to prove by induction that all other components are surjective as required in 2.1. Thus we have defined a system (Ψ, e) of equations.

b) For every $\boldsymbol{T}$-algebra $A = (X, \{\omega^A\})$, define $A' = FX \xrightarrow{\delta} X$ by

$$\delta(\omega) = \omega^A(1_X) \quad \text{for all } \omega \in FX \ (= \boldsymbol{B}(X, 1)).$$

c) It is routine to verify that the assignement $A \mapsto A'$ defines an isomorphism of categories $\boldsymbol{T}$-alg and (F, Ψ, e)-alg. Notice that the inverse isomorphism sends each (F, Ψ, e)-algebra $A = FX \xrightarrow{\delta} X$ to a $\boldsymbol{T}$-algebra $A = (X, \{\omega^A\})$ defined by

$$\omega^A(\alpha) = \varepsilon_i(\omega\alpha^*) \quad \text{for all } \omega \in \boldsymbol{T}(n, 1),\ \alpha \in X^n,$$

where ε is as in 2.1 h), and i is such that $\omega\alpha^* \in X_i$.

References

[1] J. Adámek, *Free algebras and automata realizations in the language of categories*, Comment. Math. Univ. Caroline 15 (1974), pp. 589–602.

[2] M. Barr, *Coequalizers and free triples*, Math. Z. 116 (1970), pp. 307–322.

[3] W. Felscher, *Birkhoffsche und kategorische Algebra*, Math. Ann. 180 (1969), pp. 1–25.

[4] A. W. Hales, *On the non-existence of free complete Boolean algebras*, Fund. Math. 54 (1964), pp. 45–66.

[5] J. R. Isbell, *General functorial semantics* I, Amer. J. Math. 94 (1972), pp. 535–596.

[6] G. M. Kelly, *Notes on the transfinite induction*, preprint.

[7] V. Kurková-Pohlová and V. Koubek, *When a generalized algebraic category is monadic*, Comment. Math. Univ. Carolinae 15 (1974), pp. 577–602.

[8] F. W. Lawvere, *Functorial semantics of algebraic theories*, Thesis, Columbia University 1963.

[9] F. E. J. Linton, *Some aspects of equational categories*, in: Proceedings of the conference on categorical algebra La Jolla 1965, pp. 84–94, Berlin–Heidelberg–New York 1966.

[10] S. Mac-Lane, *Categories for the working mathematician*, Springer, Berlin–Heidelberg–New York, 1971.

[11] E. G. Manes, *Algebraic Theories*, Berlin–Heidelberg–New York 1976.

[12] J. Reiterman, *Large algebraic theories with small algebras*, Bull. Austral. Math. Soc. 19 (1978), pp. 371–380.

[13] — *One more categorial model of universal algebra*, Math. Z. 161 (1978), pp. 137–146.

[14] J. Słomiński, *The theory of abstract algebras with infinitary operations*, Dissertationes Math. 18 (1959).

[15] V. Trnková, J. Adámek, V. Koubek and J. Reiterman, *Free algebras, input processes and free monads*, Comment. Math. Univ. Carolinae 16 (1975), pp. 339–452.

DEPARTMENT OF MATHEMATICS
FACULTY OF NUCLEAR SCIENCE AND TECHNICAL ENGINEERING
TECHNICAL UNIVERSITY OF PRAGUE
Prague 1, Husova 5
Czechoslovakia

Accépté par la Rédaction le 3. 2. 1981

On approximate n-connectedness

by

Thomas J. Sanders* (Annapolis, MD)

Abstract. The concept of approximate n-connectedness has been given by K. Borsuk in his book *Theory of Shape* as a property of topological spaces to correspond in the theory of shape to the concept of n-connectedness in homotopy theory. In this paper, this concept is characterized using the homotopy bi-groups. Also, a Vietoris–Smale type theorem in compactly generated shape theory is proven, conditions are given under which the shape groups are isomorphic to the usual homotopy groups, and a result on lifting CG-shape maps and some of its applications to the theory of decomposition spaces are given.

1. Introduction. Let $\underline{K}$ be a category. There are associated categories $\mathrm{inv}(\underline{K})$ whose objects are inverse systems $\underline{X} = \{X_\alpha, p_{\alpha\alpha'}, A\}$ in $\underline{K}$ and whose morphisms are morphisms of inverse systems $\underline{f} = (f, f_\beta)\colon \underline{X} \to \underline{Y} = \{Y_\beta, q_{\beta\beta'}, B\}$, and $\mathrm{pro}(\underline{K})$ which is a quotient category $\mathrm{inv}(\underline{K})/\simeq$ (see for example [Mar]). Dually, there are associated categories $\mathrm{dir}(\underline{K})$ whose objects are direct systems $X^* = [X^\alpha, p^{\alpha\alpha'}, A]$ in $\underline{K}$ and whose morphisms are morphisms of direct systems $f^* = (f, f^\alpha)\colon X^* \to Y^* = [Y^\beta, q^{\beta\beta'}, B]$, and $\mathrm{ind}(\underline{K})$ which is a quotient category $\mathrm{dir}(\underline{K})/\simeq$ (see for example [S–4]). If the function on indicees f of a morphism in either $\mathrm{inv}(\underline{K})$ or $\mathrm{dir}(\underline{K})$ is a bijection, that morphism will be called a *special* morphism. If $\underline{L}$ is a category and $F\colon \underline{K} \to \underline{L}$ is a functor, then F induces functors $\mathrm{pro}(F)\colon \mathrm{pro}(\underline{K}) \to \mathrm{pro}(\underline{L})$ and $\mathrm{ind}(F)\colon \mathrm{ind}(\underline{K}) \to \mathrm{ind}(\underline{L})$.

It can be verified (cf. [S–4]) that the following holds.

(1.1) THEOREM. *If f^* is a special* $\mathrm{dir}(\underline{K})$ *morphism, then*

(a) *if each f^α is an isomorphism in $\underline{K}$, then the equivalence class $[f^*]$ is an isomorphism in* $\mathrm{ind}(\underline{K})$,

(b) *if each f^α is an epimorphism in $\underline{K}$, then $[f^*]$ is an epimorphism in* $\mathrm{ind}(\underline{K})$, *and*

(c) *a similar statement holds for monomorphisms.*

(1.2) NOTATION. If X is a metrizable space and $x_0 \in X$, then $\pi_n(X, x_0)$ denotes the usual homotopy groups, $\underline{\pi}_n(X, x_0)$ denotes the shape groups [S–2], $\mathrm{pro}(\pi_n)(X, x_0)$ denotes the homotopy pro-groups (whenever X is

* Supported by a grant from the Naval Academy Research Council.

compact), and ind-pro(π_n) (X, x_0) denote the homotopy bi-groups [S–4]. The dimension of a metrizable space X is the covering dimension (cf. [Mor]). A continuous function $f\colon (X, x_0) \to (Y, y_0)$ is said to be *proper* if for each compact set $B \subset Y$, $f^{-1}(B)$ is a compact subset of X.

2. Approximate n-connectedness. K. Borsuk [Bor] has given the following definition: A pointed space (Y, y_0) is said to be *approximately n-connected* if, for any AR(M)-space N containing Y as a closed subset and for any compactum $B \subset Y$ containing y_0 there exists a compactum $\hat{B} \subset Y (B \subset \hat{B})$ such that for every neighborhood $\hat{V}$ of $\hat{B}$ (in N) there is a neighborhood V_0 of B (in N) with the property that every map of the pointed n-sphere $(S^n, *)$ into (V_0, y_0) is null-homotopic in $(\hat{V}, y_0)$. If (Y, y_0) is approximately k-connected for $0 \leqslant k \leqslant n$, then the notation $(Y, y_0) \in \mathrm{AC}^n$ will be used. The homotopy bi-groups characterize this concept.

(2.1) THEOREM. *A pointed metrizable space (Y, y_0) is approximately n-connected iff the homotopy bi-group ind-pro(π_n) (Y, y_0) is a zero object in the ind-pro-group category.*

Proof. The assertion follows since the homotopy bi-group ind-pro (π_n) (Y, y_0) is a zero object iff for each compactum $B \subset Y$ containing y_0 there is a compactum $\hat{B} \subset Y$, $B \subset \hat{B}$, such that the morphism induced by the inclusion map pro(π_n) $(q^{\hat{B}B})$: pro(π_n) $(B, y_0) \to$ pro(π_n) $(\hat{B}, y_0)$ is the zero morphism in the pro-group category.

(2.2) COROLLARY (cf. Theorem 8.11, p. 145 of [Bor]). *If (Y, y_0) is approximately n-connected, then the shape group $\underline{\pi}_n(Y, y_0)$ is trivial.*

As a consequence of the Whitehead theorem in CG-shape (Theorem 8.2 of [S–4]) one has the following.

(2.3) COROLLARY. *A σ-compact (locally compact) n-dimensional $(n < \infty)$ metrizable space (Y, y_0) has trivial CG-shape iff $(Y, y_0) \in \mathrm{AC}^n$.*

(2.4) NOTE. According to [S–3], one may replace CG-shape with weak shape in (2.3).

3. A Vietoris–Smale theorem. Borsuk notes ([Bor], 8.10 p. 145) that approximate n-connectedness for compacta reduces to the following statement.

(3.1) NOTE. If (Y, y_0) is a pointed compactum lying in an AR(M)-space N, then (Y, y_0) is approximately n-connected iff for every neighborhood V of Y in N there is a neighborhood V_0 of Y in N such that every map $f\colon (S^n, *) \to (V_0, y_0)$ is null homotopic in (V, y_0).

J. Dydak [Dyd] used (3.1) to give a Vietoris–Smale theorem for shape theory. As a consequence of his Theorem 8.5, one has the following.

(3.2) LEMMA. *If $f\colon (X, x_0) \to (Y, y_0)$ is a surjective map of compacta such that $f^{-1}(y) \in \mathrm{AC}^n$ for each $y \in Y$, then the induced morphism of pro-groups*

pro(π_k) (f): pro(π_k) $(X, x_0) \to$ pro(π_k) (Y, y_0) is an isomorphism for $k \leqslant n$ and an epimorphism for $k = n+1$.

An analogue in CG-shape theory is as follows.

(3.3) THEOREM. *If $f\colon (X, x_0) \to (Y, y_0)$ is a proper surjection of metrizable spaces such that $f^{-1}(y) \in \mathrm{AC}^n$ for each $y \in Y$, then the induced morphism of homotopy bi-groups*

$$\text{ind-pro}(\pi_k)\,(f)\colon\ \text{ind-pro}(\pi_k)\,(X, x_0) \to \text{ind-pro}(\pi_k)\,(Y, y_0)$$

is an isomorphism for $k \leqslant n$ and an epimorphism for $k = n+1$.

Proof. Suppose $f\colon (X, x_0) \to (Y, y_0)$ is such a map. Let $\underline{F}$ be the compact cover of (X, x_0) whose elements are of the form $f^{-1}(B)$ where B is a compact subset of Y and $y_0 \in B$. Note that $\underline{F}$ is CS-cofinal [R–S] and that f induces a special ind-pro-morphism

$$f^*\colon [A, \underline{p}^{AA'}, \underline{F}] \to (Y, y_0)^* = [B, \underline{q}^{BB'}, c(B)].$$

The result then follows from Theorem 1.1 and Lemma 3.2.

(3.4) COROLLARY. *Under the hypothesis of Theorem 3.3, the induced homomorphism*

$$\underline{\pi}_k(f)\colon \underline{\pi}_k(X, x_0) \to \underline{\pi}_k(Y, y_0)$$

is an isomorphism of groups for $k \leqslant n$.

4. Local n-connectedness. A consequence of Theorem 8.7 of [Dyd] is as follows.

(4.1) LEMMA. *If Z is a compact LC^n metrizable space and $z_0 \in Z$, then the natural homomorphism*

$$p_k\colon \pi_k(Z, z_0) \to \underline{\pi}_k(Z, z_0)$$

is an isomorphism of groups for $k \leqslant n$.

It then follows that

(4.2) THEOREM. *If (Y, y_0) is a metrizable space having a CS-cofinal cover consisting of LC^n spaces, then the homomorphism [S–2] $p_k\colon \pi_k(Y, y_0) \to \underline{\pi}_k(Y, y_0)$ is an isomorphism of groups for $k \leqslant n$.*

The principal result of this section is as follows.

(4.3) THEOREM. *If (Y, y_0) is a LC^n metrizable space, then the homomorphism*

$$p_k\colon \pi_k(Y, y_0) \to \underline{\pi}_k(Y, y_0)$$

is an isomorphism of groups for $k \leqslant n$.

The proof of this theorem uses the following definitions and lemmas.

(4.4) DEFINITION (cf. [Dyd]). Let $\underline{U}$ and $\underline{V}$ be open covers of a metrizable space Y. A realization (partial or full) $f\colon L \to Y$ of a complex K in Y is said to be *relative* to $\underline{U}$ provided that for each (closed) simplex σ of K there is a $U \in \underline{U}$ such that $f(\sigma \cap L) \subset U$. The open cover $\underline{V}$ is said to be an *n-refinement* of $\underline{U}$ if every partial realization of any (at most) $(n+1)$-dimensional complex K relative to $\underline{V}$ extends to a full realization of K relative to $\underline{U}$.

A consequence of Lemma 8.2 of [Dyd] is that

(4.5) Lemma. *If $Y \in \mathrm{LC}^n$, then each open cover $\underline{U}$ of Y has an open n-refinement $\underline{V}$.*

Suppose Y is a metrizable space and $B \subset Y$ is compact. Select sequences $\underline{U}_k$ and $\underline{V}_k$ of locally finite (normal) open covers of Y such that the mesh of $\underline{U}_k$ is less than $1/k$, $\underline{U}_{k+1}$ star refines $\underline{V}_k$, and $\underline{V}_k$ n-refines $\underline{U}_k$. Let $\underline{U}_k(B) = \{U \cap B | \; U \in \underline{U}$ and $U \cap B \neq \emptyset\}$ be the induced covers of B and let $K_k(B)$ denote the nerve of $\underline{U}_k(B)$, $p^B_{k,k+1}\colon K_{k+1}(B) \to K_k(B)$ canonical projections, and $p^B_k\colon B \to K_k(B)$ canonical maps (cf. [Spa], p. 152). A similar sequence of open covers was used in the proof of the Whitehead theorem for CG-shape. Y. Kodama [Kod] has also used a similar sequence of open covers to obtain a Δ-space having the same shape (both weak and Fox) as a finite dimensional locally compact metric space.

(4.6) Lemma. *Suppse $Y \in \mathrm{LC}^n$ is metrizable, $B \subset Y$ is compact, $f\colon K \to K_{k+1}(B)$ is a map of an (at most) $(n+1)$-dimensional complex, and $g\colon L \to B$ is a map of a subcomplex L of K relative to $\underline{U}_{k+1}(B)$ such that $p^B_{k+1}g = f$. Then there is a map $g'\colon K \to Y$ which is an extension of g relative to $\underline{U}_k$ such that for each point $x \in K$, there is a $U \in \underline{U}_k$ with $(p^B_{k+1})^{-1}f(x) \cup g'(x) \subset \mathrm{st}(U, \underline{U}_k)$.*

Proof. Let $\mathrm{Sd}(K)$ be a subdivision of K and $\tau\colon \mathrm{Sd}(K) \to K_{k+1}(B)$ a simplicial approximation of f (i.e. $f(\mathrm{st}(v)) \subset \mathrm{st}(\tau(v), K_{k+1}(B))$ for v a vertex of $\mathrm{Sd}(K)$). Extend g to $g_1\colon L \cup \mathrm{Sd}(K)^\circ \to X$ by defining $g_1(v)$ as an arbitrary element of $(p^B_{k+1})^{-1}(\tau(v))$ for v a vertex of $\mathrm{Sd}(K) - L$. Since $\underline{U}_{k+1}$ is a star refinement of $\underline{V}_k$, for each simplex $\sigma \in \mathrm{Sd}(K)$, there is a $V \in \underline{V}_k$ such that if v is a vertex of σ, then $g_1(v) \in V$. Thus $g_1\colon L \cup \mathrm{Sd}(K)^\circ \to Y$ is a partial realization of K relative to $\underline{V}_k$. Since $\underline{V}_k$ is an n-refinement of $\underline{U}_k$, g_1 extends to a full realization $g'\colon \mathrm{Sd}(K) \to X$ of K relative to $\underline{U}_k$. Let $x \in K$ and choose $\sigma \in \mathrm{Sd}(K)$ with $x \in \sigma$. If v is a vertex of σ, then

$$(p^B_{k+1})^{-1} f(x) \cup g'(x) \subset \mathrm{st}((p^B_{k+1})^{-1}\tau(v), \underline{U}_k).$$

Note that a homotopy of an n-dimensional complex will have an associated map. Thus we have as a corollary the following lemma.

(4.7) Lemma. *Each map $f\colon K \to K_{k+1}(B)$ of an (at most) $(n+1)$-dimensional complex has an associated map $f'\colon K \to Y$. This relationship is such that if K has dimension $\leqslant n$ and if f and g are homotopic as maps of K into $K_{k+1}(B)$, then the associated maps f' and g' are homotopic as maps of K into Y.*

Proof of 4.3. Let Y be an LC^n metrizable space. By Lemma 4.7, for each compact subset B of Y, $y_0 \in B$, there is a homomorphism

$$\alpha_B\colon \underline{\pi}_k(B, y_0) \to \pi_k(Y, y_0)$$

such that if $B \subset B'$, then $\alpha_{B'}, i_k = \alpha_B$. Here $i_k\colon \underline{\pi}_k(B, y_0) \to \underline{\pi}_k(B', y_0)$ denotes the homomorphism induced by the inclusion map $i\colon (B, y_0) \to (B', y_0)$. The universal mapping property of $\varinjlim$ gives a unique homomorphism $\alpha\colon \underline{\pi}_k(Y, y_0) \to \pi_k(Y, y_0)$ such that $\alpha i_k = \alpha_B$ for all compact subsets B of Y, $y_0 \in B$. Here $i_k\colon \underline{\pi}_k(B, y_0) \to \underline{\pi}_k(Y, y_0)$ denotes the homomorphism induced by the inclusion map $i\colon (B, y_0) \to (Y, y_0)$. It remains only to verify that $\alpha = p_k^{-1}$.

5. Lifting CG-shape maps. The principal result of this section is as follows (cf. [Dyd], Theorem 8.13).

(5.1) Theorem. *Let X and Y be metrizable spaces. If $f\colon (X, x_0) \to (Y, y_0)$ is a proper surjective map such that $f^{-1}(y) \in \mathrm{AC}^n$ for each $y \in Y$, then for each CG-shape map $G\colon (Z, z_0) \to (Y, y_0)$, where Z is a metrizable space having dimension $\leqslant n$, there is a unique CG-shape map $H\colon (Z, z_0) \to (X, x_0)$ with $f^*H = G$. Here $f^*\colon (X, x_0) \to (Y, y_0)$ denotes the CG-shape map induced by f.*

The first of the following corollaries is immediate, the second follows using the Whitehead theorem in CG-shape.

(5.2) Corollary. *Suppose X and Y are metrizable spaces and* $\dim Y \leqslant n$. *If $f\colon (X, x_0) \to (Y, y_0)$ is a proper surjective map such that $f^{-1}(y) \in \mathrm{AC}^n$ for each $y \in Y$, then $f^*\colon (X, x_0) \to (Y, y_0)$ is a CG-shape domination and* $\mathrm{Sh}_{\mathrm{CG}}(X, x_0) \geqslant \mathrm{Sh}_{\mathrm{CG}}(Y, y_0)$.

(5.3) Corollary. *If X and Y are σ-compact (locally compact) metrizable spaces connected and finite dimensional and $f\colon (X, x_0) \to (Y, y_0)$ is a proper surjective map such that $f^{-1}(y) \in \mathrm{AC}^n$ for each $y \in Y$ where* $\dim Y \leqslant n$, *then $f^*\colon (X, x_0) \to (Y, y_0)$ is a CG-shape equivalence and* $\mathrm{Sh}_{\mathrm{CG}}(X, x_0) = \mathrm{Sh}_{\mathrm{CG}}(Y, y_0)$.

(5.4) Note. According to [S–3], one can replace CG-shape with weak shape in (5.3) and also in (5.2) and (5.1) whenever the spaces are locally compact metrizable spaces.

Proof of 5.1. Let $G = [g, \underline{g}^C]\colon (Z, z_0)^* \to (Y, y_0)^*$ be a CS-morphism [R–S] representative of the CG-shape map G. Then for each compact subset C of Z, $z_0 \in C$, $\underline{g}^C\colon (C, z_0) \to (g(C), y_0)$ is a compact shape map. Let $h(C) = f^{-1}(g(C))$ and note that $h(C)$ is a compact subset of X, $f|_{h(C)}\colon h(C) \to g(C)$ is a closed surjective map, and if $y \in g(C)$, then $(f|_{h(C)})^{-1}(y) = f^{-1}(y) \in \mathrm{AC}^n$. By Theorem 8.13 of [Dyd], there is a unique compact shape map $\underline{h}^C\colon (C, z_0) \to (h(C), x_0)$ with $\underline{f}^{h(C)} = \underline{h}^C = \underline{g}^C$. Here $\underline{f}^{h(C)}\colon (h(C), x_0) \to (g(C), y_0)$ denotes the compact shape map induced by $f|_{h(C)}$. An application of part 2 of Lemma 8.12 of [Dyd] verifies that $H = [h, \underline{h}^C]\colon (Z, z_0)^* \to (X, x_0)^*$ is a well defined CS-morphism such that $f^*H = G$. The uniqueness follows from a similar application of Lemma 8.12 of [Dyd].

References

[Bor] K. Borsuk, *Theory of Shape*, Warszawa 1975.
[Dug] J. Dugundji, *Elements of Modern Topology*, McGraw-Hill, New York 1968.
[Dyd] J. Dydak, *The Whitehead and Smale theorems in shape theory*, Dissertationes Mat. 87, Warszawa 1976.
[Kee] J. W. Keesee, *An Introduction to Algebraic Topology*, Brooks/Cole, Belmont, Ca, 1970.
[Kod] Y. Kodama, *Decomposition spaces and shape in the sense of Fox*, Fund. Math. 97 (1977), pp. 199–208.
[Mar] S. Mardešić, *On the Whitehead theorem in shape theory* I, Fund. Math. 91 (1976), pp. 51–64.
[Mor] K. Morita, *On shapes of topological spaces*, Fund. Math. 86 (1975), pp. 251–259.
[Mos] M. Moszyńska, *The Whitehead theorem in the theory of shapes*, Fund. Math. 80 (1973), pp. 221–263.
[R–S] L. Rubin and T. Sanders, *Compactly generated shape*, Gen. Top. and Appl. 4 (1974), pp. 73–83.
[S–1] T. Sanders, *Shape groups and products*, Pacific J. Math. 48 (1973), pp. 485–496.
[S–2] — *Shape groups for Hausdorff spaces*, Glasnik Matematički 8 (28) (1973), pp. 297–304.
[S–3] — *Compactly generated shape theories*, Fund. Math. 93 (1976), pp. 37–40.
[S–4] — *A Whitehead theorem in CG-shape*, Fund. Math. 103 (1981), pp. 131–140.
[Sma] S. Smale, *A Vietoris mapping theorem for homotopy*, AMS Proceedings 8 (1975), pp. 604–610.
[Spa] E. H. Spanier, *Algebraic Topology*, McGraw-Hill, New York 1966.

DEPARTMENT OF MATHEMATICS
U. S. NAVAL ACADEMY
Annapolis, MD 21402

Accépté par la Rédaction le 2. 3. 1981