Superstable graphs

by

Heinrich Herre (Berlin), **Alan H. Mekler** (London, Ontario)
and **Kenneth W. Smith** (Toronto)

Abstract. We define a natural class of graphs which includes the planar graphs and all graphs of finite bounded valency. Every member of this class has a superstable theory.

Introduction. One sometimes knows a graph has a relatively "simple" structure; because the graph does not contain certain "complicated" graphs. We shall show how this structural "simplicity" is connected to a sort of logical "simplicity". In [7] Podewski and Ziegler call a graph Γ superflat if for every natural number m there is a natural number n such that no subdivision – by fewer than m many points on each edge – of the complete graph on n vertices is contained in Γ. Note that all trees, all graphs of finite bounded valency, all n-separated graphs embeddable in a surface of finite genus are superflat. All these graphs (but not all superflat graphs) are among those we shall call ultraflat.

We show all ultraflat graphs are superstable. This result is due independently to Herre [2] and Mekler and Smith [5] [6]. The proof presented here is from [6]. In [2], the main idea is the same. However, systems of partial isomorphisms replace automorphisms of saturated models. Similar methods to those of [2] appear in [1].

In [9], it is shown (inter alia) that every planar lattice of bounded height is superstable. Since a partial order with a finite bound on the length of chains can be viewed as a directed graph, our results include the stability results from [9]. The motivation for [6] was to understand the relation between [7] and [9]. (It should be noted that directed graphs are dealt with in [7].)

We conclude this paper with an example of a non-directed super flat graph which is not superstable.

Preliminaires. Our notation is standard (cf. [8] for example). Even our periodic failure to distinguish between a structure and its universe is common. If A is a structure and $A \supseteq X$ a *type of A over X* is a maximal consistent set of formulae in the free variable v including $\mathrm{Th}(A_X)$. Let $S_A(X)$ denote the set of all types of A over X. A countable first order theory T is *$\varkappa$-stable*, if whenever

$A \models T$ and $|A| \leqslant \varkappa$ then $|S_A(A)| \leqslant \varkappa$. Here $|X|$ denotes the cardinality of X and $\varkappa$ is a cardinal. If T is $\varkappa$-stable for some infinite cardinal $\varkappa$, then T is *stable*. If T is $\varkappa$-stable for all $\varkappa \geqslant 2^{\omega}$, then T is *superstable*. A structure A is said to be ($\varkappa$-) (*super*)*stable*, if $\mathrm{Th}(A)$ is ($\varkappa$-) (super)stable.

A graph is a structure $\Gamma = \langle G, R\rangle$ where R is a binary relation on G. A graph $\Gamma = \langle G, R\rangle$ is non-directed if R is symmetric. Given a graph $\Gamma = \langle G, R\rangle$, let Γ^* denote the graph $\langle G, R^*\rangle$ where $(g, g') \in R^*$ if either $(g, g') \in R$ or $(g', g) \in R$. Note: if Γ is non-directed, then $\Gamma^* = \Gamma$. A sequence $\langle g_0, \ldots, g_n\rangle$ is a path of length n from g_0 to g_n, if $(g_i, g_{i+1}) \in R$ for all $i < n$. If $\Gamma = \langle G, R\rangle$ and $\Delta = \langle D, P\rangle$, then Δ is a *subgraph* of Γ when $D \subseteq G$ and $P \subseteq R$. A graph Γ *omits* Δ if Γ^* has no subgraph isomorphic to Δ.

Stability of graphs. For natural numbers m and n, let $\mathscr{K}_n^m$ be the class of graphs which can be constructed from the complete graph on n vertices by inserting no more than m vertices on each edge. For example the graph in figure 1 is in $\mathscr{K}_3^2$.

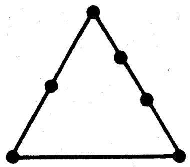

Fig. 1

A graph Γ omits $\mathscr{K}_n^m$, if Γ^* omits each element of $\mathscr{K}_n^m$. Following [7] we call a graph Γ *superflat*, if for each m there is an n such that Γ omits $\mathscr{K}_n^m$. Call Γ *ultraflat*, if there is an n such that for all m Γ omits $\mathscr{K}_n^m$.

THEOREM 1. *If Γ is ultraflat, then Γ is superstable.*

Proof. Suppose Γ is infinite and Γ omits $\mathscr{K}_n^m$ for all m. Choose $\Gamma_1 \succ \Gamma$ such that Γ_1 is saturated and $|\Gamma_1| > |\Gamma|$. (The assumption that Γ_1 is saturated can be eliminated. The argument can be carried out using $|\Gamma|^+$-saturation. We could also force to get a large saturation cardinal; carry out the proof in this new universe; and then use absoluteness facts to complete the proof.) We will show by a series of lemmas that $|S(\Gamma)| \leqslant |\Gamma| + 2^{\omega}$.

LEMMA 1. *If $b \in \Gamma_1 - \Gamma$ then $|\{g \in \Gamma\colon$ b is connected to g by a path through $\Gamma_1^* - \Gamma^*\}| \leqslant n-1$.*

Proof. Suppose $b \in \Gamma_1 - \Gamma$ is connected to $g_0, \ldots, g_{n-1} \in \Gamma$ by paths through $\Gamma_1^* - \Gamma^*$. Choose $b_0, \ldots, b_m \in \Gamma_1 - \Gamma$ such that $b = b_0$ and the paths can be formed from among these vertices. A straightforward argument about the

consistency of types gives $b_0^j, \ldots, b_m^j \in \Gamma_1$ for $j \leqslant \binom{n}{2}$ such that: $\langle b_0^j, \ldots, b_m^j\rangle$ satisfy the same type over Γ as $\langle b_0, \ldots, b_m\rangle$ and $\{b_0^j, \ldots, b_m^j\} \cap \{b_0^i, \ldots, b_m^i\} = \emptyset$, if $i \neq j$. So Γ_1^* and hence Γ does not omit $\mathscr{K}_n^m$. ∎

For $g_0, \ldots, g_k \in \Gamma$, let $\Gamma(g_0, \ldots, g_k) = \{g \in \Gamma_1\colon$ any path in Γ_1^* from g to an element of Γ contains some g_i where $i \leqslant k\}$. Choose $X(g_0, \ldots, g_k) \subseteq \Gamma(g_0, \ldots, g_k)$ such that each type in $S_\Gamma(\{g_0, \ldots, g_k\})$ which is realized by an element of $\Gamma(g_0, \ldots, g_k)$ is realized by a unique element of $X(g_0, \ldots, g_k)$.

Note that $|X(g_0, \ldots, g_k)| \leqslant 2^{\omega}$. Let X be the union of these sets. Since $|X| \leqslant |\Gamma| + 2^{\omega}$, the following lemma completes the proof.

LEMMA 2. *Every type in $S_\Gamma(\Gamma)$ is realized by an element of X.*

Proof. Since Γ_1 is saturated, it suffices to find for each $g \in \Gamma_1$ an automorphism φ of Γ_1 which fixes Γ such that $\varphi(g) \in X$. Consider $g \in \Gamma_1$. If $g \in \Gamma$, there is nothing to prove. Assume $g \in \Gamma_1 - \Gamma$. Choose $g_0, \ldots, g_{n-2}$ such that $g \in \Gamma(g_0, \ldots, g_{n-2})$. Let g' be the element of $X(g_0, \ldots, g_{n-2})$ which realizes the same type over $\{g_0, \ldots, g_{n-2}\}$ as g.

Choose φ an automorphism of Γ such that $\varphi(g) = g'$ and $\varphi(g_i) = g_i$ for $i < n-1$. For any $a \in \Gamma_1$ let

$$\langle a\rangle = \{x\colon x \text{ is connected to } a \text{ in } \Gamma_1^* - \{g_0, \ldots, g_{n-2}\}\} \cup \{g_0, \ldots, g_{n-2}\}.$$

So $\varphi \restriction \langle g\rangle$ maps onto $\langle g'\rangle$. (Recall $\varphi \restriction \langle g\rangle$ denotes the restriction of φ to $\langle g\rangle$.) If $\langle g\rangle = \langle g'\rangle$, let $\varphi_1 = \varphi \restriction \langle g\rangle$. If $\langle g\rangle \cap \langle g'\rangle = \{g_0, \ldots, g_{n-2}\}$, let $\varphi_1 = \varphi \restriction \langle g\rangle \cup (\varphi \restriction \langle g\rangle)^{-1}$. These are the only cases. Define φ_2 on Γ_1 by

$$\varphi_2(x) = \begin{cases} \varphi_1(x), & \text{if } x \in \mathrm{dom}(\varphi_1),\\ x & \text{otherwise.} \end{cases}$$

It is easy to verify φ_2 is the required automorphism. ∎

If P is a partially ordered set (poset) with a finite bound on the length of chains, we can view P as a directed graph.

COROLLARY 2. *If P is a poset with a finite bound on the length of chains and P is ultraflat, then P is superstable.* ∎

Our method also yields a proof of the result in [7].

THEOREM 3. *If Γ is a superflat, then Γ is stable.*

Proof. The proof is similar to that of Theorem 1. In place of Lemma 1, we use the following result.

LEMMA 3. *Suppose Γ is superflat, $\Gamma < \Gamma_1$, $|\Gamma| < |\Gamma_1|$ and Γ_1 is saturated. If $b \in \Gamma_1 - \Gamma$, then b is connected by paths through $\Gamma_1^* - \Gamma^*$ to at most ω elements of Γ.* ∎

COROLLARY 4. *If P is a poset with a finite bound on the length of chains and P is superflat, then P is stable.* ■

It is easy to construct ultraflat graphs which are not ω-stable. It is also easy to construct superflat graphs which are not ultraflat. It is perhaps less obvious that there is a superflat graph which is not superstable.

EXAMPLE. There is a non-directed superflat graph which is not superstable.

Proof. Let $T = \bigcup_{m \leqslant \omega} {}^m\omega$. (Here ${}^m\omega$ denotes the functions from m to ω.) For each $s \in {}^\omega\omega$ and $m < \omega$ place an edge between s and $s \upharpoonright m$, then add m new vertices to this edge. Call this graph Γ.

We will call a vertex a *branching* vertex, if it is adjacent to at least 3 other vertices (i.e., has valence at least 3). For $s \in T$, $ht(s)$ denotes the domain of s. Suppose $s_0, \ldots, s_n$ are the branching vertices of a path in Γ. If s_i is such that $ht(s_i) = k$ is minimal, then $s_0 \upharpoonright k = s_i$.

CLAIM. *For all $m < \omega$, Γ omits $\mathscr{K}^m_{m+3}$.*

Proof (of claim). Suppose $\Gamma \supseteq K$, where $K \in \mathscr{K}^m_{m+3}$. Let the branching vertices of K be $s_0, \ldots, s_{m+2}$. Assume that $ht(s_0)$ is maximal. If $0 < i \leqslant m+2$, there is a path from s_0 to s_i through the non-branching vertices of K. Let $s_i' \in T$ be the element of this path such that $ht(s_i') = k_i$ is minimal. If $i \neq j$ then $s_0 \neq s_i' \neq s_j'$. Since $s_i' = s_0 \upharpoonright k_i$, there is i such that $k_i > m$. This is a contradiction, since the shortest path from s_i' to any branching vertex of Γ has length $\geqslant k_i$.

CLAIM. *Γ is not superstable.*

Proof (of claim). The element s of ${}^\omega\omega$ are those branching vertices from which there are paths of length 1 and 2 through non-branching vertices to a branching vertex. So ${}^\omega\omega$ is a definable subset of Γ. Suppose $s, t \in {}^\omega\omega$. For any $m < \omega$, $s \upharpoonright m = t \upharpoonright m$ if and only if s and t are connected by a path of length m through non-branching vertices to the same branching vertex. Hence in Γ we can define the structure $\langle {}^\omega\omega, E_m \rangle_{m<\omega}$, where $(s, t) \in E_m$ if and only if $s \upharpoonright m = t \upharpoonright m$. This last structure is well known to be stable but not superstable. ■

References

[1] H. Herre, *Modeltheoretische Eigenschaften endlichvalenter Graphen*, Zeitschr. Math. Logic Grundl. Math. 26 (1980), pp. 51–58.

[2] — *Superstable graphs*, preprint.

[3] I. Korec, M. G. Peretiatkin and W. Rautenberg, *Definability in structures of finite valency*, Fund. Math. 81 (1974), pp. 173–181.

[4] — and W. Rautenberg, *Model-interpretability into trees and applications*, Arch. Math. Logic 17 (1976), pp. 97–104.

[5] A. H. Mekler and K. W. Smith, *Superstable graphs*, Notices Amer. Math. Soc. 26 (1979) A-226, # 79T-E15.

[6] — — *Superstable graphs*, preprint.

[7] K. Podewski and M. Ziegler, *Stable graphs*, Fund. Math. 100 (1978), pp. 101–107.

[8] G. Sacks, *Saturated model theory*, Benjamin, New York 1972.

[9] K. W. Smith, *Stability of lattices*, Ph. D. Thesis, University of Toronto 1979.

AKADEMIE DER WISSENSCHAFTEN DER DDR
ZENTRALINSTITUT FÜR MATHEMATIK UND MECHANIK
DDR 108 Berlin, Mohrenstrasse 39

DEPARTMENT OF MATHEMATICS
THE UNIVERSITY OF WESTERN ONTARIO
London, Canada N6A 5B9

current address:
DEPARTMENT OF MATHEMATICS
SIMON FRASER UNIVERSITY
Burnaby, Canada V5A 1S6

SYSTEMS AND COMPUTER SERVICES
IMPERIAL OIL LIMITED
Toronto, Canada M5W 1K3

Accepté par la Rédaction le 16.1.1981