappa$ is locally compact, this implies that

$$Z - \bigcup_{\alpha\leqslant\varkappa} T_\alpha = \big((Z - \bigcup_{\alpha<\varkappa} T_\alpha) - \bar{T}_\varkappa\big) \cup (\bar{T}_\varkappa - T_\varkappa)$$

is σ-compact or, equivalently, $\bigcup_{\alpha\leqslant\varkappa} T_\alpha$ is complete. ∎

4. A description of S. In this section we will show that spaces with the properties of S exist.

4.1. LEMMA. *Let X be a zero-dimensional compactification of $Q \times P$. Then $X - (Q \times P)$ is nowhere complete, nowhere σ-compact, and is the union of a complete and a σ-compact space.*

Proof. For each $q \in Q$ let $A_q = (\{q\} \times P)^-$ and $B_q = A_q - (Q \times P)$. Then $E = X - \bigcup_{q\in Q} A_q$ is complete and $F = \bigcup_{q\in Q} B_q$ is σ-compact since each B_q is σ-compact. Clearly $X \setminus (Q \times P) = E \cup F$.

If $U \subset X$ is clopen and $U \cap \big(X - (Q \times P)\big)$ is σ-compact, then $U \cap (Q \times P)$ is complete. This implies that $X - (Q \times P)$ is nowhere σ-compact, since $Q \times P$ is complete. Similarly $X - (Q \times P)$ is nowhere complete, since $Q \times P$ is nowhere σ-compact. ∎

Before we give a space with the properties of S, observe that P is the unique zero-dimensional, complete and nowhere locally compact space, [1].

4.2. LEMMA. *There is a space S with the following properties:*

(1) *S is zero-dimensional;*
(2) *S is the union of a complete and a σ-compact subspace;*
(3) *S is nowhere σ-compact;*
(4) *if $U \subset S$ is open and nonempty and if $D \subset S$ is countable, then $U - D$ is not complete.*

Proof. Let $K \subset C$ be the union of countable many nowhere dense Cantor sets which is dense. By [1], $K \approx Q \times C$ and $C - K \approx P$. In addition, let $E \subset C$ be countable and dense. Then $E \approx Q$, [11], and $C - E \approx P$. Put

$$S = (C \times C) - \big((C - K) \times E\big).$$

Since $(C-K)\times E \approx Q\times P$, by Lemma 4.1, S satisfies (1), (2) and (3). Now let $D\subset S$ be countable and let $U\subset S$ be clopen and nonempty. Since $K\times E$ is dense in S, there is an $e\in E$ such that $U\cap(K\times\{e\})\neq\emptyset$. Observe that $U\cap(K\times\{e\})\approx Q\times C$ since $Q\times C$ is strongly homogeneous. Since $Q\times C$ clearly contains uncountably many pairwise disjoint closed copies of Q, there is an uncountable family $\mathscr{K}$ of pairwise disjoint closed copies of Q in $U\cap(K\times\{e\})$. Since $\mathscr{K}$ is uncountable, some $G\in\mathscr{K}$ misses D. Then G is a closed (in S) copy of Q in $U\cap(K\times\{e\})$ (notice that $U\cap(K\times\{e\})$ is closed in S). Consequently, $U-D$ is not complete. ∎

Let $\mathscr{S}$ denote the class of spaces satisfying (1), (2), (3) and (4) of Lemma 4.2.

4.3. Lemma. *If $S\in\mathscr{S}$ and if $U\subset S$ is nonempty and clopen, then $U\in\mathscr{S}$.*

Proof. Obvious. ∎

Since we will show later that, up to homeomorphism, $\mathscr{S}$ contains precisely one space, which we call S, Lemma 4.3 implies that S is strongly homogeneous. As a consequence, S is homogeneous. In fact, any homeomorphism between closed and nowhere dense subsets of S extends to a homeomorphism of S, [9, 3.1].

4.4. Lemma. *Let $S\in\mathscr{S}$ and let aS be a zero-dimensional compactification of S. Then $aS-S\approx Q\times P$.*

Proof. By using the same technique as in the proof of Lemma 4.1 it easily follows that $aS-S$ is nowhere complete and nowhere σ-compact. Since $S\in\mathscr{S}$, $S = E\cup F$, where E is complete and F is σ-compact. Let $aS-E=\bigcup_{i=1}^{\infty}K_i$, where each K_i is compact. For each $i\in N$ the space K_i-F is clearly complete and closed in $aS-S$. Since $\bigcup_{i=1}^{\infty}(K_i-F) = aS-S$ we conclude that $aS-S$ is the union of countably many closed and complete subspaces. Since $Q\times P$ is the unique nowhere complete, nowhere σ-compact space which is the union of countably many closed and complete subspaces, [9], this implies that $aS-S\approx Q\times P$. ∎

We conclude this section with the following result which will be of crucial importance in the proof of our characterization of S.

4.5. Theorem. *Let $S\in\mathscr{S}$. Then S contains a closed copy of $Q\times C$.*

Proof. Let aS be a zero-dimensional compactification of S. By Lemma 4.4, $aS-S\approx Q\times P$ hence $aS-S=\bigcup_{i=1}^{\infty}P_i$, where each P_i, is complete and closed in $aS-S$. For each $i\in N$ put $B_i=\bar{P}_i-P_i$. Then B_i is a closed and σ-compact subspace of S. Suppose that no B_i contains a closed copy of $Q\times C$. We will derive a contradition. Then, by Theorem 3.1, we can find disjoint $A_i, F_i\subset B_i$ with $A_i\cup F_i = B_i$ such that A_i is countable and F_i is complete ($i\in N$). For each $i\in N$, we have that $\bar{P}_i-F_i$ is σ-compact and consequently $\bigcup_{i\in N}(\bar{P}_i-F_i)$ is σ-compact. Therefore

$$aS-\bigcup_{i\in N}(\bar{P}_i-F_i) = S-\bigcup_{i\in N}A_i$$

is complete, which is a contradiction since $\bigcup_{i\in N}A_i$ is countable, and $S\in\mathscr{S}$. ∎

5. A characterization of S. We will show that, topologically, $\mathscr{S}$ contains at most one space. Since any $S\in\mathscr{S}$ has a compactification homeomorphic to C, it suffices to show that whenever $S, T\in\mathscr{S}$ are dense in C, then there is a homeomorphism $\varphi\colon C\to C$ such that $\varphi(S)=T$. If $A\subset C$ then $\bar{A}$ denotes the closure of A in C.

5.1. Lemma. *Let $S\subset C$ be dense such that $S\in\mathscr{S}$. If $T\subset C-S$ is closed (in $C-S$), nowhere locally compact and complete, then there exists a closed and complete $F\subset C-S$ such that*

(1) $T\subset F$ *and* T *is nowhere dense in* F;

(2) $\bar{F}-F\approx Q\times C$.

Proof. Let $E=\bar{T}-T$. In addition, let $\mathscr{D}=\{D_n\colon n\in N\}$ be a family of nonempty clopen subsets of S such that

(3) $D_n\subset S-E$ for all $n\in N$;

(4) $\operatorname{diam} D_n<1/h$ for all $n\in N$;

(5) if $n\neq m$ then $D_n\cap D_m=\emptyset$;

(6) $\mathrm{cl}_S(\bigcup\mathscr{D}) = \bigcup\mathscr{D}\cup E$.

It is clear that such a family exists since E is a nowhere dense closed subset of S. Since, by Lemma 4.3 and Theorem 4.5, each nonempty clopen subspace of S contains a closed (in S) copy of $Q\times C$, for each $n\in N$ we can find a closed copy K_n of $Q\times C$ in D_n. Clearly

$$\mathrm{cl}_S\Big(\bigcup_{n=1}^{\infty}K_n\Big) = \bigcup_{n=1}^{\infty}K_n\cup E$$

and since E is σ-compact. $A=\bigcup_{n=1}^{\infty}K_n\cup E\approx Q\times C$. Notice that E is closed and nowhere dense in A. Put $F=\bar{A}\cap(C-S)$. Since A is nowhere locally compact, F is dense in $\bar{A}$, i.e. $\bar{F}-F=A\approx Q\times C$. Since E is nowhere locally compact, T is dense in $\bar{E}$, consequently $T\subset F$. Since E is nowhere dense in A, it easily follows that T is nowhere dense in F. ∎

5.2. Corollary. *Let $S\subset C$ be dense such that $S\in\mathscr{S}$. Then there is a sequence P_i ($i\in N$) of closed and complete subspaces of $C-S$ such that for all $i\in N$*

(1) $P_i\subset P_{i+1}$ *and* $\bigcup_{i=1}^{\infty}P_i=C-S$;

(2) P_i *is nowhere dense in* P_{i+1};
(3) $\bar{P}_i - P_i \approx Q\times C$.

Proof. By Lemma 4.4, $C-S\approx Q\times P$. Consequently, there is a family $\{E_i\colon i\in N\}$ of closed copies of P in $C-S$ such that $\bigcup_{i\in N} E_i = C-S$. From Lemma 5.1 it is clear that inductively one can construct closed copies P_i ($i\in N$) of P in $C-S$ such that for each $i\in N$

$$\bigcup_{j=1}^{i} P_j \cup \bigcup_{j=1}^{i} E_j$$

is a closed nowhere dense subset of P_{i+1} while moreover $\bar{P}_i - P_i\approx Q\times C$. ■

We now come to the main result in this paper.

5.3. THEOREM. *Let* $S, T\subset C$ *be dense such that* $S, T\in\mathscr{S}$. *Then for each* $\varepsilon>0$ *there is a homeomorphism* $\varphi\colon C\to C$ *such that* $\varphi(S)=T$ *and* $d(\varphi, \mathrm{id})<\varepsilon$.

Proof. Inductively we will construct a sequence of homeomorphisms $h_i\colon C\to C$ and a sequence of homeomorphisms $g_i\colon C\to C$ such that

$$\varphi = \lim_{n\to\infty} g_n^{-1}\circ h_n\circ\ldots\circ g_1^{-1}\circ h_1$$

is as required. Without further mentioning, it is understood that at each stage the next homeomorphism is constructed in accordance with the Inductive Convergence Criterion 1.6 while moreover

$$\sum_{i=1}^{\infty} d(h_i, \mathrm{id}) + \sum_{i=1}^{\infty} d(g_i, \mathrm{id}) < \varepsilon.$$

By Corollary 5.2 there are sequences P_i ($i\in N$) and R_i ($i\in N$) of closed and complete subspaces of, respectively, $C-S$ and $C-T$ such that for all $i\in N$

(1) $P_i\subset P_{i+1}$, $R_i\subset R_{i+1}$, $\bigcup_{i=1}^{\infty} P_i = C-S$ *and* $\bigcup_{i=1}^{\infty} R_i = C-T$;

(2) P_i is nowhere dense in P_{i+1} and R_i is nowhere dense in R_{i+1};

(3) $\bar{P}_i - P_i\approx \bar{R}_i - R_i\approx Q\times C$.

Suppose that we have to construct h_1 in such a way that $d(h_1, \mathrm{id})<\delta_1$ for certain $\delta_1>0$. Find an integer n_1 such that $d(x, \bar{R}_{n_1})<\frac{1}{3}\delta_1$ for all $x\in\bar{P}_1$. It is clear that there is a finite disjoint clopen cover $E_1, \ldots, E_k$ of $\bar{P}_1$ such that each E_j has diameter at most $\frac{1}{3}\delta_1$. For each $1\leqslant j\leqslant k$ pick a point $x_j\in E_j$ and a point $y_j\in\bar{R}_{n_1}-\{y_1, \ldots, y_{j-1}\}$ such that $d(x_j, y_j)<\frac{1}{3}\delta_1$. In addition, for each $1\leqslant j\leqslant k$ pick a clopen (in $\bar{R}_{n_1}$) set $F_j\subset\bar{R}_{n_1}$ of diameter at most $\frac{1}{3}\delta_1$ such that $y_j\in F_j$ and such that the family $\{F_1, \ldots, F_k\}$ is pairwise disjoint. Fix $1\leqslant j\leqslant k$. Notice that $E_j\approx F_j\approx C$ and that

$$E_j - P_j\approx F_j - R_j\approx Q\times C,$$

consequently, by Theorem 2.3, we can find a homeomorphism $\xi_j\colon E_j\to F_j$ such that $\xi_j(E_j\cap P_j) = F_j\cap R_j$. Define $\psi\colon \bar{P}_1\to\bar{R}_{n_1}$ by $\psi(x)=\xi_j(x)$ if $x\in E_j$. Notice that ψ is an embedding such that $d(\psi, \mathrm{id})<\frac{1}{3}\delta_1+\frac{1}{3}\delta_1+\frac{1}{3}\delta_1=\delta_1$. In addition, $\psi(P_1) = R_{n_1}\cap\psi(\bar{P}_1)$. By Theorem 1.2 we can extend ψ to a homeomorphism $h_1\colon C\to C$ with $d(h, \mathrm{id})<\delta_1$. Notice that h_1 has the following properties

(4) $$d(h, \mathrm{id})<\delta_1;$$

(5) $$h_1(P_1) = R_{n_1}\cap h_1(\bar{P}_1).$$

This defines h_1.

Suppose that we have to construct g_1 in such a way that $d(g_1, \mathrm{id})<\mu_1$. Find an integer $m_1>1$ such that $d(x, h_1(\bar{P}_{m_1}))<\frac{1}{4}\mu_1$ for all $x\in\bar{R}_{n_1}$. We wish to construct g_1 in such a way that $g_1(\bar{R}_{n_1})\subset h_1(\bar{P}_{m_1})$ and $g_1(R_{n_1})\subset h_1(P_{m_1})$ while moreover $g_1|h_1(\bar{P}_1) = g_1^{-1}|h_1(\bar{P}_1)=\mathrm{id}$. Then, since $h_1(P_1)\subset R_{n_1}$, also $g_1^{-1}h_1(P_1)\subset R_{n_1}$. To this end, let $\mathscr{D}$ be an open cover of $\bar{R}_{n_1}-h_1(\bar{P}_1)$ by disjoint, compact, nonempty subsets such that for each $D\in\mathscr{D}$

$$\operatorname{diam} D<\min\{d(D, h_1(\bar{P}_1)),\ \tfrac{1}{4}\mu_1\}.$$

For each $D\in\mathscr{D}$ pick a point $x(D)\in h_1(\bar{P}_{m_1})-h_1(\bar{P}_1)$ such that $d(x(D), D)<\min\{2d(D, h_1(\bar{P}_1)), \frac{1}{2}\mu_1\}$, and with $x(D)\neq x(D')$ if $D\neq D'$. This is possible since $h_1(\bar{P}_1)$ is nowhere dense in $h_1(\bar{P}_{m_1})$. In addition, for each D, choose a clopen Cantor set (clopen in $h_1(\bar{P}_{m_1})$) $C(D)\subset h_1(\bar{P}_{m_1})-h_1(\bar{P}_1)$ containing $x(D)$ such that

$$\operatorname{diam}(C(D))<\min\{d(x(D), h_1(\bar{P}_1)), \tfrac{1}{4}\mu_1\},$$

and such that $C(D)\cap C(D')=\emptyset$ if $D\neq D'$. Now, for each $D\in\mathscr{D}$ choose an arbitrary homeomorphism $f_D\colon D\to C(D)$. By Theorem 2.3 we may assume that $f_D(D\cap R_{n_1}) = C(D)\cap h_1(P_{m_1})$. Define an embedding $f\colon \bar{R}_{n_1}\to h_1(\bar{P}_{m_1})$ by

$$f(x) = \begin{cases} f_D(x) & \text{if } x\in D\in\mathscr{D},\\ x & \text{if } x\in h_1(\bar{P}_1).\end{cases}$$

Clearly f is an embedding and $d(f, \mathrm{id})<\frac{1}{4}\mu_1+\frac{1}{2}\mu_1+\frac{1}{4}\mu_1=\mu_1$. By Theorem 1.2 we can extend f to a homeomorphism $g_1\colon C\to C$ such that $d(g_1, \mathrm{id})<\frac{1}{4}\mu_1$. Notice that g_1 has the following properties

(6) $$g_1|h_1(\bar{P}_1)=\mathrm{id};$$

(7) $$g_1(R_{n_1})\subset h_1(P_{m_1}).$$

Continuing in this way inductively, it is easy to construct a sequence of homeomorphisms $\{h_n\colon C\to C\}_n$ and a sequence of homeomorphisms $\{g_n\colon C\to C\}_n$ and sequences of integers $m_0<m_1<\ldots$ and $n_1<n_2<\ldots$, with

$m_0 = 1$, such that for all $i \in N$

(8) $$g_i^{-1} \circ h_i(P_{m_{i-1}}) \subset R_{n_i};$$

(9) $$g_i^{-1} \circ h_i(P_{m_i}) \supset R_{n_i};$$

(10) if $k \geq i+1$ then $g_k^{-1} \circ h_k | (g_i^{-1} \circ h_i)(P_{m_i}) = \mathrm{id}$.

Therefore, if we put

$$\varphi = \lim_{n \to \infty} g_n^{-1} \circ h_n \circ \ldots \circ g_1^{-1} \circ h_1$$

then, by (8), (9) and (10), $\varphi(C-S) = C-T$. ∎

5.4. Remark. In the proof of the above theorem we have used ideas from [7, 2.1] and [6, 2.1].

5.5. Corollary. *Let X be zero-dimensional and both complete and σ-compact. Then $X \times S \approx S$.*

Proof. Clearly, $X \times S$ satisfies the properties of S which characterize S. ∎

5.6. Corollary. *Let $a(Q \times P)$ and $b(Q \times P)$ be zero-dimensional compactifications of $Q \times P$ and let X be any nowhere countable zero-dimensional space which is both complete and σ-compact. Then*

$$\big(a(Q \times P)-(Q \times P)\big) \times X \approx \big(b(Q \times P)-(Q \times P)\big) \times X.$$

Proof. Let $U_1 \subset a(Q \times P)-(Q \times P)$ and $U_2 \subset X$ be nonempty and open and let $D \subset U_1 \times U_2$ be countable. Since U_2 is uncountable, we can find a point $e \in U_2$ such that $(U_1 \times \{e\}) \cap D = \emptyset$. Let $E \subset Q \times P$ be a closed homeomorphism of P such that $\bar{E}-E \subset U_1$. Then $\bar{E}-E$ is closed in U_1 and is not complete. Consequently

$$(\bar{E}-E) \times \{e\} \subset U_1 \times U_2$$

is a closed subset of $U_1 \times U_2$ which is not complete and which obviously misses D. This easily implies that

$$\big(a(Q \times P)-(Q \times P)\big) \times X \approx S.$$

Similarly $\big(b(Q \times P)-(Q \times P)\big) \times X \approx S$. Consequently,

$$\big(a(Q \times P)-(Q \times P)\big) \times X \approx \big(b((Q \times P)-(Q \times P))\big) \times X. \ ∎$$

6. Perfect images of S. A closed map $f\colon X \to Y$ is called *perfect* if f is onto and if preimages of compact subsets of Y are compact. In addition, a map $f\colon X \to Y$ is called *irreducible* if f is onto and if $f(A) \neq Y$ for any proper closed $A \subset X$. In this section we show that if X is any nowhere complete, nowhere σ-compact space which is the union of a complete and a σ-compact subspace, then there is a perfect map $f\colon S \to X$.

The following lemma is well-known.

6.1. Lemma. *Let X be a space. Then there is a zero-dimensional space Y and a perfect irreducible map $\pi\colon Y \to X$.*

Proof. Let γX be a compactification of X and let $f\colon C \to \gamma X$ be irreducible. It is easily seen that $Y = f^{-1}(X)$ and $\pi = f|Y$ is as required. ∎

We now come to the main result in this section.

6.2. Theorem. *Let X be nowhere complete, nowhere σ-compact space which is the union of a σ-compact and a complete subspace. Then there is a perfect map $f\colon S \to X$.*

Proof. Let Y be zero-dimensional and let $\pi\colon Y \to X$ be perfect and irreducible, Lemma 6.1. Using the fact that π is both perfect and irreducible it easily follows that Y is also nowhere complete, nowhere σ-compact, and the union of a complete and a σ-compact subspace. Hence by Theorem 5.3, $Y \times C \approx S$. The rest is routine. ∎

6.3. Remark. By a more complicated construction, using ideas from van Mill & Woods [10], it can be shown that there even is a perfect irreducible map $\pi\colon S \to X$.

References

[1] P. Alexandroff and P. Urysohn, *Über nulldimensionale Punktmengen*, Math. Ann. 98 (1928), pp. 89–106.
[2] R. D. Anderson, *On topological infinite deficiency*, Michigan Math. J. 14 (1967), pp. 365–383.
[3] — *On sigma-compact subsets of infinite-dimensional spaces*, unpublished manuscript.
[4] — D. W. Curtis and J. van Mill, *A fake topological Hilbert space*, Trans. Amer. Math. Soc. 272 (1982), pp. 311–321.
[5] L. E. J. Brouwer, *On the structure of perfect sets of points*, Proc. Acad. Amsterdam 12 (1910), pp. 785–794.
[6] C. Bessaga and A. Pełczyński, *Selected Topics in Infinite-dimensional Topology*, Polish Scientific Publishers, Warszawa 1975.
[7] D. W. Curtis and J. van Mill, *Zero-dimensional countable dense unions of Z-sets in the Hilbert cube*, Fund. Math. 118 (1983), pp. 103–108.
[8] B. Knaster and M. Reichbach, *Notion d'homogénéité et prolongements des homéomorphies*, Fund. Math. 40 (1953), pp. 180–193.
[9] J. van Mill, *Characterization of some zero-dimensional separable metric spaces*, Trans. Amer. Math. Soc. 264 (1981), pp. 205–215.
[10] — and R. G. Woods, *Perfect images of zero-dimensional separable metric spaces*, Canad. Math. Bull. 25 (1982), pp. 41–47.
[11] W. Sierpiński, *Sur une propriété topologique des ensembles denombrables denses en soi*, Fund. Math. 1 (1920), pp. 11–16.

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Accepté par la Rédaction le 2.2.1981