References

[1] J. Cerf, *Topologie de certains espaces de plongements*, Bull. Soc. Math. de France 89 (1961), pp. 227–380.
[2] A. Douady, *Variétés a bord anguleux et voisinages tubulaires*, Seminaire Cartan 1961/62.
[3] A. Haefliger, *Lissage des Immersions I*, Topology 6 (1967), pp. 221–240.
[4] J. F. P. Hudson, *Piecewise Linear Topology*, Benjamin, 1969.
[5] F. E. A. Johnson, *Triangulation of stratified sets and singular varieties*, Trans. Amer. Math. Soc. 275 (1983), pp. 333–343.
[6] S. Lang, *Differential Manifolds*, Addison–Wesley, 1972.
[7] R. K. Lashof and M. Rothenberg, *Microbundles and smoothing*, Topology 3 (1965), pp. 357–388.
[8] J. R. Munkres, *Elementary Differential Topology*, Ann. of Math. Studies no. 54, Princeton Univ. Press, Princeton, N. J. 1966.
[9] H. Putz, *Triangulation of fibre bundles*, Canad. Math. 19 (1967), pp. 499–513.
[10] J. R. Stallings, *Lectures on Polyhedral Topology*, Tata Institute, Bombay 1968.
[11] J. H. C. Whitehead, *On C^1-complexes*, Ann. of Math. 41 (1940), pp. 809–824.

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Algebraic theories and varieties of functor algebras

by

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Abstract. We prove that the concept of a variety of functor algebras [13] is equivalent to that of a Linton's equational theory [9] satisfying a certain condition called locally small basedness. We show that this condition ensures reasonable properties of algebras.

0. Introduction. We shall compare two categorical approaches to algebras in the category of sets: Linton's equational theories [9] and author's varieties of functor algebras [13] restricted to the case that the base category is the category of sets. Both approaches are more general than triples in sets including also algebraic theories not admiting free algebras, such as that of complete Boolean algebras and that of complete lattices.

Linton's equational theories provide a natural and efficient generalization of Lawvere's theories [8]. The price which equational theories pay for their generality and elegance is that they include also theories which are not of nature. For instance, for the theory generated by a proper class of operations subject to no equations, no non-trivial algebra can be described by a set of data and the number of all algebras exceeds the cardinality of the universum we work in.

On the other hand, dealing with functor algebras does not lead to any non-legitimacy of that kind. Categories of functor algebras have been investigated in a lot of papers (see [11] for references) as a categorical generalization of categories of algebras of a given type. The disadvantage of the approach is that selection of varieties ([13], see section 1) in a category of functor algebras is complicated.

The basic concept of our paper is as follows: an equational theory is locally small based if it is generated by a subcategory which is locally small. The main result states that a concrete category can be represented as a variety of functor algebras iff it can be represented as the category of algebras for a locally small based equational theory, see 3.3 and 3.9. This solves the problem of the relation between the two approaches. Further, this gives a simple characterization of varieties of functor algebras. Finally, this provides a natural restrictive condition on an equational theory to ensure reasonable properties of algebras: If a theory is locally small based then for every set X there is a *set* of basic operations such that each algebra A whose underlying set is X is determined by these basic operations; other operations are obtained by a canonical procedure, see 3.5; homomorphisms from A are characterized by compatibility with basic operations, cf. 3.7; subalgebras of A are characterized by closedness under basic operations, see 3.8; the category of algebras is small fibred (see 3.6). Notice that relations between smallness conditions of that type are investigated in [12].

1. Equational theories. In the first section, we recall elements and examples of Linton's equational theories [9].

1.1. We shall work in the Bernais–Gödel set heory involving sets and proper classes. The class of all sets is denoted by $\mathscr{U}$, the category of sets by Set while Set* is the dual of Set. Sometimes, when dealing with proper classes, we shall suppose that our universum $\mathscr{U}$ is embedded into a universum $\mathscr{U}'$ in such a way that our classes coincide with $\mathscr{U}'$-subsets of $\mathscr{U}$ (thus, in fact, we accept the approach of [10]). However, terms *set* and *class* are meant relatively $\mathscr{U}$; this applies, in particular, to the notion of a *category*: if not stated otherwise, objects of a category, and also maps between any two objects, form a (possibly proper) class.

1.2. An *equational theory* in the sense of Linton [9] is a category $\boldsymbol{T}$ whose objects are sets, equipped with a functor Set* $\to \boldsymbol{T}$ which is identical on objects and preserves products. If each class $\boldsymbol{T}(n, k)$ with $n \neq 0$ has just one element then $\boldsymbol{T}$ is called *degenerate*. If $\boldsymbol{T}$ is non-degenerate then the functor Set* $\to \boldsymbol{T}$ is necessarily faithful and so we may and shall assume that it is an embedding of a subcategory. A theory $\boldsymbol{T}$ is *varietal* if each class $\boldsymbol{T}(n, k)$ is a set.

1.3. If $\boldsymbol{T}$ is an equational theory then a $\boldsymbol{T}$*-algebra* is a product preserving functor A: $\boldsymbol{T} \to$ Set; it can be expressed as a couple $A = (X, \{\omega^A\})$ where X is the *underlying set*, the ω^A's (where ω runs over all $\boldsymbol{T}$-maps) are *operations*, ω^A: $X^n \to X^k$ if $\omega \in \boldsymbol{T}(n, k)$, such that

(i) $(\omega\delta)^A = \omega^A \delta^A$ whenever $\omega\delta$ is defined in $\boldsymbol{T}$,

(ii) $\omega^A(\alpha) = \alpha f$ if $\alpha \in X^n$ and $\omega = f^* \in \mathrm{Set}^*(n, k)$ for some $f \in \mathrm{Set}(k, n)$.

Homomorphisms between $\boldsymbol{T}$-algebras, defined as natural transformations, can be identified with maps between underlying sets which compatible with all operations.

$\boldsymbol{T}$-alg denotes the category of all $\boldsymbol{T}$-algebras and their homomorphisms. Notice that, to form $\boldsymbol{T}$-alg, one needs a higher universum (see 1.1). They are two reasons for it:

1) Each $\boldsymbol{T}$-algebra is a proper class and the set theory does not admit to form collections whose members are proper classes.

2) It can happen that $\boldsymbol{T}$-alg is illegitimate in the sense that it is even not isomorphic with any category within $\mathscr{U}$ because the number of $\boldsymbol{T}$-algebras exceeds the cardinality of $\mathscr{U}$.

1.4. Equational presentation. An equational theory is often given in the form of an equational presentation. Namely, a possibly proper class $\Omega = \bigcup_n \Omega_n$ of *operation symbols* is given where n runs over sets; each Ω_n is the class of symbols of *arity* n. Further, there is given a class E of *equations* between terms formed by transfinite induction from operation symbols and variables.

An (Ω, E)*-algebra* is a system $A = (X, \{\omega^A\})$ where X is the underlying set, ω runs over Ω, each ω^A is an operation on X, n-ary if $\omega \in \Omega_n$, such that all E-equations are satisfied. *Homomorphisms* between (Ω, E)-algebras are maps between underlying sets which are compatible with all operations. Up to the same formal difficulties as in 1.3 we can form the category (Ω, E)-alg of (Ω, E)-algebras and their homomorphisms. The couple (Ω, E) yields an equational theory $\boldsymbol{T}$ such that categories (Ω, E)-alg and $\boldsymbol{T}$-alg can be naturally identified. The theory $\boldsymbol{T}$ is obtained as follows: we accept a higher universum $\mathscr{U}'$, see 1.1. Then Ω and E are sets from the point of view of $\mathscr{U}'$ and we can construct (within $\mathscr{U}'$), for every $\mathscr{U}$-set n, an (Ω, E)-free algebra over n, using the Słomiński construction [14]. Then $\boldsymbol{T}$ is the dual of the category of free (Ω, E)-algebras over $\mathscr{U}$-sets. The category $\boldsymbol{T}$ is isomorphic to a category within $\mathscr{U}$. In fact, $\boldsymbol{T}$ can be constructed within $\mathscr{U}$ using transfinite induction and the axiom of choice for classes: $\boldsymbol{T}(n, 1)$ is obtained from the class of Ω-terms with variables in n by identifying terms whose equality is derivable from E, and $\boldsymbol{T}(n, k) = (\boldsymbol{T}(n, 1))^k$.

One remark more. It is well-known [3] that if the underlying functor (Ω, E)-alg $\to$ Set has a left adjoint F then the dual $\boldsymbol{T}'$ of the full subcategory of (Ω, E)-alg consisting of the $F(n)$'s is a theory such that $\boldsymbol{T}'$-alg $\simeq (\Omega, E)$-alg. But $\boldsymbol{T}'$ can differ from the above $\boldsymbol{T}$. The reason is that $\boldsymbol{T}'$ respects not only the equations which can be derived from E but also those which cannot be derived but hold in every (Ω, E)-algebra. E.g., in the example mentioned in [5, p. 558], $\boldsymbol{T}$ is non-varietal while $\boldsymbol{T}'$ is degenerate. The example is as follows: Ω consists of n-ary symbols ω_n, $n \in$ Card and unary symbols α_{in}, $i \in n$, $n \in$ Card and E consists of equations

$$\omega_n(\alpha_{in}(x);\ i \in n) = x, \qquad \alpha_{in}(\omega_n(x_j;\ j \in n)) = x_i.$$

1.5. Examples.

a) *Classical algebraic theories*, such as the theory of groups, of semigroups ect. are always given in the form of an equational presentation (Ω, E) with Ω a set. The corresponding theory $\boldsymbol{T}$ is then varietal.

b) Each *category of algebras for a triple* over Set can be resented as

T-alg for a suitable varietal theory T and conversely, every category T-alg with T varietal is tripleable [9].

c) *Complete lattices* represent the most familiar example of a category of algebraic nature which does not admit free algebras [4] (and thus is not tripleable) but can be regarded as the category of T-algebras for a suitable (non-varietal) theory T [9]. Analogously for *complete Boolean algebras.*

d) *Categories of functor algebras* [1], [2], [6], [7], [11], [13], [15]. Let K be a category and $F\colon K\to K$ a functor. An F-algebra is a K-map $FX\xrightarrow{\delta}X$. A homomorphism from $A=FX\xrightarrow{\delta}X$ to $B=FY\xrightarrow{\sigma}Y$ is a K-map $f\colon X\to Y$ such that $f\delta=\sigma Ff$. All F-algebras and their homomorphisms form a category denoted by $K(F)$.

PROPOSITION. *For every functor* $F\colon \mathrm{Set}\to\mathrm{Set}$, *the category* $\mathrm{Set}(F)$ *can be identified with the category* (Ω, E)-alg *where* $F=\bigcup_n \Omega_n$, $\Omega_n=\{\omega_x;\ x\in Fn\}$ *and* E *consists of equations*

$$\omega_x(x_{f(i)};\ i\in m)=\omega_{Ff(x)}(x_j;\ j\in n)$$

for all sets m, n, *every mapping* $f\colon m\to n$ *and every* $x\in Fm$.

The proof is straightforward.

e) *Varieties of functor algebras* ([13], cf. the next section) provide another examples of categories of the form T-alg (to be proved bellow).

2. Varieties of functor algebras. These varieties were introduced to show that the functorial calculus is rich enough to cover all examples of algebraic categories from 1.5 a)–c); notice that categories $\mathrm{Set}(F)$ do not suffice; e.g., the category of semigroups is not of the form $\mathrm{Set}(F)$.

We recall the basic definitions concerning varieties of functor algebras [13] restricted to the Set case. The aim of the section is to prove that each of these varieties is equivalent to T-alg for some equational theory T.

2.1. In what follows, F is a fixed functor, $F\colon \mathrm{Set}\to\mathrm{Set}$.

a) An algebraized F-chain (or an approximate F-algebra in terms of [6]) is a chain $\{X_i\}=X_0\to X_1\ldots\to X_i\to\ldots$ (i runs over all ordinals) where X_i are sets, $X_i\to X_j$ are mappings with

$$X_i\to X_i=1,\qquad X_i\to X_j\to X_k=X_i\to X_k\qquad (i<j<k),$$

equipped with a transformation $\{FX_i\}\xrightarrow{\gamma}\{X_{i+1}\}$, that is, with a family $\{FX_i\xrightarrow{\gamma_i}X_{i+1}\}$ of mappings such that $FX_i\xrightarrow{\gamma_i}X_{i+1}\to X_{j+1}=FX_i\to FX_j\xrightarrow{\gamma_j}X_{j+1}$ (here $FX_i\to FX_j=F(X_i\to X_j)$).

b) The *free algebraized* F*-chain* ([1], [13]) $\{\tilde X_i\}$ over a set X is defined by

$$\tilde X_0=X,\qquad \tilde X_i=\tilde X_0\vee \operatorname*{colim}_{j<i} F\tilde X_j;$$

in particular, $\tilde X_{i+1}=X\vee F\tilde X_i$ which gives the required maps $F\tilde X_i\to\tilde X_{i+1}$ ($\vee$ denotes the coproduct, i.e. the disjoint union).

c) A *homomorphism* from an algebraized F-chain $\{X_i\}$ to an algebraized F-chain $\{Y_i\}$ is a family $\{X_i\xrightarrow{h_i}Y_i\}$ of mappings such that all diagrams

$$\begin{array}{ccccccc} X_i & \xrightarrow{h_i} & Y_i & \qquad & FX_i & \xrightarrow{Fh_i} & FY_i\\ \downarrow & & \downarrow & & \downarrow & & \downarrow\\ X_j & \xrightarrow{h_j} & Y_j & & X_{i+1} & \xrightarrow{h_{i+1}} & Y_{i+1}\end{array}$$

commute.

d) The free algebraized chain $\{\tilde X_i\}$ over X has the property that for each algebraized chain $\{Y_i\}$, every mapping $h_0\colon \tilde X_0\to Y_0$ admits a unique extension to a homomorphism $\{h_i\}\colon\{\tilde X_i\}\to\{Y_i\}$.

e) We shall consider assignements Ψ sending each set X to an algebraized chain ΨX and each mapping $f\colon X\to Y$ to a homomorphism $\Psi f\colon \Psi X\to\Psi Y$. The assignements are required to be functorial (preserving identities and composition).

We shall use the following notation: if X is a set then X_i will denote the ith component of ΨX, if $f\colon X\to Y$ is a mapping then $f_i\colon X_i\to Y_i$ is the ith component of Ψf. The mappings $FX_i\to X_{i+1}$ in ΨX will be denoted by Ψ_i.

The *free assignement*, sending each set X to the free algebraized chain $\{\tilde X_i\}$ over X, and each mapping $f\colon X\to Y$ to the unique homomorphism $\{\tilde f_i\}\colon\{\tilde X_i\}\to\{\tilde Y_i\}$ with $f_0=f$, will be denoted by Φ. The mappings $FX_i\to X_{i+1}$ will be denoted by φ_i.

f) Each F-algebra $A=FX\xrightarrow{\delta}X$ can be identified with the algebraized chain $\{X_i\}$ where $X_i=X$, $X_i\to X_j=1$, $FX_i\to X_{i+1}=\delta$ for every i, j ($i<j$). By d), the identity mapping $1\colon X\to X$ yields a unique homomorphism $\{\tilde X_i\to X\}$ from ΦX to A, called the *canonical homomorphism* of A.

g) Let Ψ be an assignement and $e\colon \Phi\to\Psi$ an epitransformation. More in detail, for every set X, a homomorphism $e^X\colon \Phi X\to\Psi X$ is given such that $\Phi X\xrightarrow{\Phi f}\Phi Y\xrightarrow{e^Y}\Psi Y=\Phi X\xrightarrow{e^X}\Psi X\xrightarrow{\Psi f}\Psi Y$ for every mapping $f\colon X\to Y$, and $e^X=\{e_i^X\}$ where each mapping e_i^X is surjective. Then the couple (Ψ, e) is called a *system of equations.*

h) An F-algebra $A=FX\to X$ is said to *satisfy* (Ψ, e) if the canonical homomorphism of A factors as $\Phi X\xrightarrow{e^X}\Psi X\to A$. It follows from d) that A satisfies (Ψ, e) iff there exists a homomorphism $\varepsilon^A\colon \Psi X\to A$, $\varepsilon^A=\{\varepsilon_i^A\}$, such that $\varepsilon_0^A=e_0^{X-1}$.

i) The full subcategory of $\mathrm{Set}(F)$ whose objects are F-algebras satisfying (Ψ, e) will be called the *variety of functor algebras* determined by (Ψ, e) and denoted by (F, Ψ, e)-alg.

2.2. Theorem. *Every variety* (F, Ψ, e)*-alg of functor algebras is equivalent to* (Ω, E)*-alg for some equational presentation* (Ω, E).

Proof. Given (F, Ψ, e), define (Ω, E) as follows.

Ω consists of symbols ω_{nit} where n runs over sets, i runs over ordinals and t runs over n_i, the ith component of Ψn; the symbol ω_{nit} is n-ary.

E consists of all equations of the form (i), (ii), (iii) bellow.

(i) $$\omega_{nit}(x_k;\ k \in n) = \omega_{miu}(x_{f(j)};\ j \in m)$$

for every $f\colon m \to n$, $t \in m_i$, $u = f_i(t) \in n_i$;

(ii) $$\omega_{nit} \sim \omega_{nju}$$

for every n, i, $t \in n_i$, $j > i$ and $u \in n_j$ where u is the image of t under the chain map $n_i \to n_j$ in the chain $\Psi n = \{n_i\}$;

(iii) $$\omega_{n_i,1,b_1}(\omega_{nit}(\alpha);\ t \in n_i) = \omega_{n,i+1,b_2}(\alpha)$$

for every n, i, $t \in n_i$, $b_1 \in (n_i)_1$, $b_2 \in n_{i+1}$ such that

$$b_1 = \psi_0^{n_i} F e_0^{n_i}(b), \qquad b_2 = \psi_i^n(b)$$

for some $b \in Fn_i$ and for every collection $\alpha = (x_k;\ k \in n)$ of variables.

Given an F-algebra $A = FX \xrightarrow{\delta} X$ satisfying (Ψ, e), define an (Ω, E)-algebra $A' = (X, \{\omega_{nit}^{A'}\})$ as follows. Let

$$\{\tilde{X}_i \xrightarrow{e_i^X} X_i \xrightarrow{\varepsilon_i^A} X\}$$

be the factorization of the canonical homomorphism of A, see 2.1 h). Put

$$\omega_{nit}^{A'}(\alpha) = \varepsilon_i^A \alpha_i(t) \qquad (\alpha \in X^n)$$

(remember that α_i is the ith component of $\Psi\alpha$).

It is rather lengthy but quite routine to verify that the assignement $A \mapsto A'$ defines an isomorphism of categories (F, Ψ, e)-alg and (Ω, E)-alg; notice that the inverse isomorphism sends every (Ω, E)-algebra $\bar{A}$ to $A = FX \xrightarrow{\delta} X$ where $\delta(x) = \omega_{X1t}^{\bar{A}}(1_X)$ where $t = \psi_0^X F e_0^X(x)$.

3. Locally small based equational theories. In the third section, we introduce the concept of a locally small based equational theory. We show that all theories from 1.5 (including theories corresponding to varieties of functor algebras) are locally small based. We prove that if the theory is locally small based then the category of algebras is legitimate (in fact, small fibred), and each algebra is determined by a *set* of operations; homomorphisms are selected by compatibility with a *set* of operations; subalgebras are carried by subsets closed under a *set* of operations. The main result is that the category T-alg with T locally small based be represented as a variety of functor algebras. Consequently, the concept of a locally small based equational theory is equivalent to that one of a variety of functor algebras.

3.1. Locally small based equational theories. An equational theory T is said to be *locally small based* if either T is degenerate or there exists a category B with $\mathrm{Set}^* \subset B \subset T$ which is locally small (that is, each class $B(n, k)$ is a set) and which *generates* T in the sense that the only subcategory of T closed under products of maps in T and containing B is all of T. The category B is called a *base* of T.

3.2. Remark. Let a theory T be given by an equational presentation (Ω, E) where $\Omega = \bigcup_n \Omega_n$, n runs over sets and Ω_n is the class of n-ary symbols in Ω. Let us suppose that

(i) each Ω_n is a set,

(ii) for each $\omega \in \Omega_m$ and $f\colon m \to n$ there is $\sigma \in \Omega_n$ such that the equation

$$\omega(x_{f(i)};\ i \in m) = \sigma(x_j;\ j \in n)$$

can be derived from E.

(The condition (ii) says, roughly speaking, that basic operations are closed under the composition with the Set*-ones.) Then T is locally small based. Indeed, recall (cf. 1.4) that T-maps are tuples of terms taken modulo E; so the least subcategory B of T which contains Set* and all terms $\omega(x_i;\ i \in n)$ generates T and, if (i), (ii) hold, it is locally small.

3.3. *All theories corresponding to examples in* 1.5 *are locally small based.*

As for 1.5 a)-d), one could show that these are only special cases of 1.5 e), but direct proofs are simpler:

Classical algebraic theories 1.5 a) and, more generally, theories corresponding to triples 1.5 b) are varietal. So they are locally small and hence locally small based.

Complete lattices lead to a locally small based theory via the equational presentation (Ω, E), $\Omega = \bigcup_n \Omega_n$, $\Omega_n = \{\sigma_{nA};\ A \subset n\} \cup \{\iota_{nA};\ A \subset n\}$ where the σ_{nA}'s correspond to sups and the ι_{nA}'s to infs ($\sigma_{nA}(x_i;\ i \in n) = \sup\{x_i;\ i \in A\}$ and analogously for infs) and E consists of the usual laws. Analogously for complete Boolean algebras.

Concerning categories $\mathrm{Set}(F)$, apply Remark 3.2 and Proposition 1.5 d).

Proposition. *The theory corresponding to a variety* (F, Ψ, e)-alg *of functor algebras* (cf. 2.2) *is locally small based.*

Proof. Let us consider the equational presentation (Ω, E) of 2.2. Put $\Omega'_n = \{\omega_{n1t};\ t \in n_1\}$ for every n and $\Omega' = \bigcup_n \Omega'_n$. Each Ω'_n is a set and Ω' is closed under the composition with Set*-operations in the sense of 3.2 (ii) by 2.2 (i). Thus, to conclude the proof, it suffices to show that Ω'-operations generate all Ω-operations. This is provided by the lemma bellow.

Lemma. *For every* $\omega \in \Omega_n$ *there exists an* Ω'*-term* w *such that the equation* $\omega(x_i;\ i \in n) = w$ *can be derived from* E.

Proof. Let $\omega = \omega_{nit}$. We proceed by induction on i. For $i=0$, $\omega_{n0t} = \omega_{n1u}$ for some u by 2.2 (ii). Let the assertion hold for some $i \geqslant 1$. Let $\omega = \omega_{n,i+1,t}$. Consider the homomorphism $e^n\colon \Phi n \to \Psi n$ to obtain the following commutative diagram

$$\begin{array}{ccc} \tilde{n}_0 & \xrightarrow{\;e_0^n\;} & n_0 \\ \downarrow & & \downarrow \\ \tilde{n}_{i+1} & \xrightarrow{\;e_{i+1}^n\;} & n_{i+1} \\ {\scriptstyle \varphi_i^n}\downarrow & & {\scriptstyle \psi_i^n}\downarrow \\ F\tilde{n}_i & \xrightarrow{\;Fe_i^n\;} & Fn_i \end{array}$$

Recall that $\tilde{n}_{i+1} = \tilde{n} \vee F\tilde{n}_i$, that maps $\tilde{n}_0 \to \tilde{n}_{i+1}$, $F\tilde{n}_i \to \tilde{n}_{i+1}$ are summand embeddings; hence their images cover $\tilde{n}_{i+1}$. Finally, the map e_{i+1}^n is surjective. It follows that t is either in the image of $n_0 \to n_{i+1}$ or in the image of ψ_i^n. In the former, case, $\omega_{nit} = \omega_{n0u}$ for some u, see 2.2 (ii). In the latter case, $t = \psi_i^n(b)$ for some $b \in Fn_i$. Then $t = b_2$ in terms of 2.2 (iii) and we can use 2.2 (iii) and the induction assumption.

Let the assertion hold for all $i<j$ where j is a limit ordinal. Let $\omega = \omega_{njt}$. As the images of maps $\tilde{n}_i \to \tilde{n}_j$ $(i<j)$ cover $\tilde{n}_j$ and the map $e_j^n\colon \tilde{n}_j \to n_j$ is surjective, t is necessarily in the image of some map $n_i \to n_j$ and so $\omega = \omega_{niu}$ for some $u \in n_i$ by 2.2 (ii). Thus we can use the induction assumption. The proof is finished.

3.4. Hierarchy of operations. Let $\boldsymbol{T}$ be a theory and $\boldsymbol{B}$ a locally small category with $\mathrm{Set}^* \subset \boldsymbol{B} \subset \boldsymbol{T}$. For every set X, define sets $X_0 \subset X_1 \subset X_2 \subset \ldots \subset X_i \subset \ldots \subset \boldsymbol{T}(X,1)$ by induction using rules

(i) $X_0 = \mathrm{Set}^*(X,1)$,

(ii) $X_{i+1} = \{\beta\langle\omega;\ \omega\in X_i\rangle;\ \beta \in \boldsymbol{B}(X_i,1)\}$,

(iii) $X_i = \bigcup_{j<i} X_j$ for i limit.

Define a category $\boldsymbol{T}' \subset \boldsymbol{T}$ by

$$\boldsymbol{T}'(X,1) = \bigcup_i X_i, \qquad \boldsymbol{T}'(X,Y) = \{\langle\omega_y;\ y\in Y\rangle;\ \omega_y \in \boldsymbol{T}'(X,1) \text{ for every } y \in Y\}.$$

Lemma. *$\boldsymbol{T}'$ is the smallest subcategory of $\boldsymbol{T}$ that is closed under products of maps in $\boldsymbol{T}$ and contains $\boldsymbol{B}$.*

Corollary. *$\boldsymbol{B}$ is a base of $\boldsymbol{T}$ iff $\boldsymbol{T}(X,1) = \bigcup_i X_i$ for every set X.*

Proof of the lemma is routine.

3.5. Let $A = (X, \{\omega^A\})$ be a $\boldsymbol{T}$-algebra where $\boldsymbol{T}$ is a locally small based theory with a base $\boldsymbol{B}$. Then

(i) each ω^A with $\omega \in \boldsymbol{B}(n,1)$ for some n is determined by the operations β^A with $\beta \in \boldsymbol{B}(X,1)$ by $\omega^A(\alpha) = (\omega\alpha^*)^A(1_X)$,

(ii) each δ^A with $\delta \in \boldsymbol{T}(X,1)$ is obtained by succesive composition (see 3.4 (i), (ii), (iii) and Corollary) from the β^A's with $\beta \in \bigcup_n \boldsymbol{B}(n,1)$,

(iii) each ω^A with ω in $\boldsymbol{T}$ is determined by the operations δ^A with $\delta \in \boldsymbol{T}(X,1)$ (as in (i)).

These observations show that all operations of the algebra A are obtained by a canonical procedure from the operations β^A with $\beta \in \boldsymbol{B}(X,1)$. As a consequence we have the following three propositions.

3.6. Proposition. *If $\boldsymbol{T}$ is a locally small based theory with a base $\boldsymbol{B}$ then every $\boldsymbol{T}$-algebra $A = (X, \{\omega^A\})$ is uniquely determined by a set of data, viz by $\{\beta^A;\ \beta \in \boldsymbol{B}(X,1)\}$.*

Corollary. *If $\boldsymbol{T}$ is locally small based then the category $\boldsymbol{T}$-alg is small fibred (that is, for every set X, all $\boldsymbol{T}$-algebras whose underlying set is X form a set).*

3.7. Proposition. *Let $\boldsymbol{T}$ be a locally small based theory with a base $\boldsymbol{B}$. Let $A = (X, \{\omega^A\})$, $B = (Y, \{\omega^B\})$ be $\boldsymbol{T}$-algebras. Then a mapping $f\colon X \to Y$ is a homomorphism from A to B iff it is compatible with a set of operations, viz by all $\beta \in \boldsymbol{B}(X,1)$.*

3.8. Proposition. *Let $\boldsymbol{T}$ be a locally small based theory with a base $\boldsymbol{B}$. Let $B = (Y, \{\omega^B\})$ be a $\boldsymbol{T}$-algebra and let $X \subset Y$. Then X is a subalgebra if and only if it is closed under all β^B with $\beta \in \boldsymbol{B}(X,1)$.*

3.9. Theorem. *Let $\boldsymbol{T}$ be a locally small based theory. Then there exists a functor F: Set $\to$ Set and a system (Ψ, e) of equations such that the categories $\boldsymbol{T}$-alg and (F, Ψ, e)-alg are equivalent.*

Proof. The case that $\boldsymbol{T}$ is degenerate is trivial. Let $\boldsymbol{T}$ be non-degenerate and let $\boldsymbol{B}$ be a base of $\boldsymbol{T}$.

a) The functor F: Set $\to$ Set is defined by

$$FX = \boldsymbol{B}(X,1), \qquad Ff = \boldsymbol{B}(f,1).$$

The system (Ψ, e) of equations is defined as follows. For every set X, let $\{X_i\}$ be the chain from 3.4 where the connecting maps $X_i \to X_j$ are inclusions. For every i we have a natural mapping $\psi_i^X\colon FX_i \to X_{i+1}$ sending each $\beta \in FX_i = \boldsymbol{B}(X_i,1)$ to $\beta\langle\omega;\ \omega\in X_i\rangle$. Maps ψ_i^X make $\{X_i\}$ an algebraized chain which will be denoted by ΨX. If $f\colon X \to Y$ is a mapping then the mappings $f_i\colon X_i \to Y_i$ defined by

$$f_i(\omega) = \omega f^*$$

form a homomorphism Ψf from ΨX to ΨY. Thus, we have defined an assignement Ψ which is easily seen to be functorial.

For every set X, there is a natural bijection $e_0^X\colon X \to X_0$ where $e_0^X(x) = j_x^*$ where $j_x\colon 1 \to X$ is the mapping with value x. By 2.1 d) there is a unique homomorphism $e^X\colon \Phi X \to \Psi X$ whose 0th component is e_0^X. It is easy to prove by induction that all other components are surjective as required in 2.1. Thus we have defined a system (Ψ, e) of equations.

b) For every $\boldsymbol{T}$-algebra $A = (X, \{\omega^A\})$, define $A' = FX \xrightarrow{\delta} X$ by

$$\delta(\omega) = \omega^A(1_X) \quad \text{for all } \omega \in FX \ (= \boldsymbol{B}(X, 1)).$$

c) It is routine to verify that the assignement $A \mapsto A'$ defines an isomorphism of categories $\boldsymbol{T}$-alg and (F, Ψ, e)-alg. Notice that the inverse isomorphism sends each (F, Ψ, e)-algebra $A = FX \xrightarrow{\delta} X$ to a $\boldsymbol{T}$-algebra $A = (X, \{\omega^A\})$ defined by

$$\omega^A(\alpha) = \varepsilon_i(\omega\alpha^*) \quad \text{for all } \omega \in \boldsymbol{T}(n, 1),\ \alpha \in X^n,$$

where ε is as in 2.1 h), and i is such that $\omega\alpha^* \in X_i$.

References

[1] J. Adámek, *Free algebras and automata realizations in the language of categories*, Comment. Math. Univ. Caroline 15 (1974), pp. 589–602.
[2] M. Barr, *Coequalizers and free triples*, Math. Z. 116 (1970), pp. 307–322.
[3] W. Felscher, *Birkhoffsche und kategorische Algebra*, Math. Ann. 180 (1969), pp. 1–25.
[4] A. W. Hales, *On the non-existence of free complete Boolean algebras*, Fund. Math. 54 (1964), pp. 45–66.
[5] J. R. Isbell, *General functorial semantics* I, Amer. J. Math. 94 (1972), pp. 535–596.
[6] G. M. Kelly, *Notes on the transfinite induction*, preprint.
[7] V. Kurková-Pohlová and V. Koubek, *When a generalized algebraic category is monadic*, Comment. Math. Univ. Carolinae 15 (1974), pp. 577–602.
[8] F. W. Lawvere, *Functorial semantics of algebraic theories*, Thesis, Columbia University 1963.
[9] F. E. J. Linton, *Some aspects of equational categories*, in: Proceedings of the conference on categorical algebra La Jolla 1965, pp. 84–94, Berlin–Heidelberg–New York 1966.
[10] S. Mac-Lane, *Categories for the working mathematician*, Springer, Berlin–Heidelberg–New York, 1971.
[11] E. G. Manes, *Algebraic Theories*, Berlin–Heidelberg–New York 1976.
[12] J. Reiterman, *Large algebraic theories with small algebras*, Bull. Austral. Math. Soc. 19 (1978), pp. 371–380.
[13] — *One more categorial model of universal algebra*, Math. Z. 161 (1978), pp. 137–146.
[14] J. Słomiński, *The theory of abstract algebras with infinitary operations*, Dissertationes Math. 18 (1959).
[15] V. Trnková, J. Adámek, V. Koubek and J. Reiterman, *Free algebras, input processes and free monads*, Comment. Math. Univ. Carolinae 16 (1975), pp. 339–452.

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On approximate n-connectedness

by

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Abstract. The concept of approximate n-connectedness has been given by K. Borsuk in his book *Theory of Shape* as a property of topological spaces to correspond in the theory of shape to the concept of n-connectedness in homotopy theory. In this paper, this concept is characterized using the homotopy bi-groups. Also, a Vietoris–Smale type theorem in compactly generated shape theory is proven, conditions are given under which the shape groups are isomorphic to the usual homotopy groups, and a result on lifting CG-shape maps and some of its applications to the theory of decomposition spaces are given.

1. Introduction. Let $\underline{K}$ be a category. There are associated categories $\operatorname{inv}(\underline{K})$ whose objects are inverse systems $\underline{X} = \{X_\alpha, p_{\alpha\alpha'}, A\}$ in $\underline{K}$ and whose morphisms are morphisms of inverse systems $\underline{f} = (f, f_\beta)\colon \underline{X} \to \underline{Y} = \{Y_\beta, q_{\beta\beta'}, B\}$, and $\operatorname{pro}(\underline{K})$ which is a quotient category $\operatorname{inv}(\underline{K})/\simeq$ (see for example [Mar]). Dually, there are associated categories $\operatorname{dir}(\underline{K})$ whose objects are direct systems $X^* = [X^\alpha, p^{\alpha\alpha'}, A]$ in $\underline{K}$ and whose morphisms are morphisms of direct systems $f^* = (f, f^\alpha)\colon X^* \to Y^* = [Y^\beta, q^{\beta\beta'}, B]$, and $\operatorname{ind}(\underline{K})$ which is a quotient category $\operatorname{dir}(\underline{K})/\simeq$ (see for example [S–4]). If the function on indicees f of a morphism in either $\operatorname{inv}(\underline{K})$ or $\operatorname{dir}(\underline{K})$ is a bijection, that morphism will be called a *special* morphism. If $\underline{L}$ is a category and $F\colon \underline{K} \to \underline{L}$ is a functor, then F induces functors $\operatorname{pro}(F)\colon \operatorname{pro}(\underline{K}) \to \operatorname{pro}(\underline{L})$ and $\operatorname{ind}(F)\colon \operatorname{ind}(\underline{K}) \to \operatorname{ind}(\underline{L})$.

It can be verified (cf. [S–4]) that the following holds.

(1.1) THEOREM. *If f^* is a special* $\operatorname{dir}(\underline{K})$ *morphism, then*

(a) *if each f^α is an isomorphism in $\underline{K}$, then the equivalence class $[f^*]$ is an isomorphism in* $\operatorname{ind}(\underline{K})$,

(b) *if each f^α is an epimorphism in $\underline{K}$, then $[f^*]$ is an epimorphism in* $\operatorname{ind}(\underline{K})$, *and*

(c) *a similar statement holds for monomorphisms.*

(1.2) NOTATION. If X is a metrizable space and $x_0 \in X$, then $\pi_n(X, x_0)$ denotes the usual homotopy groups, $\underline{\pi}_n(X, x_0)$ denotes the shape groups [S–2], $\operatorname{pro}(\pi_n)(X, x_0)$ denotes the homotopy pro-groups (whenever X is

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