According to Theorem 2 there must be at least two unknotted fibers. Since h is invariant on each of these, it follows from our earlier remark that h is equivalent to a standard rotation. This verifies Theorem 4.

THEOREM 5. *If h is a piecewise linear homeomorphism of period $n > 1$ on S^3 with fixed point set a simple closed curve J, then J cannot be a non-trivial torus knot.*

Proof. Suppose J is a non-trivial torus knot with $h(J) = J$. Then there is a regular neighborhood $N(J)$ of J such that $h(N(J)) = N(J)$. Using the argument given in the proof of Theorem 4, we may fiber S^3 such that J is a fiber and every fiber remains invariant under the action of h.

If F is a fiber in $\partial N(J)$, then we may choose a meridian disc D for $N(J)$ such that $D \cap J$ is a point, $h(D) = D$ and the number of points in $F \cap \partial D$ equals the winding number of F on $N(J)$. By our previous observation, this winding number must be greater than zero. Since the only fixed point of h on D is $D \cap J$ and $h(F \cap \partial D) = F \cap \partial D$, the winding number of F on $N(J)$ must be greater than 1. Thus J is an exceptional fiber and, according to Theorem 2, cannot be a non-trivial torus knot. But this contradicts our supposition and proves Theorem 5.

In view of Theorem 5 and the fact that every torus knot is a fibered knot, it seems feasible that a purely geometric proof of the Smith Conjecture for fibered knots can be developed.

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On the triangulation of smooth fibre bundles

by

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Abstract. In this paper we prove that if $p: (E, \partial E) \to B$ is a smooth fibre bundle, where $(E, \partial E)$ is a smooth compact manifold with corners in the boundary E, then p admits a piecewise differentiable (= P.D.) triangulation by a PL bundle, and moreover that any such triangulation of $p: \partial E \to B$ extends to one on the whole of E. This generalises the theorems of Putz, in the case where ∂E is smooth, and Lashof and Rothenberg, in the case of a vector bundle.

The main technical result is that if $\alpha: K \to M$ is a P.D. triangulation of a smooth manifold m-ad M by a PL manifold m-ad K, then the simplicial set $\mathrm{PL}(K) \backslash \mathrm{PD}(K, M)$ is contractible, where $\mathrm{PL}(K)$ is the simplicial group of PL automorphisms of K, and $\mathrm{PD}(K, M)$ is the simplicial set whose n-simplices are P.D. triangulations $\Delta^n \times K \to \Delta^n \times M$ commuting with projection onto Δ^n.

§0. Introduction. In its simplest form, the main theorem proved in this paper is that a smooth fibre bundle with compact fibre is triangulable by a PL bundle, and that, if the fibre is bounded, such a triangulation of the subfibre space determined by the fibre boundary can be extended to one of the whole bundle. In its most general form, we wish, in addition, for the fibre to have corners, and to be given a labelled collection of transversely intersecting submanifolds of codimension zero in the boundary, and a compatible collection of PL bundles triangulating the subfibre spaces corresponding to the labelled faces. The theorem then asserts the existence of a PL bundle triangulating the whole bundle and extending the given triangulations.

Our main theorem, in its greatest degree of generality, is a necessary ingredient of our paper [5], in which we prove that compact stratified sets in the sense of Thom are triangulable by simplicial complexes. A proof of the simplest form of our main theorem has been given by Putz [9]. Were this form sufficient for our application, this present paper would be unnecessary. However, Putz gives no consideration to the case where the fibre boundary has corners and since we definitely require the theorem in this degree of generality, and since it is also not clear how to modify Putz' somewhat adhoc argument to give the result, we are forced to give an independent treatment.

In fact, the published result which is closest to ours, and from which ours derives, is the Lashof–Rothenberg functorial triangulation theorem for vector bundles [7]. Whenever we could, we have followed [7] and wish to thank Lashof and Rothenberg for having written it.

There are really only two published sources for the basic material of this paper, namely, Whitehead's original paper [11], and Munkres' monograph [8], and unfortunately neither deals with triangulations preserving projections onto a simplex. Consequently we expend some effort in setting out those details which are not immediately accessible. Much the hardest part of writing this paper lay in deciding what to include and what to omit. For example, the PD Isotopy Extension Theorem is omitted since it is derivable from the PL Isotopy Extension Theorem, as remarked in [3].

This work originated in the author's 1972 University of Liverpool Thesis. The author would like to thank Professor C. T. C. Wall for some helpful conversations in the early stages of this work. Thanks also to the referee for some constructive comments which have improved the style and content of the paper.

§1. Mappings of simplicial complexes into manifolds. We explain briefly our usage concerning smooth manifolds, simplicial complexes and polyhedra.

With regard to smooth manifolds, we generally follow Cerf [1], with the difference that our local models are slightly more complicated: whereas Cerf's local models are the spaces $\boldsymbol{R}^n \times (\boldsymbol{R}_+)^m$, we allow extra factors of the form $\overline{\boldsymbol{R}^n - (\boldsymbol{R}_+)^n}$. Precisely, by a local model we shall mean a product of the form $C_1 \times \ldots \times C_k$, where each C_i is one of $\boldsymbol{R}^n$, $(\boldsymbol{R}_+)^m$ or $\overline{\boldsymbol{R}^p - (\boldsymbol{R}_+)^p}$. As might be expected, a smooth manifold is then a pair (X, Λ) where X is a paracompact Hausdorff space and Λ is a maximal collection of charts of the form $h\colon U \to V$ with U open in X, V an open subset of a local model, such that the domains of charts in Λ cover X, and such that any two charts are C^∞ compatible in the usual sense, namely that if $h_i\colon U_i \to V_i$ are charts in Λ, $i = 1, 2$, then $h_2 \circ h_1^{-1}\colon h_1(U_1 \cap U_2) \to h_2(U_1 \cap U_2)$ is a C^∞ diffeomorphism. In particular, we take "smooth" to mean "C^∞ smooth", lower classes of smoothness being indicated by "C^k smooth" for appropriate k. If M is a smooth manifold and $x \in M$, then it is easy to see that x has a neighbourhood diffeomorphic to a local model by a diffeomorphism taking x to 0.

We denote by DIFF the category of smooth manifolds and C^∞ smooth maps.

If K is a finite simplicial complex, by its formal tangent space TK we mean the subset $\bigcup_{x \in |K|} \{x\} \times \overline{St(x, K)}$ of $|K| \times |K|$, where $|K|$ is the geometrical realisation of K, and $\overline{St(x, K)}$ is the closed star neighbourhood of x in $|K|$. If M is a smooth manifold and $f\colon |K| \to M$ is a simplex-wise C^1 map, we get an induced continuous tangent map $Tf\colon TK \to TM$. This is a coordinate-free reformulation of the usual treatment [8], [11]. We denote by $C^1(K, M)$ the set of all continuous maps $|K| \to M$ which are C^1 smooth on each simplex of $|K|$. If $f \in C^1(K, M)$ and $\varepsilon > 0$, let $N_\varepsilon(f)$ be the subset of $C^1(K, M)$ consisting of all strong C^1 ε-approximations to f in the sense of Munkres ([8], p. 83). $C^1(K, M)$ will have the topology generated by all such $N_\varepsilon(f)$, and, without further mention, subsets of $C^1(K, M)$ will have the subspace topology. Note that the tangent map $T\colon C^1(K, M) \to C^0(TK, TM)$ is continuous when K is finite and $C^0(TK, TM)$ has the compact-open topology.

Still supposing K is finite, let $T(K, M)$ be the set of C^1-triangulations of M by K. If $T(K, M)$ is non-empty, then K is a combinatorial manifold with a well defined boundary ∂K. We make the convention that, for any subset $F(K, M) \subset C^1(K, M)$, $F^\partial(K, M) = \{f \in F(K, M)\colon f(|\partial K|) \subset \partial M\}$. From Munkres [8], we obtain

Proposition 1.1. *If K is a finite simplicial complex and M a smooth manifold then $T(K, M)$ is open in $C^1_\partial(K, M)$.*

It is ultimately more useful to consider mappings of polyhedra into manifolds. By a polyhedron (X, Ω), we mean a locally compact Hausdorff space X, together with a maximal class Ω of triangulations (K, h), where K is a locally finite simplicial cnmplex and $h\colon |K| \to X$ a homeomorphism, any two elements of Ω admitting an isomorphic simplicial subdivision. Let $X = (X, \Omega)$ be a compact polyhedron and M a smooth manifold. By a piecewise C^1 map $f\colon X \to M$, we mean one for which $f \circ h \in C^1(K, M)$, for some admissible triangulation $(K, h) \in \Omega$. We denote by $C^1(X, M)$ the set of all piecewise C^1 maps from X to M, topologised by the finest topology for which all the maps $h^*\colon C^1(K, M) \to C^1(X, M)$, $f \mapsto f \circ h^{-1}$, are continuous where (K, h) runs through Ω. If M has boundary, we denote by $C^1_\partial(X, M)$ the subset of $C^1(X, M)$ consisting of all maps which take ∂X into ∂M, and topologise it with the subspace topology. Also we write $\mathrm{PD}(X, M)$ for the union of $h^*(T(K, M))$ as (K, h) runs through Ω. $\mathrm{PD}(X, M)$ is the set of piecewise differentiable triangulations of M by X. From (1.1), we get

Proposition 1.2. *Let X be a compact polyhedron and M a smooth manifold. Then $\mathrm{PD}(X, M)$ is open in $C^1_\partial(X, M)$.*

We relativise this by considering mappings which preserve projection onto a simplex. Fix (i) a smooth manifold M (ii) a compact polyhedron X (iii) a subpolyhedron, Y, of $\Delta^m \times X$ and (iv) a piecewise C^1 map $\alpha\colon Y \to \Delta^m \times M$ which commutes with projection onto Δ^m. Define $C^1(X, M, \alpha, m)$ to be the following set with its subspace topology;

$$\{f \in C^1(\Delta^m \times X, \Delta^m \times M)\colon f \text{ extends } \alpha \text{ and commutes with projection onto } \Delta^m\}.$$

We write $C^1_\partial(X, M, \alpha, m)$ for the subset of $C^1(X, M, \alpha, m)$ consisting of those maps which in addition preserve the total boundary over t, for each $t \in \Delta^m$; i.e. which map $\partial(\Delta^m \times X) \cap \{t\} \times X$ into $\partial(\Delta^m \times M) \cap \{t\} \times M$.

THEOREM 1.3. *With the above notation, if* $f \in C^1(X, M, \alpha, m)$, *there exists a fundamental system of neighbourhoods* $(U_\lambda)_{0<\lambda<1}$ *of* f *in* $C^1(X, M, \alpha, m)$ *such that, for each* λ *and each* $g \in U_\lambda$, *there exists a piecewise* C^1 *map* F: $I \times \Delta^m \times X \to I \times \Delta^m \times M$ *commuting with projection onto* $I \times \Delta^m$ *and such that* (i) $F_0 = f$ (ii) $F_1 = g$ (iii) $F_s \in U_\lambda$ *for all* $s \in I$. *Moreover, if* f *and* g *are in* $C^1_\partial(X, M, \alpha, m)$, *then we can ensure that* $F_s \in U_\lambda \cap C^1_\partial(X, M, \alpha, m)$ *for all* $s \in I$.

Proof. Fix a smooth manifold N and a Riemannian metric on N, adapted to the boundary, so that each face of N is totally geodesic. If L is a compact Hausdorff space and if $f: L \to N$ is continuous, we will describe a fundamental system of neighbourhoods of f in $C^0(L, N)$ with the compact open topology. $\widetilde{TN}$ will denote the tangent bundle of N, and TN the subspace of $\widetilde{TN}$ whose fibre over x is a copy of the standard model for the corner at x. In Cerf's treatment [1], TN is fibrewise convex since the standard corner models in [1] are convex. For us, TN is fibrewise starlike from 0. Choose an open neighbourhood D of the zero section in TN which is fibrewise starlike from 0 and such that the exponential map defines a diffeomorphism Exp: $D \to N \times N$ onto an open neighbourhood of the diagonal. Here $\mathrm{Exp}(x) = (\pi(x), \exp(x))$, where π is the tangent bundle projection, and exp is the 'usual' exponential map. If $0 < \mu < 1$, define an open subset $\hat{W}(f, \mu)$ of $C^0(L, N)$ thus;

$$\hat{W}(f, \mu) = \{g \in C^0(L, N) \colon \forall\, x \in L,\ (f(x), g(x)) \in \mathrm{Exp}(D_\mu)\},$$

where, if '$\|v\|$' denotes the norm of a tangent vector v in the Riemannian metric, $D_\mu = \{v \in D \colon \|v\| < \mu\}$. It is easy to see that $(\hat{W}(f, \mu))_{0<\mu<1}$ is a fundamental system of neighbourhoods of f in $C^0(L, N)$.

Next take a smooth manifold M with a complete Riemannian metric adapted to the boundary and let $\Delta^m \times M$ have the product metric of the standard Euclidean metric on Δ^m with the above. Put $N = T(\Delta^m \times M)$ and let N have a complete Riemannian metric adapted to its boundary in which geodesics in fibres over $\Delta^m \times M$ are linear, and which extends that on $\Delta^m \times M$. If K is a finite simplicial complex and if $f: |K| \to \Delta^m \times M$ is C^1 on each simplex of K, and if $0 < \mu < 1$, let $W(f, \mu, K)$ be the open subset $T^{-1}(W(Tf, \mu))$ of $C^1(K, \Delta^m \times M)$, where we are putting $L = TK$, $N = T(\Delta^m \times M)$, and where T: $C^1(K, -) \to C^0(TK, T(-))$ is the tangent map.

Let X be a compact polyhedron and $f \in C^1(X, \Delta^m \times M)$. Let $(K_a, h_a)_{a \in A}$ be the collection of admissible triangulations of X such that $f \circ h_a \in C^1(K_a, \Delta^m \times M)$. Define

$$V(f, \lambda) = \varinjlim_{a \in A} h_a^*(W(f \cdot h_a, K_a, \lambda)) \quad \text{where} \quad h^*(g) = g \circ h^{-1}.$$

It is easy to see that $(V(f, \lambda))_{0<\lambda<1}$ is a fundamental system of neighbourhoods of f in $C^1(X, \Delta^m \times M)$.

Finally, replace X by $\Delta^m \times X$, and define

$$U(f, \lambda) = V(f, \lambda) \cap C^1(X, M, \alpha, m).$$

Let $g \in U(f, \lambda)$. For each $x \in \Delta^m \times X$, there is a unique $v_x \in D \cap (T(\Delta^m \times M))_{f(x)}$ such that $\mathrm{Exp}(v_x) = (f(x), g(x))$. Define F: $I \times \Delta^m \times X \to I \times \Delta^m \times M$ by $F(s, x) = (s, \exp(sv_x))$, for $s \in I$. Each F_s will be C^1 on each simplex of any admissible triangulation of $\Delta^m \times X$ on which both f and g are simplexwise C^1. By choice of metrics, it is easy to see that each F_s commutes with projection onto Δ^m and belongs to $V(f, \lambda)$. Moreover, if f and g are boundary preserving, then so is each F_s since ∂M is totally geodesic. In addition, if $x \in Y$ then $v_x = 0$ since $g(x) = f(x)$, hence each $F_s(x) = f(x)$, and each $F_s \in U(f, \lambda)$. ∎

Note that for $0 < \mu < \lambda < 1$, we have $V(f, \mu) \subset V(f, \lambda)$. Consequently $(U(f, 1/r))_{r \in \mathbb{Z}_+}$ is a countable fundamental system of neighbourhoods of f.

§2. Relativisation by means of space-valued functors. We relativise the theory of §1 by introducing an indexing category $\mathscr{C}$, and replacing M by a functor $\mathscr{M}$: $\mathscr{C} \to$ DIFF, and X by a functor $\mathscr{X}$: $\mathscr{C} \to$ PL, where PL denotes the category of polyhedra and piecewise linear maps. A piecewise C^1 map f: $\mathscr{X} \to \mathscr{M}$ is then a natural transformation of functors for which, at each $x \in \mathrm{Obj}(\mathscr{C})$, $f(x)$: $\mathscr{X}(x) \to \mathscr{M}(x)$ is piecewise C^1. Similarly, f is called a PD *triangulation* if each $f(x)$ is a PD triangulation. $C^1(\mathscr{X}, \mathscr{M})$ (resp. $\mathrm{PD}(\mathscr{X}, \mathscr{M})$) will denote the set of all piecewise C^1 maps (resp. PD triangulations) from $\mathscr{X}$ to $\mathscr{M}$. $C^1(\mathscr{X}, \mathscr{M})$ is the subset of $\prod_{x \in \mathrm{Obj}(\mathscr{C})} C^1(\mathscr{X}(x), \mathscr{M}(x))$ consisting of all $\mathscr{C}$-tuples which commute with the structure maps. We give $C^1(\mathscr{X}, \mathscr{M})$ the subspace topology from the product topology, and give $\mathrm{PD}(\mathscr{X}, \mathscr{M})$ the subspace topology from $C^1(\mathscr{X}, \mathscr{M})$.

In practice, we shall resort to a very limited range of indexing categories, all of them finite. For a positive integer n, 2^n will denote the category whose objects are the subsets of $\{1, \ldots, n\}$, with at most one morphism between two objects, and $\mathrm{Hom}(x, y) \neq \emptyset$ iff $x \supset y$. K_n will denote the full subcategory of 2^{n+1} obtained by omitting the empty set. We take 2^0 to be the category of 2^{n+1} obtained by omitting the empty set. We take 2^0 to be the trivial category with one object. The nerves of 2^n, K_n are respectively the n-cube and the barycentric subdivision of the n-simplex.

DEFINITION. By a *smooth manifold* n-ad we mean a functor $\mathscr{M}$: $2^n \to$ DIFF such that

(i) if $x \supset y$ and $\mathrm{card}(x) = \mathrm{card}(y) + 1$, then $\mathscr{M}(x)$ is a compact smooth submanifold of codimension 1 in $\mathscr{M}(y)$, with $\mathscr{M}(x)$ actually contained in the boundary of $\mathscr{M}(y)$, and such that the structural map $\mathscr{M}(x) \to \mathscr{M}(y)$ is an imbedding, and

(ii) for all x, y, $\mathcal{M}(x \cup y) = \mathcal{M}(x) \cap \mathcal{M}(y)$, the intersection being transverse.

The restriction of an n-ad to K_{n-1} is called its *formal boundary*. In general, by a *formal boundary of type* $(n-1)$ we mean a functor $\delta\mathcal{M}\colon K_{n-1} \to \mathrm{DIFF}$ such that, for some smooth manifold n-ad $\mathcal{M}$, $\mathcal{M}_{|K_{n-1}} = \delta\mathcal{M}$. Questions about formal boundaries are reducible to questions about n-ads thus; let 2_i^n be the full subcategory of K_n consisting of subsets containing i. 2_i^n is isomorphic to 2^n by $x \mapsto x \setminus \{i\}$. Restriction of a formal boundary of type n to 2_i^n, gives, with this re-indexing, a smooth n-ad. Hence a formal boundary of type n can be regarded as a union of $(n+1)$ n-ads, corresponding to the values $1 \leqslant i \leqslant n+1$.

One can mimic the above to give definitions of PL n-ads and PL formal boundaries. An n-ad in either sense has a well defined total topological boundary, which contains the formal boundary, in general properly. A formal boundary consists of a number of n-ads, as described above, each of which has its own boundary. Still writing subscript '∂' for sets of boundary-preserving mappings, we can deduce the following from (1.2), using continuity of restriction maps.

Proposition 2.1. *Let $\mathcal{M}$ be either a smooth manifold n-ad or else a smooth manifold formal boundary, and let $\mathcal{X}$ be either a PL n-ad or a PL formal boundary, over the same diagram scheme as $\mathcal{M}$. Then $\mathrm{PD}(\mathcal{X}, \mathcal{M})$ is open in $C_\partial^1(\mathcal{X}, \mathcal{M})$.*

Finally, we combine the above sort of relativisation with relativisation by means of taking products with simplices as in §1. If $\mathcal{N}\colon \mathcal{C} \to \mathrm{DIFF}$ is a functor and N is a smooth manifold, $N \times \mathcal{N}$ will denote the functor $x \mapsto N \times \mathcal{N}(x)$. Likewise, if $\mathcal{X}$ is a functor $\mathcal{X}\colon \mathcal{C} \to \mathrm{PL}$ and X is a polyhedron, $X \times \mathcal{X}$ will denote the functor $x \mapsto X \times \mathcal{X}(x)$. Let $\mathcal{M}$ be either a smooth n-ad or else a smooth formal boundary. Let $\mathcal{X}$ be either a PL n-ad or a PL formal boundary, over the same diagram scheme as $\mathcal{M}$. Let $\mathcal{Y}$ be a subfunctor of $\Delta^m \times \mathcal{X}$ and let $\alpha\colon \mathcal{Y} \to \Delta^m \times \mathcal{M}$ be a piecewise C^1 natural transformation commuting with projection onto Δ^m. Define $C^1(\mathcal{X}, \mathcal{M}, \alpha, m)$ (resp. $\mathrm{PD}(\mathcal{X}, \mathcal{M}, \alpha, m)$) to be the subset of $C^1(\Delta^m \times \mathcal{X}, \Delta^m \times \mathcal{M})$ (resp. $\mathrm{PD}(\Delta^m \times \mathcal{X}, \Delta^m \times \mathcal{M})$) consisting of these elements which extend α and commute with projection onto Δ^m. $C_\partial^1(\mathcal{X}, \mathcal{M}, \alpha, m)$ will consist of those elements which, additionally, preserve boundaries in fibres over Δ^m. With this notation we have, remembering that our n-ads and formal boundaries are, by choice, compact

Theorem 2.2. *If $f \in C^1(\mathcal{X}, \mathcal{M}, \alpha, m)$, there exists a fundamental system of neighbourhoods $(U_\lambda)_{\lambda \in \Lambda}$ of f in $C^1(\mathcal{X}, \mathcal{M}, \alpha, m)$ such that, for each $g \in U_\lambda$, there is a piecewise C^1 map $F\colon I \times \Delta^m \times \mathcal{X} \to I \times \Delta^m \times \mathcal{M}$ commuting with projection onto $I \times \Delta^m$ such that* (i) $F_0 = f$ (ii) $F_1 = g$ *and* (iii) $F_s \in U_\lambda$ *for all* $s \in I$. *Moreover, if f and g are in $C_\partial^1(\mathcal{X}, \mathcal{M}, \alpha, m)$ then we can ensure that $F_s \in U_\lambda \cap C_\partial^1(\mathcal{X}, \mathcal{M}, \alpha, m)$ for all $s \in I$.*

Proof. Suppose that $\mathcal{M}$ is an n-ad. The same proof as (1.3) will give us the result provided that we choose the Riemannian metric on $\mathcal{M}(\emptyset)$ to have the additional property that each $\mathcal{M}(x)$ is totally geodesic. Deformation along geodesics then preserves the functorial subspaces.

If $\mathcal{M}$ is a formal boundary of type n, then its total space is a union of $(n+1)$ n-ads, as previously indicated, and we may prove the theorem one n-ad at a time using the above. ∎

§3. Simplicial sets and a homotopy equivalence. If $\mathcal{X}\colon \mathcal{C} \to \mathrm{PL}$, $\mathcal{M}\colon \mathcal{C} \to \mathrm{DIFF}$ are functors, we get a simplicial set $\mathrm{PD}(\mathcal{X}, \mathcal{M})$ whose set of n-simplices is the subset of $\mathrm{PD}(\Delta^n \times \mathcal{X}, \Delta^n \times \mathcal{M})$ consisting of those elements which commute with projection onto Δ^n, face and degeneracy maps being defined in the obvious way. Similarly, we have simplicial groups $\mathrm{PL}(\mathcal{X})$ (resp. $\mathrm{DIFF}(\mathcal{M})$) whose n-simplices are PL (resp. DIFF) natural self-equivalences of $\Delta^n \times \mathcal{X}$ (resp. $\Delta^n \times \mathcal{M}$) commuting with projection onto Δ^n. Composition defines a free right (resp. left) action of $\mathrm{PL}(\mathcal{X})$ (resp. $\mathrm{DIFF}(\mathcal{M})$) on $\mathrm{PD}(\mathcal{X}, \mathcal{M})$. If $\mathcal{X}$ and $\mathcal{M}$ are n-ads or formal boundaries, then each of the above complexes satisfies the Kan extension condition. In the case of $\mathrm{PD}(\mathcal{X}, \mathcal{M})$, this requires Whitehead's theorem on extension of PD triangulations from boundaries (see Munkres [8], §10). We leave the reader to fill in the extra complication of requiring the extension to commute with projection as an exercise using "openness of PD triangulations".

Now let $\mathcal{M}$ be either a DIFF n-ad or a DIFF formal boundary, and let $\mathcal{X}$ be either a PL n-ad or a PL formal boundary, though having the same diagram scheme as $\mathcal{M}$. Suppose in addition that there exists a PD triangulation $\alpha\colon \mathcal{X} \to \mathcal{M}$. The existence of such an $\mathcal{X}$ follows from Whitehead's original theorem thus; firstly suppose that $\mathcal{M}$ is an n-ad. Although $\mathcal{M}(\emptyset)$ is given as a manifold with corners, it is well known that the total boundary of $\mathcal{M}(\emptyset)$ admits a smoothing. For the details we refer the reader to Douady [2]. Under any such smoothing, the formal boundary of $\mathcal{M}$ consists of a union of smooth closed submanifolds. A triangulation of the formal boundary can then be extended to one of the whole boundary and thence to one of $\mathcal{M}(\emptyset)$. Moreover, such a triangulation of the "smoothed" version of $\mathcal{M}(\emptyset)$ triangulates equally well the unsmoothed version. Hence the problem for n-ads reduces to that for formal boundaries of type $(n-1)$. But this is a union of problems for $(n-1)$-ads. Hence we proceed by induction from Whitehead's orginal theorem corresponding to the case $n = 0$.

We abbreviate $\mathrm{PL}(\mathcal{X})$ to PL, $\mathrm{PD}(\mathcal{X}, \mathcal{M})$ to PD, and write $\mathrm{PL}(\partial)\ [m]$ for the subgroup of $\mathrm{PL}[m]$ consisting of elements which extend the identity on $\partial\Delta^m \times \mathcal{X}$. A PD triangulation $\alpha\colon \mathcal{X} \to \mathcal{M}$ induces a simplicial map

α_*: PL $\to$ PD by $\alpha_*(f) = (1_{\Delta^n} \times \alpha) \circ f$ for $f \in \mathrm{PL}[n]$. We wish to show that α_* is a homotopy equivalence.

PROPOSITION 3.1. *Let β: $\partial\Delta^m \times \mathscr{X} \to \partial\Delta^m \times \mathscr{M}$ be a PD triangulation, commuting with projection onto $\partial\Delta^m$. Then every orbit of* PL$(\partial)[m]$ *is dense in* PD$(\mathscr{X}, \mathscr{M}, \beta, m)$.

Proof. In the first case take $\mathscr{M}$ to be an n-ad. Let $\xi, \gamma \in \mathrm{PD}(\mathscr{X}, \mathscr{M}, \beta, m)$. If g is a PL automorphism of $\Delta^m \times \mathscr{X}$, which we do not assume commutes with projection onto Δ^m, define $\psi(g)$ by $\psi(g)(t, x) = (t, g_2(t, x))$, where $g = (g_1, g_2)$. We have

$$(\xi \cdot g)(t, x) = \big(g_1(t, x), \xi_2(g_1(t, x), g_2(t, x))\big),$$

$$(\xi \cdot \psi(g))(t, x) = \big(t, \xi_2(t, g_2(t, x))\big),$$

$$\gamma(t, x) = (t, \gamma_2(t, x)).$$

Computing Jacobians, defined piecewise on simplices, we get

$$J(\xi \cdot g) = \begin{bmatrix} \dfrac{\partial g_1}{\partial t} & \dfrac{\partial \xi_2}{\partial t}\dfrac{\partial g_1}{\partial t} + \dfrac{\partial \xi_2}{\partial x}\dfrac{\partial g_2}{\partial t} \\ \dfrac{\partial g_1}{\partial x} & \dfrac{\partial \xi_2}{\partial t}\dfrac{\partial g_1}{\partial x} + \dfrac{\partial \xi_2}{\partial x}\dfrac{\partial g_2}{\partial x} \end{bmatrix}, \quad J(\gamma) = \begin{bmatrix} 1 & \dfrac{\partial \gamma_2}{\partial t} \\ 0 & \dfrac{\partial \gamma_2}{\partial x} \end{bmatrix},$$

$$J(\xi \cdot \psi(g)) = \begin{bmatrix} 1 & \dfrac{\partial \xi_2}{\partial t} + \dfrac{\partial \xi_2}{\partial x}\dfrac{\partial g_2}{\partial t} \\ 0 & \dfrac{\partial \xi_2}{\partial x}\dfrac{\partial g_2}{\partial x} \end{bmatrix}.$$

Let U be a neighbourhood of γ in PD$(\mathscr{X}, \mathscr{M}, \beta, m)$. We must find $a \in \mathrm{PL}(\partial)[m]$ such that $\xi \cdot \alpha \in U$. Let $(V_r)_{r \in N}$ be a countable fundamental system of neighbourhoods of γ in PD$(\Delta^m \times \mathscr{X}, \Delta^m \times \mathscr{M}, \beta, 0)$ (see the remark at the end of §1). By Whitehead's original theorem on uniqueness of PD triangulations, for each r we can choose a PL automorphism g^r of $\Delta^m \times \mathscr{X}$ such that $\xi \cdot g^r \in V_r$. To achieve this we can use the methods of Munkres ([8], 10.5). Consideration of the above expressions for $\xi \cdot g^r$, $\xi \cdot \psi(g^r)$, γ and their Jacobians shows that since the sequence $(\xi \cdot g^r)_{r \in N}$ converges to γ in $C^1(\Delta^m \times \mathscr{X}, \Delta^m \times \mathscr{M}, \beta, 0)$, so also does $(\xi \cdot \psi(g^r))_{r \in N}$. Now each $\xi \cdot \psi(g^r)$ belongs to $C^1_\partial(\mathscr{X}, \mathscr{M}, \beta, m)$. Since PD$(\mathscr{X}, \mathscr{M}, \beta, m)$ is open in $C^1_\partial(\mathscr{X}, \mathscr{M}, \beta, m)$, then, for all sufficiently large r, $\xi \cdot \psi(g^r)$ is a homeomorphism, hence $\psi(g^r)$ is a PL automorphism. However each g^r must be the identity on $\partial\Delta^m \times \mathscr{X}$, hence $\psi(g^r)$ is also the identity on $\partial\Delta^m \times \mathscr{X}$ and when $\psi(g^r)$ is a PL automorphism, it belongs to PL$(\partial)[m]$. Thus if we choose $a = \psi(g^r)$ for large enough r, we will ensure that $\xi \cdot a \in U$. This completes the proof in the case where $\mathscr{M}$ is an n-ad.

If $\mathscr{M}$ is a formal boundary of type $n-1$, then we can find a smooth n-ad $\mathscr{N}$ whose formal boundary is $\mathscr{M}$. In fact, by restricting our attention to collar neighbourhoods of $\mathscr{M}$ in $\mathscr{N}$, we can assume that $\mathscr{N}$ is topologically a product $\mathscr{M} \times I$. Since β: $\partial\Delta^m \times \mathscr{X} \to \partial\Delta^m \times \mathscr{M}$ is a PD triangulation commuting with projection onto $\partial\Delta^m$, then by employing the usual method of extending triangulations from boundaries (Munkres [8], 10.6), we may find a PL n-ad $\mathscr{Z}$, with formal boundary $\mathscr{X}$ and total space combinatorially equivalent to $\mathscr{X} \times I$, such that there exists a PD triangulation $\hat{\beta}$: $\partial\Delta^m \times \mathscr{Z} \to \partial\Delta^m \times \mathscr{N}$ which extends β and commutes with projection onto $\partial\Delta^m$. The set PD$(\mathscr{Z}, \mathscr{N}, \hat{\beta}, m)$ is then well defined, being the set of all PD triangulations τ: $\Delta^m \times \mathscr{Z} \to \Delta^m \times \mathscr{N}$ which extend $\hat{\beta}$ and commute with projection onto Δ^m. Restriction to $\Delta^m \times \mathscr{X}$ defines a continuous map ϱ: PD$(\mathscr{Z}, \mathscr{N}, \hat{\beta}, m) \to$ PD$(\mathscr{X}, \mathscr{M}, \beta, m)$ and an easy argument using essentially only the Kan condition for PD$(\mathscr{X}, \mathscr{M})$ shows that ϱ is surjective. Let PL$(\mathscr{Z}, \partial)[m]$ denote the set of all PL automorphisms of $\Delta^m \times \mathscr{Z}$ which extend the identity on $\partial\Delta^m \times \mathscr{Z}$ and commute with projection onto Δ^m. There is also a restriction map ϱ: PL$(\mathscr{Z}, \partial)[m] \to$ PL$(\partial)[m]$. PL$(\mathscr{Z}, \partial)[m]$ acts freely on the right of PD$(\mathscr{Z}, \mathscr{N}, \hat{\beta}, m)$ and the restriction maps give a morphism of transformation groups thus;

$$\begin{array}{ccc} \mathrm{PD}(\mathscr{Z}, \mathscr{N}, \hat{\beta}, m) \times \mathrm{PL}(\mathscr{Z}, \partial)[m] & \to & \mathrm{PD}(\mathscr{Z}, \mathscr{N}, \hat{\beta}, m) \\ \downarrow{\scriptstyle \varrho \times \varrho} & & \downarrow{\scriptstyle \varrho} \\ \mathrm{PD}(\mathscr{X}, \mathscr{M}, \beta, m) \times \mathrm{PL}(\partial)[m] & \to & \mathrm{PD}(\mathscr{X}, \mathscr{M}, \beta, m) \end{array}$$

Now let $\xi, \gamma \in \mathrm{PD}(\mathscr{X}, \mathscr{M}, \beta, m)$ and let U be an open neighbourhood of γ in PD$(\mathscr{X}, \mathscr{M}, \beta, m)$. We seek an element $a \in \mathrm{PL}(\partial)[m]$ such that $\xi \cdot a \in U$. Choose $\bar{\xi}$, $\bar{\gamma}$ in PD$(\mathscr{Z}, \mathscr{N}, \hat{\beta}, m)$ such that $\varrho(\bar{\xi}) = \xi$, $\varrho(\bar{\gamma}) = \gamma$. Since ϱ is continuous, $\varrho^{-1}(U)$ is an open neighbourhood of $\bar{\gamma}$ in PD$(\mathscr{Z}, \mathscr{N}, \hat{\beta}, m)$. Hence by the case already proved for n-ads, we may choose $\bar{a} \in \mathrm{PL}(\mathscr{Z}, \partial)[m]$ such that $\bar{\xi} \cdot \bar{a} \in \varrho^{-1}(U)$. Then $\xi \cdot a \in U$, so that the orbit of ξ under PL$(\partial)[m]$ is dense in PD$(\mathscr{X}, \mathscr{M}, \beta, m)$. ∎

THEOREM 3.2. PD/PL = PD$(\mathscr{X}, \mathscr{M})$/PL$(\mathscr{X})$ *is contractible.*

Proof. It is enough to show that $\pi_m(\mathrm{PD/PL}) = 0$ for all m. We begin the proof by showing that $\pi_m(\mathrm{PD/PL}) = 0$ for all m in the case where $\mathscr{M}$ and $\mathscr{X}$ are both 0-ads.

Let $\alpha \in \mathrm{PD}[0]$ be our distinguished triangulation, and let $f \in \mathrm{PD}[0]$. Since we are supposing that $\mathscr{M}$ and $\mathscr{X}$ are 0-ads, we may apply Whitehead's original uniqueness theorem and conclude that there is a PD triangulation $F(f)$: $I \times \mathscr{X} \to I \times \mathscr{M}$ commuting with projection onto I such that $F(f)_0 = f$ and $F(f)_1 \in \alpha_*(\mathrm{PL})$. Hence $[f] \simeq [\alpha]$ in PD/PL, so that $\pi_0(\mathrm{PD/PL}) = 0$.

Suppose that $\pi_r(\mathrm{PD/PL}) = 0$ for $r < m$, and let $x \in \pi_m(\mathrm{PD/PL})$. We may represent x as $[f]$, for $f \in \mathrm{PD}[m]$ such that $f_{|\partial\Delta^m \times \mathscr{X}}$ is homotopically trivial in PD/PL. By induction on the faces of $\partial\Delta^m$, and using the facts that $\pi_r(\mathrm{PD/PL}) = 0$ for $r < m$, and that $\partial\Delta^m$ has a product neighbourhood in Δ^m, we may suppose that $f_{|\partial\Delta^m\times\mathscr{X}} = 1\times\alpha$. To see this, add a collar $I\times\partial\Delta^m$ onto the boundary of Δ^m so that $\{0\}\times\partial\Delta^m$ is the new boundary, and $\{1\}\times\partial\Delta^m$ is the original boundary. Put $V = (I\times\partial\Delta^m)\underset{\partial\Delta^m}{\cup}\Delta^m$ and let $V^{(k)} = \{0\}\times\partial\Delta^m\cup I\times S^{(k)}\cup\Delta^m\subset V$, where $S^{(k)}$ is the k-skeleton of $\partial\Delta^m$, so that $V^{(-1)} = \{0\}\times\partial\Delta^m\cup\Delta^m$, and $V^{(m-1)} = V$. Define $g^{(-1)}\colon V^{(-1)}\times\mathscr{X}\to V^{(-1)}\times\mathscr{M}$ by setting $g^{(-1)}$ equal to $1\times\alpha$ on $\{0\}\times\partial\Delta^m\times\mathscr{X}$ and equal to f on $\Delta^m\times\mathscr{X}$. Then $g^{(-1)}$ is a triangulation commuting with projection onto V. Moreover, the composite of $g^{(-1)}$ and the inclusion into $V\times\mathscr{M}$ is a PD immersion. Using the fact that $\pi_r(\mathrm{PD/PL}) = 0$, we can extend $g^{(r-1)}$ to a map $g^{(r)}\colon V^{(r)}\times\mathscr{X}\to V^{(r)}\times\mathscr{M}$ which is a triangulation commuting with projection onto V and is a PD immersion into $V\times\mathscr{M}$. This is possible since the closure $\overline{V^{(r)}-V^{(r-1)}}$ is a union of cells of dimension $(r+1)$ with intersections only in the boundaries. The extension is then effected by a collection of null homotopies. Eventually, since $\pi_r(\mathrm{PD/PL})$ is zero for $r < m$, we obtain a PD triangulation $t\colon V\times\mathscr{X}\to V\times\mathscr{M}$ which commutes with projection onto V. Identifying V with Δ^m by squashing the collar into a product neighbourhood of $\partial\Delta^m$ we obtain an element $t\in\mathrm{PD}[m]$ which is $1\times\alpha$ on $\partial\Delta^m\times\mathscr{X}$, and, by construction, represents the same element of $\pi_m(\mathrm{PD/PL})$ as $[f]$.

Hence suppose that f restricts to $1\times\alpha$ on $\partial\Delta^m\times\mathscr{X}$. By (2.1) and (2.2), we can choose a neighbourhood U of f in $C^1_\partial(\mathscr{X}, \mathscr{M}, 1\times\alpha, m)$ such that $U\subset\mathrm{PD}(\mathscr{X}, \mathscr{M}, 1\times\alpha, m)$, and such that, for each $g\in U$, there exists a PD isotopy $G(f, g)\colon I\times\Delta^m\times\mathscr{X}\to I\times\Delta^m\times\mathscr{M}$ with $G(f, g)_0 = f$, $G(f, g)_1 = g$ and $G(f, g)_t\in U$ for all $t\in I$. By (3.1), we may choose $h\in\mathrm{PL}(\partial)[m]$ such that $(1\times\alpha)\circ h\in U$. Put $F(f) = G(f, (1\times\alpha)\circ h)$. Then $F(f)$ is a homotopy from $[f]$ to $[1\times\alpha]$ in PD/PL. More accurately, the simplicial subdivision of $I\times\Delta^m$ "without extra vertices" plus the extension condition for PD gives, when applied to $F(f)$, a simplicial homotopy from $[f]$ to $[1\times\alpha]$ in PD/PL. The details here are the same as in Lashof–Rothenberg ([7] pp. 363–364). Hence $\pi_m(\mathrm{PD/PL}) = 0$ for all m in the case where $\mathscr{X}$ and $\mathscr{M}$ are 0-ads.

The remainder of the proof now goes by induction on the complexity of the diagram scheme of $\mathscr{X}$ and $\mathscr{M}$. Let $P(n)$ and $Q(n)$ be the statements

$P(n)$: PD/PL is contractible if $\mathscr{X}$ and $\mathscr{M}$ are r-ads and $r\leqslant n$.

$Q(n)$: PD/PL is contractible if $\mathscr{X}$ and $\mathscr{M}$ are formal boundaries of type r and $r\leqslant n$.

$Q(-1)$ is true since it is empty. Since $P(0)$ is true, it suffices to prove the two further steps

(i) $P(n)\,\&\,Q(n-1)\Rightarrow Q(n)$ (ii) $Q(n)\Rightarrow P(n+1)$.

$P(n)\,\&\,Q(n-1)\Rightarrow Q(n)$: Let $\mathscr{X}$, $\mathscr{M}$ be formal boundaries of type n and let $\alpha\colon\mathscr{X}\to\mathscr{M}$ be our distinguished PD triangulation. If $f\colon\Delta^m\times\mathscr{X}\to\Delta^m\times\mathscr{M}$ is an m-simplex of PD we wish to show that $[f]$ is homotopic to $[1\times\alpha]$ in PD/PL. Define a sequence $\mathscr{X}\langle r\rangle$, $1\leqslant r\leqslant n+1$ as follows; $\mathscr{X}\langle 1\rangle$ is a 0-ad with total space $\mathscr{X}(1)$. $\mathscr{X}\langle r+1\rangle$ is an r-ad with total space $\mathscr{X}(r+1)$ and formal boundary of type $(r-1)$ given thus; $\delta\mathscr{X}\langle r+1\rangle\colon K_{r-1}\to\mathrm{PL}$, $b\mapsto\mathscr{X}(b\cup\{r+1\})$. It may be more useful to remember that the union of the vertex spaces of $\delta\mathscr{X}\langle r+1\rangle$ is $\mathscr{X}(r+1)\cap\big(\bigcup_{i=1}^{r}\mathscr{X}(i)\big)$. Similarly we define $\mathscr{M}\langle r\rangle$, $1\leqslant r\leqslant n+1$. Restriction in the obvious manner gives PD triangulations $\alpha\langle i\rangle\colon\mathscr{X}\langle i\rangle\to\mathscr{M}\langle i\rangle$ and $f\langle i\rangle\colon\Delta^m\times\mathscr{X}\langle i\rangle\to\Delta^m\times\mathscr{M}\langle i\rangle$, for $1\leqslant i\leqslant n+1$.

Now, using hypothesis $P(n)$, deform $[f\langle 1\rangle]$ to $[1\times\alpha]$ in $\mathrm{PD}(\mathscr{X}\langle 1\rangle, \mathscr{M}\langle 1\rangle)/\mathrm{PL}(\mathscr{X}\langle 1\rangle)$. This gives a deformation of $f_{|\Delta^m\times\delta\mathscr{X}\langle 2\rangle}$ to $[1\times\alpha]$. We need to deform $[f\langle 2\rangle]$ to $[1\times\alpha]$ compatibly with the deformation already given on $\Delta^m\times\delta\mathscr{X}\langle 2\rangle$. This situation recurs. Typically we are in the position of having deformed $[f\langle i\rangle]$ to $[1\times\alpha]$ for $1\leqslant i\leqslant r$, compatibly on the intersections of the domains of the $[f\langle i\rangle]$. This gives a deformation of $f\langle r+1\rangle_{|\Delta^m\times\delta\mathscr{X}\langle r+1\rangle}$ to $[1\times\alpha]$. To proceed, we wish to extend this deformation to a deformation of $[f\langle r+1\rangle]$ to $[1\times\alpha]$. We show how to solve this problem.

Let N be a product neighbourhood of $\delta\mathscr{X}\langle r+1\rangle$ in $\mathscr{X}\langle r+1\rangle$. By hypothesis $P(n)$, we can choose a deformation of $f\langle r+1\rangle_{|\Delta^m\times(\mathscr{X}\langle r+1\rangle-\mathrm{Int}(N))}$ to $[1\times\alpha]$. This gives a deformation to $[1\times\alpha]$ at each end of $\Delta^m\times N$, but with no deformation defined on $\Delta^m\times\mathrm{Int}(N)$. Using hypothesis $Q(n-1)$, we can extend this deformation to $[1\times\alpha]$ over $\Delta^m\times N$, giving a deformation of $[f\langle r+1\rangle]$ to $[1\times\alpha]$, which is compatible with the one already defined on $\Delta^m\times\delta\mathscr{X}\langle r+1\rangle$. Notice that the most complicated of the $\mathscr{X}\langle r\rangle$'s is $\mathscr{X}\langle n+1\rangle$, whose formal boundary is of type $(n-1)$, all other boundaries of the $\mathscr{X}\langle r\rangle$'s having lower type. Hence we can legitimately apply hypothesis $Q(n-1)$.

When we have finally deformed $[f\langle n+1\rangle]$ to $[1\times\alpha]$ in $\mathrm{PD}(\mathscr{X}\langle n+1\rangle, \mathscr{M}\langle n+1\rangle)/\mathrm{PL}(\mathscr{X}\langle n+1\rangle)$, then we have completed a deformation of $[f]$ to $[1\times\alpha]$ in $\mathrm{PD}(\mathscr{X}, \mathscr{M})/\mathrm{PL}(\mathscr{X}) = \mathrm{PD/PL}$. Hence PD/PL is contractible, and $P(n)\,\&\,Q(n-1)\Rightarrow Q(n)$.

$Q(n)\Rightarrow P(n+1)$: Let $\mathscr{X}$, $\mathscr{M}$ be $(n+1)$-ads and let $\alpha\colon\mathscr{X}\to\mathscr{M}$ be our distinguished triangulation. If $f\colon\Delta^m\times\mathscr{X}\to\Delta^m\times\mathscr{M}$ is an m-simplex of PD $= \mathrm{PD}(\mathscr{X}, \mathscr{M})$, we wish to deform $[f]$ to $[1\times\alpha]$ in PD/PL. Let N be a product neighbourhood of ∂X in $\delta\mathscr{X}$. Since $P(0)$ is true, we may deform $f_{|\Delta^m\times(\mathscr{X}-\mathrm{Int}(N))}$ to $[1\times\alpha]$ in PD/PL by considering $\mathscr{X}-\mathrm{Int}(N)$ as a 0-ad. By hypothesis $Q(n)$, we may choose a deformation of $f_{|\Delta^m\times\delta\mathscr{X}}$ to $[1\times\alpha]$ in PD/PL. This gives a deformation to $[1\times\alpha]$ at each end of $\Delta^m\times N$, but no deformation over $\Delta^m\times\mathrm{Int}(N)$. By hypothesis $Q(n)$, we may extend the deformation over the whole of $\Delta^m \times N$, thus giving a deformation of $[f]$ to $[1 \times \alpha]$. Hence PD/PL is contractible, completing the proof that $Q(n) \Rightarrow P(n+1)$, and hence also of Theorem (3.2). ∎

COROLLARY 3.3. *Let $\mathcal{M}$ be either a* DIFF *n-ad or formal boundary, and let $\alpha\colon \mathcal{X} \to \mathcal{M}$ be a* PD *triangulation. Then $\alpha_*\colon \mathrm{PL}(\mathcal{X}) \to \mathrm{PD}(\mathcal{X}, \mathcal{M})$ is a homotopy equivalence, where $\alpha_*(f) = (1_{\Delta^m} \times \alpha) \circ f$ for $f \in \mathrm{PL}(\mathcal{X})[m]$.*

Proof. Put $\mathrm{PL} = \mathrm{PL}(\mathcal{X})$ and $\mathrm{PD} = \mathrm{PD}(\mathcal{X}, \mathcal{M})$. By general simplicial nonsense, the natural map $\mathrm{PD} \to \mathrm{PD/PL}$ is a Kan fibration, so that, since PD/PL is contractible, the inclusion of the fibre over $[\alpha] \in (\mathrm{PD/PL})[0]$ is a homotopy equivalence. However, this inclusion is just $\alpha_*\colon \mathrm{PL} \to \mathrm{PD}$. ∎

At this point it is as well to remind the reader that it is part of our definition of n-ads and formal boundaries that all vertex spaces be compact. Though this is not necessary, it is technically simple. Though the following result is thus not the most general possible, it is enough for our purposes.

COROLLARY 3.4 (EXTENSION THEOREM). *Let $\mathcal{M}$ be either a (compact)* DIFF *n-ad or (compact)* DIFF *formal boundary. Let $\mathcal{X}$ be either a* PL *n-ad or formal boundary over the same diagram scheme as $\mathcal{M}$, and suppose that there exists a* PD *triangulation $\alpha\colon \mathcal{X} \to \mathcal{M}$.*

Then for each PD *triangulation $f\colon \partial\Delta^m \times \mathcal{X} \to \partial\Delta^m \times \mathcal{M}$ which commutes with projection onto $\partial\Delta^m$, there exists a* PD *triangulation $F\colon \Delta^m \times \mathcal{X} \to \Delta^m \times \mathcal{M}$, commuting with projection onto Δ^m such that $f^{-1}F_{|\partial\Delta^m \times \mathcal{X}}$ is a* PL *automorphism of $\partial\Delta^m \times \mathcal{X}$. Note that $f^{-1}F_{|\partial\Delta^m \times X}$ automatically commutes with projection onto $\partial\Delta^m$.*

Proof. Put $\mathrm{PL} = \mathrm{PL}(\mathcal{X})$ and $\mathrm{PD} = \mathrm{PD}(\mathcal{X}, \mathcal{M})$. Let $f\colon \partial\Delta^m \times \mathcal{X} \to \partial\Delta^m \times \mathcal{M}$ be a PD triangulation commuting with projection onto $\partial\Delta^m$. By taking a product neighbourhood of the boundary, we may represent Δ^m as $\partial\Delta^m \times [0, 1] \cup_{\partial\Delta} \Delta^m$. Consider a deformation of f into $\alpha_*(\mathrm{PL})$. This gives a PD triangulation $G\colon \partial\Delta^m \times [0, 1] \times \mathcal{X} \to \partial\Delta^m \times [0, 1] \times \mathcal{M}$ which commutes with projection onto $\partial\Delta^m \times [0, 1]$, and such that $G_0 = f$ and $G_1 = (1_{\partial\Delta^m} \times \alpha)\tau$ for some PL automorphism τ of $\partial\Delta^m \times \mathcal{X}$. Define a PD triangulation F as follows; $F\colon (\partial\Delta^m \times [0, 1] \cup_{\partial\Delta^m} \Delta^m) \times \mathcal{X} \to (\partial\Delta^m \times [0, 1] \cup_{\partial\Delta^m} \Delta^m) \times \mathcal{M}$,

$$F(s, t, x) = G(s, t, x)(1 \times \tau^{-1}), \quad (s, t, x) \in \partial\Delta^m \times [0, 1] \times \mathcal{X},$$

$$F(r, x) = (r, \alpha(x)), \quad (r, x) \in \Delta^m \times \mathcal{X}.$$

F commutes with projection onto $(\partial\Delta^m \times [0, 1] \cup_{\partial\Delta^m} \Delta^m)$. Equating Δ^m with $(\partial\Delta^m \times [0, 1] \cup_{\partial\Delta^m} \Delta^m)$ gives the result. ∎

§4. Triangulating smooth fibre bundles. Let $\mathscr{C}$ be either 2^n or K_n. By a DIFF $\mathscr{C}$-bundle we mean a triple $(\mathscr{E}, B, p)$ where $\mathscr{E}\colon \mathscr{C} \to \mathrm{DIFF}$ is a functor which is either a smooth manifold n-ad, if $\mathscr{C} = 2^n$, or formal boundary, if $\mathscr{C} = K_n$, where B is a smooth manifold, and $p\colon \mathscr{E} \to B$ is a natural transformation from $\mathscr{E}$ to the constant functor $x \mapsto B$, such that, for each $x \in \mathrm{Obj}(\mathscr{C})$, $p\colon \mathscr{E}(x) \to B$ is a smooth, locally trivial fibre bundle. Moreover, for this §, we relax our convention that n-ads and formal boundaries must be compact. We do not insist that each vertex space of $\mathscr{E}$ be compact, but only that each fibre of $p\colon \mathscr{E}(x) \to B$ be compact, for all $x \in \mathrm{Obj}(\mathscr{C})$. It is easy to see that the fibre over each $b \in B$ is then either a (compact) smooth manifold n-ad or formal boundary, depending upon the category $\mathscr{C}$.

If K is a compact polyhedron, then by a PL bundle with fibre K we mean a triple $\eta = (X, B, p)$ where X, B are polyhedra and $p\colon X \to B$ is a locally trivial PL map with fibre homeomorphic to K such that, for some admissible triangulation (L, h) of B, the coordinate transformations of $h^*(\eta)$ over each simplex of L lie in the simplicial group $\mathrm{PL}(K)$. As in the smooth case, if $\mathscr{C} = 2^n$ or K_n, then by a PL $\mathscr{C}$-bundle we mean a triple $(\mathcal{X}, B, p)$ where B is a polyhedron, $\mathcal{X}$ is either a PL n-ad or formal boundary, depending on $\mathscr{C}$, and $p\colon \mathcal{X} \to B$ is a natural transformation to the constant functor B, such that, for each $y \in \mathrm{Obj}(\mathscr{C})$, $p\colon \mathcal{X}(y) \to B$ is a PL bundle with compact fibre.

We shall allow ourselves to confuse the total space, i.e. the union of the vertex spaces, of an n-ad or formal boundary with the notation for the functor itself. We have already made this obfuscation in §3.

Let $\mathscr{K}$ be either a PL n-ad or formal boundary with total space K. Suppose that $p\colon X \to B$ is a PL bundle with fibre K, and that the group of coordinate transformations reduces to $\mathrm{PL}(\mathscr{K})$. Then any such reduction imposes on X the structure, $\mathcal{X}$, of either a PL n-ad or formal boundary, having the same type as $\mathscr{K}$. $\mathcal{X}$ is determined thus; let $\mathscr{K}\colon \mathscr{C} \to \mathrm{PL}$. If σ is a simplex in some admissible triangulation of B, then, given a trivialisation $h_\sigma\colon \sigma \times \mathscr{K} \to p^{-1}(\sigma)$, we get a functor $p^{-1}(\sigma)\colon \mathscr{C} \to \mathrm{PL}$ by taking $p^{-1}(\sigma)(y) = h_\sigma(\sigma \times \mathscr{K}(y))$. We then define $\mathcal{X}\colon \mathscr{C} \to \mathrm{PL}$ by $\mathcal{X}(y) = \bigcup_\sigma p^{-1}(\sigma)(y)$, where σ runs through the simplices of some admissible triangulation of B. It is easy to see that this definition depends only on the particular reduction to $\mathrm{PL}(\mathscr{K})$.

Before stating the next theorem, it will ease matters to introduce some terminology.

DEFINITION. Let $\xi = (\mathscr{E}, B, \pi)$ be a DIFF $\mathscr{C}$-bundle, where $\mathscr{C} = 2^n$ or K_n, let $\alpha\colon L \to B$ be a PD triangulation, and let $\eta = (\mathcal{X}, L, p)$ be a PL $\mathscr{C}$-bundle. By a PD triangulation of ξ by η over α we mean a PD triangulation $\beta\colon \mathcal{X} \to \mathscr{E}$ making the following commute

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\ \beta\ } & \mathscr{E} \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \pi} \\ L & \xrightarrow{\ \alpha\ } & B \end{array}$$

Suppose that $\eta_i = (\mathscr{X}_i, L, p_i)$ $(i = 0, 1)$ are PL bundles, and that we have PD triangulations over α, $h_i: \eta_i \to \xi$. We say that (η_0, h_0) and (η_1, h_1) are PD isotopic iff there is a PD triangulation $\Psi: \mathscr{X}_0 \times [0, 1] \to \mathscr{E} \times [0, 1]$ such that

(a) $(\pi \times \mathrm{Id}_{[0,1]})\Psi = (\alpha \circ p_0) \times \mathrm{Id}_{[0,1]}$,

(b) if $\Psi(x, t) = (\Psi_t(x), t)$, then $\Psi_0 = h_0$,

(c) $h_1^{-1} \circ \Psi_1: \eta_0 \to \eta_1$ is a PL equivalence of bundles.

Ψ is then called an *isotopy from* (η_0, h_0) *to* (η_1, h_1).

More generally, suppose that $L_0 \subset L$ is a subpolyhedron, and that, if $\zeta_i = \eta_{i|L_0}$, ψ is a PD isotopy from (ζ_0, h_0) to (ζ_1, h_1). We say that (η_0, h_0) and (η_1, h_1) *are isotopic relative to* ψ iff there is a PD isotopy Ψ from (η_0, h_0) to (η_1, h_1) which extends ψ.

THEOREM 4.1 (Uniqueness of PD triangulations). *Let $\mathscr{C}$ be either 2^n or K_n and let $\xi = (\mathscr{E}, B, \pi)$ be a* DIFF *$\mathscr{C}$-bundle with compact fibre $\mathscr{F}$. Let (L, L_0) be a simplicial pair, and let $\alpha: |L| \to B$ be a* PD *triangulation. Suppose that, for $i = 0, 1$, $h_i: \eta_i \to \xi$ is a* PD *triangulation over α, and that, if $\zeta_i = \eta_{i|L_0}$, ψ is a* PD *isotopy from (ζ_0, h_0) to (ζ_1, h_1). Then (η_0, h_0) and (η_1, h_1) are* PD *isotopic relative to ψ.*

In particular, taking $L_0 = \emptyset$, (η_0, h_0) and (η_1, h_1) are PD *isotopic.*

Proof. Let $P(k)$ be the statement of the theorem modified to read that the isotopy ψ can be extended over $L^{(k)} \cup L_0$, where $L^{(k)}$ is the k-skeleton of L. Let $Q(k)$ denote the statement of the theorem in the case where $L = B = \Delta^k$, $L_0 = \partial\Delta^k$, and α is the identity. It will suffice to prove the following three statements.

(I) $P(0)$ is true.

(II) Each $Q(k)$ is true.

(III) $P(k-1) \& Q(k) \Rightarrow P(k)$.

We shall confuse L with its geometrical realisation.

(I) *$P(0)$ is true*: For each 0-simplex x of L, $x \notin L_0$, put $\mathscr{F}_x = \pi^{-1}(\alpha(x))$. h_0 and h_1 each give a PD triangulation of $\mathscr{F}_x$, and we must produce PD isotopies between the two. That we can so do is the content of the original Whitehead Theorem on the uniqueness of PD triangulations ([11], [8], 10.5), once it is realised that PD triangulations which are close enough are isotopic, by Theorem (2.2) and the fact that PD triangulations are open in the space of piecewise C^1 maps, by Proposition (2.1). Hence $P(0)$ is true.

(II) *Each $Q(k)$ is true*: Here $L = \Delta^k$, $L_0 = \partial\Delta^k$, and we must extend a PD isotopy given over $\partial\Delta^k$ to one over the whole of Δ^k. Since Δ^k is contractible, by taking a DIFF trivialisation of ξ and PL trivialisations of η_0, η_1, we may without loss of generality assume that ξ is the product bundle $(\Delta^k \times \mathscr{F} \to \Delta^k)$ and that $\eta_0 = \eta_1 = \eta$ is the product bundle $(\Delta^k \times \mathscr{K} \to \Delta^k)$, that h_0, h_1, are distinct PD triangulations of ξ by η over the identity and that, if $\zeta = (\partial\Delta^k \times \mathscr{K} \to \partial\Delta^k)$, ψ is a PD isotopy from (ζ, h_0) to (ζ, h_1) thus; $\psi: \partial\Delta^k \times \mathscr{K} \times I \to \partial\Delta^k \times \mathscr{K} \times I$. Define $\tilde{\psi}: (\Delta^k \times \partial I \cup \partial\Delta^k \times I) \times \mathscr{K} \to (\Delta^k \times \partial I \cup \partial\Delta^k \times I) \times \mathscr{F}$ by

$$\tilde{\psi}(x, 0, y) = h_0(x, y), \quad \tilde{\psi}(x, 1, y) = h_1(x, y), \qquad x \in \Delta^k,\ y \in \mathscr{K},$$

$$\tilde{\psi}(x, s, y) = \psi(x, y, s), \qquad (x, s, y) \in \partial\Delta^k \times I \times \mathscr{K}.$$

Then $\tilde{\psi}$ is a PD triangulation, commuting with projection onto $(\Delta^k \times \partial I \cup \partial\Delta^k \times I) = \partial(\Delta^k \times I)$. Since $(\Delta^{k+1}, \partial\Delta^{k+1})$ is PL equivalent to $(\Delta^k \times I, \partial(\Delta^k \times I))$, we may apply the Extension Theorem (3.4) to conclude that, possibly after a PL bundle isomorphism of $(\partial(\Delta^k \times I) \times \mathscr{K} \to \partial(\Delta^k \times I))$, $\tilde{\psi}$ extends to a PD triangulation $\Psi: \Delta^k \times I \times \mathscr{K} \to \Delta^k \times I \times \mathscr{F}$ commuting with projection onto $\Delta^k \times I$. However, a bundle isomorphism of $(\partial(\Delta^k \times I) \times \mathscr{K} \to \partial(\Delta^k \times I))$ in effect changes the chosen trivialisations of η_0, η_1. Thus choosing the correct trivialisations at the outset, we can ensure that Ψ is a true extension of $\tilde{\psi}$. Ψ now provides the desired PD isotopy from (η_0, h_0) to (η_1, h_1) relative to ψ after the following sequence of aliases; $\Delta^k \times I \times \mathscr{K} \cong \Delta^k \times \mathscr{K} \times I \cong \mathscr{X}_i \times I$, where $\eta_i = (\mathscr{X}_i, \Delta^k, p_i)$ and $\Delta^k \times I \times \mathscr{F} = \mathscr{E} \times I$. Thus each $Q(k)$ is true.

(III) $P(k-1) \& Q(k) \Rightarrow P(k)$: Originally, we are given an isotopy ψ over L_0. By $P(k-1)$ we can extend this to an isotopy, still denoted by ψ, over $L^{(k-1)} \cup L_0$. Let $(\sigma_\lambda)_{\lambda \in \Lambda}$ be the set of k-simplices of L. For each σ_λ, each proper face of σ_λ lies in $L^{(k-1)} \cup L_0$. For each λ such that $\sigma_\lambda \not\subset L_0$, we are given two PL bundles over σ_λ and PD triangulations of $\xi_{|\alpha(\sigma_\lambda)}$ by each of them, related by the PD isotopy ψ defined over $\partial\sigma_\lambda$. Appealing to $Q(k)$, we may extend each of these isotopies over the whole of σ_λ. Glueing together the extended isotopies gives the required extension of ψ over $L^{(k)} \cup L_0$. Thus $P(k-1) \& Q(k) \Rightarrow P(k)$. ■

Notice that (4.1) has the following easy but important consequence.

COROLLARY 4.2. *Let $\mathscr{C}$ be either 2^n or K_n, let ξ be a trivial* DIFF *$\mathscr{C}$-bundle $\xi = (B \times \mathscr{F}, B, \pi_B)$ with compact fibre $\mathscr{F}$, and let η be a* PL *$\mathscr{C}$-bundle admitting a* PD *triangulation of ξ, $h: \eta \to \xi$. Then η is* PL *trivial.*

Proof. By (4.1), any two PL bundles which PD triangulate the same DIFF bundle are PL equivalent. However, choosing PD triangulations $L \to B$ and $\mathscr{K} \to \mathscr{F}$, we see that the trivial bundle $(L \times \mathscr{K}, L, \pi_L)$ PD triangulates the trivial DIFF $\mathscr{C}$-bundle ξ. Hence η is PL trivial. ■

To establish the existence of PD triangulations we need a criterion for recognising polyhedra. We use the following folk theorem (c. f. [10], Exercise 2.2.8, p. 42). Since the proof does not appear explicitly in the literature, we provide one.

LEMMA 4.3. *Suppose given a pushout diagram of locally compact spaces and proper maps thus*

$$\begin{array}{ccc} A & \xrightarrow{\ i\ } & B \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\psi} \\ C & \xrightarrow{\ j\ } & D \end{array}$$

in which A, B, C are polyhedra, i, φ are PL *maps, and i is injective. Then D admits a polyhedral structure for which j and ψ are* PL. *Moreover, j is injective.*

Proof. Since we are dealing with a pushout, it suffices only to show that A, B, C admit triangulations with respect to which i and φ are simultaneously simplicial. First triangulate A, C by A_0, C_0 so that $\varphi\colon A_0 \to C_0$ is simplicial. Then triangulate A, B by A_1, B_1 so that A_1 subdivides A_0 and $i\colon A_1 \to B_1$ is simplicial. Then $\varphi\colon A_1 \to C_0$ is linear on each simplex so we may choose (e.g. by [4], p. 16, Lemma 1.9) triangulations A_2, C_2 of A, C such that $\varphi\colon A_2 \to C_2$ is simplicial, and A_2 subdivides A_1. Finally, since i is injective, we can subdivide B_1 to B_2 so that $i\colon A_2 \to B_2$ is simplicial ([4], p. 10, Lemma 1.3). Then $i\colon A_2 \to B_2$ and $\varphi\colon A_2 \to C_2$ are simultaneously simplicial. ■

Finally, we have the main theorem of this §.

THEOREM 4.4 (Existence of PD triangulations). *Let $\mathscr{C}$ be either 2^n or K_n, let $\xi = (\mathscr{E}, B, \pi)$ be a* DIFF *$\mathscr{C}$-bundle with compact fibre $\mathscr{F}$, and let $\alpha\colon |L| \to B$ be a* PD *triangulation. Suppose also given the following; L_0 is a subcomplex of L; $\xi_0 = \xi_{|\alpha(|L_0|)}$; $h_0\colon \zeta \to \xi_0$ is a* PD *triangulation over α, where ζ is some* PL *$\mathscr{C}$-bundle over L_0. Then there exists a* PL *$\mathscr{C}$-bundle η, a* PD *triangulation $h\colon \eta \to \xi$ over α, and a* PL *$\mathscr{C}$-bundle isomorphism $\varphi\colon \zeta \to \eta_{||L_0|}$, such that $h_0 = h_\varphi$. In particular, taking $L_0 = \emptyset$, ξ admits a* PD *triangulation $h\colon \eta \to \xi$ over α.*

Proof. Again we shall confuse L with its geometrical realisation. Let $R(k)$ denote the statement of the theorem modified to read that the bundle η and PD triangulation h can be constructed over $L^{(k)} \cup L_0$, where $L^{(k)}$ is the k-skeleton of L.

$R(0)$ is true: Let $\beta\colon \mathscr{K} \to \mathscr{F}$ be a PD triangulation of the fibre $\mathscr{F}$ of ξ. We may extend ζ over $L^{(0)} \cup L_0$ by taking, for each vertex v of L such that $v \notin L_0$, a copy $\mathscr{K}_v$ of $\mathscr{K}$ and a PD triangulation $h_v\colon \mathscr{K}_v \to \mathscr{F}_{\alpha(v)}$, where $\mathscr{F}_{\alpha(v)}$ is the fibre of ξ over $\alpha(v) \in B$. This proves $R(0)$.

$R(k-1) \Rightarrow R(k)$: Let $(\sigma_\lambda)_{\lambda\in\Lambda}$ be the set of k-simplices of L such that $\sigma_\lambda \not\subset L_0$. Then $L^{(k)} \cup L_0$ is a pushout thus;

$$\begin{array}{ccc} \coprod_{\lambda\in\Lambda} \partial\sigma_\lambda & \xrightarrow{\ i\ } & \coprod_{\lambda\in\Lambda} \sigma_\lambda \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\psi} \\ L^{(k-1)} \cup L_0 & \xrightarrow{\ j\ } & L^{(k)} \cup L_0 = \varinjlim\,(i, \varphi) \end{array}$$

Moreover, j is injective. Now since L triangulates the finite dimensional manifold B, L is locally finite, so that $\varphi\colon \coprod_{\lambda\in\Lambda} \partial\sigma_\lambda \to L^{(k-1)} \cup L_0$ is a proper PL map. Consider the PL bundle η which, by induction, is already constructed over $L^{(k-1)} \cup L_0$. We claim that $\varphi^*(\eta)$ is PL trivial. For, by induction, η PD triangulates $j^*(\alpha^*(\xi)_{|L^{(k)} \cup L_0})$, hence $\varphi^*(\eta)$ PD triangulates $(j\varphi)^*(\alpha^*(\xi)_{|L^{(k)} \cup L_0}) = i^*(\psi^*(\alpha^*(\xi)_{|L^{(k)} \cup L_0}))$, which bundle is DIFF trivial, since each σ_λ is contractible. Thus $\varphi^*(\eta)$ is PL trivial by Corollary (4.2). Thus there is a PL bundle isomorphism $i_*\colon \varphi^*(\eta) \to (\coprod_\lambda \sigma_\lambda) \times \mathscr{K}$ such that the PD triangulation already given on $\varphi^*(\eta)$ extends over i_* to a PD bundle triangulation $\hat{h}$ of $(\coprod_\lambda \sigma_\lambda) \times \mathscr{F}$ by $(\coprod_\lambda \sigma_\lambda) \times \mathscr{K}$, by (3.4). Take Z to be defined by the pushout diagram

$$\begin{array}{ccc} X & \xrightarrow{\ i_*\ } & (\coprod_\lambda \sigma_\lambda) \times \mathscr{K} \\ \downarrow{\scriptstyle\varphi_*} & & \downarrow{\scriptstyle\psi_*} \\ Y & \xrightarrow{\ j_*\ } & Z = \varinjlim\,(i_*, \varphi_*) \end{array}$$

where X (resp. Y) is the total space of $\varphi^*(\eta)$ (resp. η). Z is a polyhedron by (4.3). Moreover, there is a natural map of pushouts $\natural\colon \varinjlim(i_*, \varphi_*) \to \varinjlim\,(i, \varphi)$ which is clearly a PL $\mathscr{C}$-bundle with fibre $\mathscr{K}$. $(Z, L^{(k)} \cup L_0, \natural)$ is the sought after extension of η over $L^{(k)} \cup L_0$. To define the extension of h to a PD triangulation over $L^{(k)} \cup L_0$, put $\xi' = \alpha^*(\xi)_{|L^{(k)} \cup L_0}$, and decompose the total space Z' of ξ' as a pushout

$$\begin{array}{ccc} X' & \longrightarrow & (\coprod_\lambda \sigma_\lambda) \times \mathscr{F} \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Z' \end{array}$$

where Y' (resp. X') is the total space of $j^*(\xi')$ (resp. $(j\varphi)^*(\xi')$). There is a PD triangulation $Z \xrightarrow{\gamma} Z'$ obtained by taking $\gamma = h$ (already constructed, by induction) on Y, and $\gamma = \hat{h}$ (constructed above) on $(\coprod_\lambda \sigma_\lambda) \times \mathscr{K}$. γ is the required extension of h over $L^{(k)} \cup L_0$. Hence $R(k-1) \Rightarrow R(k)$. ■

The final triangulation theorem we prove is a relative version of the above. Given a compact DIFF n-ad $\mathscr{M}$, its formal boundary $\delta\mathscr{M}$ has a class of product neighbourhoods, the prismatic neighbourhoods of [1]. If N is a closed prismatic neighbourhood of $\delta\mathscr{M}$, let $\mathrm{DIFF}_N(\mathscr{M})$ be the simplicial group whose k-simplices are n-ad diffeomorphisms $f\colon \Delta^k \times \mathscr{M} \to \Delta^k \times \mathscr{M}$

which commute with projection on Δ^k and satisfy $f(\Delta^k \times N) = \Delta^k \times N$. Observe that $\mathrm{DIFF}_N(\mathscr{M})$ is a simplicial subgroup of $\mathrm{DIFF}(\mathscr{M})$. We first prove

PROPOSITION 4.5. *The inclusion $\mathrm{DIFF}_N(\mathscr{M}) \hookrightarrow \mathrm{DIFF}(\mathscr{M})$ is a homotopy equivalence.*

Proof. Dealing with 0-simplices first, suppose given a diffeomorphism $f: \mathscr{M} \to \mathscr{M}$, then $f(N)$ is a prismatic neighbourhood of $f(\delta\mathscr{M}) = \delta\mathscr{M}$. By the uniqueness theorem for prismatic neighbourhoods, we can ambient isotop $f(N)$ onto N by an isotopy $h_t: \mathscr{M} \to \mathscr{M}$ such that $h_0 = 1$ and $h_1(f(N)) = N$. Then f is isotopic to $h_1 \circ f \in \mathrm{DIFF}_N(\mathscr{M})$. For the main step, take a k-simplex $f: \Delta^k \times \mathscr{M} \to \Delta^k \times \mathscr{M}$ such that $f(\partial\Delta^k \times N) = \partial\Delta^k \times N$. We wish to show that f is isotopic relative to $\partial\Delta^k \times N$ to an element of $\mathrm{DIFF}_N(\mathscr{M})[k]$ via an isotopy which at each stage commutes with projection on Δ^k. Consider $f(\Delta^k \times N)$. It is certainly a prismatic neighbourhood of $f(\Delta^k \times \delta\mathscr{M}) = \Delta^k \times \delta\mathscr{M}$. Since f commutes with projection onto Δ^k then, for each $x \in \Delta^k$, $f(\{x\} \times N)$ is a prismatic neighbourhood of $\{x\} \times \delta\mathscr{M}$. Equivalently, putting $N_x = \pi_2 f(\{x\} \times N)$ where π_2 is projection onto $\mathscr{M}$, $(N_x)_{x \in \Delta^k}$ is a continuous family of prismatic neighbourhoods of $\delta\mathscr{M}$, with $N_x = N$ for all $x \in \partial\Delta^k$.

Now prismatic neighbourhoods of $\delta\mathscr{M}$ are associated with Riemannian metrics on $\mathscr{M}$ which are adapted to $\delta\mathscr{M}$. We refer to [1] (3.3.2, Théorème 3, p. 262 or its analogue in our setting). Thus we may replace $(N_x)_{x \in \Delta^k}$ by a continuous family of Riemannian metrics $(R_x)_{x \in \Delta^k}$ such that each R_x is adapted to $\delta\mathscr{M}$ and gives rise to the prismatic neighbourhood N_x, and such that R_x takes the constant value R for $x \in \partial\Delta^k$. However, the space of all Riemannian metrics adapted to $\delta\mathscr{M}$ is a convex set in the space of all symmetric bilinear forms on $T\mathscr{M}$ (see [6], Chapter IV, for the case where $\mathscr{M}$ is unbounded), and hence is contractible. Thus the family $(R_x)_{x \in \Delta^k}$ is homotopic relative to $\partial\Delta^k$ to the constant R. At the level of prismatic neighbourhoods, this gives an isotopy of $f(\Delta^k \times N)$ to $\Delta^k \times N$ by means of an isotopy of imbeddings into $\Delta^k \times \mathscr{M}$, the isotopy being constant on $\partial\Delta^k \times N$ and commuting with projection onto Δ^k at each stage. By the Isotopy Extension Theorem, we may extend this to an isotopy $(h_t): \Delta^k \times \mathscr{M} \to \Delta^k \times \mathscr{M}$ such that $h_0 = 1$, $h_1(f(\Delta^k \times N)) = \Delta^k \times N$, and such that each h_t commutes with projection onto Δ^k. (h_t) now provides the necessary deformation of the simplex $f \in \mathrm{DIFF}(\mathscr{M})[k]$ to $h_1 \circ f \in \mathrm{DIFF}_N(\mathscr{M})[k]$, relative to its boundary. ∎

Finally we get our main theorem

THEOREM 4.6 (Relative Triangulability of smooth fibre bundles). *Let $\xi = (\mathscr{E}, B, p)$ be* DIFF *2^n-bundle with compact fibre $\mathscr{F}$. Let $\xi_0 = (\delta\mathscr{E}, B, p)$ be the* DIFF *K_{n-1}-bundle determined by the formal boundary of $\mathscr{E}$, and let $\alpha: |L| \to B$ be a* PD *triangulation. If η_0 is a* PL *K_{n-1}-bundle over L for which there exists a* PD *triangulation $\beta_0: \eta_0 \to \xi_0$ over α, then there exists a* PL *2^n-bundle $\eta = (\mathscr{X}, |L|, p)$ with formal boundary $\delta\eta = (\delta\mathscr{X}, |L|, p)$, and a* PD *triangulation $\beta: \eta \to \xi$ over α whose restriction to $\delta\eta$ is* PL *equivalent to (η_0, β_0).*

Proof. It suffices to triangulate the bundle $\alpha^*(\xi)$ relative to the triangulation already given on $\alpha^*(\xi_0)$. Let N be a prismatic neighbourhood of $\delta\mathscr{F}$. N is a product neighbourhood. Specifically, there is a homeomorphism $\delta\mathscr{F} \times [0, 1] \to N$ which is the identity on $\delta\mathscr{F} \times \{0\}$ and is a smooth immersion on each face of $\delta\mathscr{F} \times [0, 1]$, recalling that faces of $\delta\mathscr{F} \times [0, 1]$ are of the form $F \times [0, 1]$, where F is a face of $\delta\mathscr{F}$. Considering $\alpha^*(\xi) = (\mathscr{V}, |L|, \pi)$ as a bundle with structure group $\mathrm{DIFF}(\mathscr{F})$, we see from (4.5) that $\alpha^*(\xi)$ admits a reduction to $\mathrm{DIFF}_N(\mathscr{F})$. This means that we can decompose $\mathscr{V}$, the total space of $\alpha^*(\xi)$, as $\mathscr{V} = \mathscr{V}_0 \cup \mathscr{V}_1$, where $\pi: \mathscr{V}_0 \to |L|$ is a $\mathrm{DIFF}(N)$ bundle, $\pi: \mathscr{V}_1 \to |L|$ is a $\mathrm{DIFF}(\mathscr{F} - N)$ bundle, and $\pi: \mathscr{V}_0 \cap \mathscr{V}_1 \to |L|$ is a $\mathrm{DIFF}(\delta\mathscr{F})$ bundle, corresponding to the $\delta\mathscr{F} \times \{1\}$ end of N, and isomorphic to $\alpha^*(\xi_0)$. Put $\pi_i = \pi|\mathscr{V}_i$. We are given a triangulation of the $\delta\mathscr{F} \times \{0\}$ end of $\pi_0: \mathscr{V}_0 \to |L|$. By (4.4), we can construct a triangulation of $\pi_1: \mathscr{V}_1 \to |L|$. We now have a triangulation at each end of $\pi_0: \mathscr{V}_0 \to |L|$. Since the fibre of π_0 is $\delta\mathscr{F} \times [0, 1]$, then by uniqueness of PD triangulations up to isotopy, i.e. (4.1), we can extend the triangulations given at each end of π_0 to a triangulation of the whole of $\pi_0: \mathscr{V}_0 \to |L|$. Glueing the triangulations of $\pi_i: \mathscr{V}_i \to |L|$ for $i = 0, 1$, together along $\mathscr{V}_0 \cap \mathscr{V}_1$, we achieve the desired result. ∎

In proving (4.4) and (4.6) we have attained the stated goals of this paper. However, in conclusion we relate our results to those of Lashof and Rothenberg [7]. The main result of [7] is the construction of a map of classifying spaces $\mathrm{BO}_n \to \mathrm{BPL}_n$, even though there is no corresponding homomorphism of simplicial groups $\mathrm{O}_n \to \mathrm{PL}_n$. This is the famous "functorial triangulation of vector bundles" theorem. The analogous result in our context is the construction of a map $\alpha_*: \mathrm{BDIFF}(\mathscr{F}) \to \mathrm{BPL}(\mathscr{X})$, where $\alpha: \mathscr{X} \to \mathscr{F}$ is a PD triangulation of n-ads or formal boundaries. The details are very similar to the Lashof–Rothenberg construction. Put $\mathrm{PL} = \mathrm{PL}(\mathscr{X})$, $\mathrm{DIFF} = \mathrm{DIFF}(\mathscr{F})$, and $\mathrm{PD} = \mathrm{PD}(\mathscr{X}, \mathscr{F})$. Let $(\mathrm{PL} \to \mathrm{EPL} \to \mathrm{BPL})$ be the universal principal simplicial bundle with structure group PL acting freely on the right of EPL. If PL acts diagonally on $\mathrm{PD} \times \mathrm{EPL}$, then since PD/PL is contractible, so also is $(\mathrm{PD} \times \mathrm{EPL})/\mathrm{PL}$. DIFF acts freely on the left of $(\mathrm{PD} \times \mathrm{EPL})/\mathrm{PL}$ via the trivial action on EPL and composition on PD. We get a universal principal simplicial DIFF bundle $(\mathrm{DIFF} \to \mathrm{EDIFF} \to \mathrm{BDIFF})$ where $\mathrm{EDIFF} = (\mathrm{PD} \times \mathrm{EPL})/\mathrm{PL}$ and $\mathrm{BDIFF} = \mathrm{DIFF} \backslash (\mathrm{PD} \times \mathrm{EPL})/\mathrm{PL}$. The projection $\mathrm{PD} \times \mathrm{EPL} \to \mathrm{EPL}$ induces a simplicial fibre bundle $(\mathrm{DIFF} \backslash \mathrm{PD} \to \mathrm{BDIFF} \to \mathrm{BPL})$.

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Algebraic theories and varieties of functor algebras

by

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Abstract. We prove that the concept of a variety of functor algebras [13] is equivalent to that of a Linton's equational theory [9] satisfying a certain condition called locally small basedness. We show that this condition ensures reasonable properties of algebras.

0. Introduction. We shall compare two categorical approaches to algebras in the category of sets: Linton's equational theories [9] and author's varieties of functor algebras [13] restricted to the case that the base category is the category of sets. Both approaches are more general than triples in sets including also algebraic theories not admiting free algebras, such as that of complete Boolean algebras and that of complete lattices.

Linton's equational theories provide a natural and efficient generalization of Lawvere's theories [8]. The price which equational theories pay for their generality and elegance is that they include also theories which are not of nature. For instance, for the theory generated by a proper class of operations subject to no equations, no non-trivial algebra can be described by a set of data and the number of all algebras exceeds the cardinality of the universum we work in.

On the other hand, dealing with functor algebras does not lead to any non-legitimacy of that kind. Categories of functor algebras have been investigated in a lot of papers (see [11] for references) as a categorical generalization of categories of algebras of a given type. The disadvantage of the approach is that selection of varieties ([13], see section 1) in a category of functor algebras is complicated.

The basic concept of our paper is as follows: an equational theory is locally small based if it is generated by a subcategory which is locally small. The main result states that a concrete category can be represented as a variety of functor algebras iff it can be represented as the category of algebras for a locally small based equational theory, see 3.3 and 3.9. This solves the problem of the relation between the two approaches. Further, this gives a simple characterization of varieties of functor algebras. Finally, this provides a natural restrictive condition on an equational theory to ensure