The degree sets of connected infinite graphs

by

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Abstract. The degree set of a (finite or infinite) graph is the set of degrees of its vertices. We determine precisely which sets of cardinals constitute the degree set of some connected graph.

Throughout this note the concept of graph is used in the sense [2] of a graph without loops and multiple edges, except that also infinite graphs are studied here. The degree of a vertex v of a graph G is the cardinality of the set of all vertices of G adjacent to v. It is denoted by $\deg(v)$, and the set of all cardinals $\deg(v)$, where v is a vertex of G is called the *degree set of* G. The degree sequence of a denumerable graph G is the nondecreasing sequence of the cardinals $\deg(v)$. Such degree sequences were studied in [1].

A basic question immediately suggests itself: Given any set D of cardinals, which graphs (or which graphs of a special class of graphs) have D as degree set? Several results in the case of finite degree sets of finite numbers were obtained by Kapoor, Polimeni, and Wall [4] and Sipka [5].

It is trivial that every set of cardinals is the degree set of a graph. For if $D = \{a_i \mid i \in I\}$ is a set of cardinals, then the union of complete graphs G_i, $i \in I$, having pairwise disjoint vertex sets, and where G_i has a_i+1 vertices has D as its degree set. Our object is to determine precisely which sets D of cardinal numbers are the degree sets of connected graphs.

In the following, small Greek letters denote ordinals, ω the least infinite ordinal and n a nonnegative integer. We shall always make free use of the Axiom of Choice. So if D is any set of cardinal numbers we can represent it in the form,

(1) $D = \{a_\nu \mid \nu < \alpha\}$ for an ordinal α, where $\mu < \nu < \alpha$ implies $a_\mu < a_\nu$.

Using transfinite induction, (1) easily yields

$$a_\nu \geqslant |\nu| \quad \text{for} \quad \nu < \alpha. \tag{2}$$

We partition D into the subsets D_f and D_i where D_f contains all finite cardinals of D and D_i the infinite ones. The case where D_i is empty is very easy. It is settled by the following immediate observation.

LEMMA 1. *If a_ν, $0 < \nu < \omega$, are positive integers (not necessarily pairwise different) with $a_\nu > 1$ for $\nu > 1$, then there exists a rooted tree such that the root vertex r has degree a_1, all vertices of the 1-sphere of r have degree a_2, all of the 2-sphere of r have degree a_3 and so on.*

COROLLARY. *Every set D of positive integers is the degree set of some tree.*

If D is finite this follows from the lemma because we can repeat an element > 1 of D infinitely often in a_ν, $0 < \nu < \omega$. Of course, for finite D, one can derive more special results (see [4]).

It remains to discuss the situation $D_i \neq \emptyset$. We split it into two cases, namely where D_i (and hence D) has a greatest element or not.

LEMMA 2. *Let D be a set of cardinals which has a greatest infinite cardinal. Then there exists a connected graph G having D as degree set.*

Proof. Let D be represented as in (1) and $D_i = \{a_\nu | \ \beta \leqslant \nu < \alpha\}$. Then there exists an ordinal τ such that $\tau + 1 = \alpha$. For every ν satisfying $\beta \leqslant \nu < \alpha$ let G_ν be a complete graph whose vertex set V_ν has cardinality a_ν. Further we can assume that the sets V_ν, $\beta \leqslant \nu < \alpha$, are pairwise disjoint. We choose a fixed element $u_\nu \in V_\nu$. Then we construct the union of the graphs G_ν, $\beta \leqslant \nu < \alpha$, and add to it an edge linking u_τ with u_ν for every ν with $\beta \leqslant \nu < \tau$. The resulting graph G^* is connected and still has D_i as its degree set since $a+1 = a$ for infinite cardinals a and since the degree of u_τ in G^* is increased at most by $|\tau| \leqslant a_\tau$ (see (2)!). Then it is easy, using the method of Lemma 1, to implant G^* in a suitable tree for D_f by replacing a branch of this tree in such a way that the resulting graph G is connected and has D as its degree set.

For the discussion of the remaining case where D_i is non-empty but has no greatest element we have to recall some basic concepts and theorems of set theory which were stated by Hausdorff [2, p. 130–131].

According to von Neumann's approach, an ordinal α is the set of all ordinals β satisfying $\beta < \alpha$. A subset S of an ordinal α is called *cofinal* in α if for every $x \in \alpha$ there exists an element $s \in S$ such that $x \leqslant s$. If λ is a given limit ordinal, then among the (well-ordered) sets $S \subset \lambda$ which are cofinal in λ there is one whose corresponding ordinal type is minimal. This latter is denoted by $\mathrm{cf}(\lambda)$. A limit ordinal λ is called *regular* if $\mathrm{cf}(\lambda) = \lambda$, otherwise λ is *singular*. A regular limit ordinal must be an initial ordinal ω_α. An initial number ω_α with $\alpha > 0$ is said to be *weakly inaccessible* if ω_α is regular and α a limit number. All ω_α where α is not a limit number are regular. If $\alpha > 0$ is a limit number then ω_α is regular only if $\omega_\alpha = \alpha$.

LEMMA 3. *Let $D = \{a_\nu | \ \nu < \lambda\}$ be an infinite set of cardinals where $a_\mu < a_\nu$ for $\mu < \nu < \lambda$ and D has at least one infinite cardinal, but no greatest cardinal, so that λ is a limit ordinal. We put $\omega_\gamma = \mathrm{cf}(\lambda)$. If there exists an ordinal $\tau < \lambda$ such that $a_\tau \geqslant \aleph_\gamma$, then there exists a connected graph having D as its degree set.*

Proof. The set D_i of infinite cardinals of D is representable as $D_i = \{a_\nu | \ \beta \leqslant \nu < \lambda\}$ for a suitable β. Let $B = \{\nu | \ \beta \leqslant \nu < \lambda\}$. Then there exists a subset $C \subset B$ of order-type ω_γ which is cofinal in λ. For $\nu \in B$ let G_ν be a complete graph of order a_ν. We can assume that the corresponding vertex sets V_ν, $\nu \in B$, are pairwise disjoint. For every $\nu \in B$ we choose an element $u_\nu \in V_\nu$. We write U for the union graph of the graphs G_ν, $\nu \in B$. Now let τ be an ordinal satisfying our assumption. Without loss of generality we can assume that $\tau \in C$. Then for every $c \in C$ which is different from τ we join u_c with u_τ. So there are $|C| = \aleph_\gamma$ additional edges having u_τ as vertex. But this does not increase the degree $a_\tau \geqslant \aleph_\gamma$ of u_τ in U.

Further, if $\mu \in B \setminus C$, we join u_μ with u_ϱ, where ϱ is the first ordinal in C which satisfies $\mu < \varrho$. For every $\varrho \in C$ this introduces at most $|\varrho| \leqslant a_\varrho$ new edges having u_ϱ as vertex. But this does not increase the degree a_ϱ of u_ϱ. Also the degrees of the vertices u_μ with $\mu \in B \setminus C$ are not changed by one additional edge. So we have constructed a connected graph having D_i as degree set. As in Lemma 2 we can now accomplish the proof for the entire degree set D.

Now we can give a complete characterization of those sets of cardinals which are the degree set of a connected graph as our main result.

THEOREM 1. *Let $D = \{a_\nu | \ \nu < \lambda\}$ be a set of cardinals, where $\mu < \nu < \lambda$ implies $a_\mu < a_\nu$. If λ is a regular initial number ω_α with $\alpha > 0$ and $a_\nu < \aleph_\alpha$ for all $\nu < \lambda$ then D is not the degree set of a connected graph. In all other cases D is the degree set of a connected graph.*

Proof. According to the three lemmas, we can restrict ourselves to the discussion of the following case:

λ is a limit ordinal and D contains infinite cardinals.

Now, if λ is singular, then because of (2) there exists an ordinal $\tau < \lambda$ such that $a_\tau \geqslant \aleph_\gamma = |\mathrm{cf}(\lambda)|$. Hence by Lemma 3 there exists a connected graph with degree set D.

Suppose now that λ is regular. Then λ is a regular initial number ω_α. If $\alpha = 0$ there is an ordinal $\tau < \lambda = \omega$ such that $a_\tau \geqslant \aleph_0$ and again Lemma 3 can be applied.

So finally we have to handle the case where λ is a regular initial ordinal ω_α with $\alpha > 0$ and $a_\nu < \aleph_\alpha$ for all $\nu < \lambda$.

Assume that G is a connected graph having D as degree set. Its vertex set V clearly satisfies $|V| \geqslant \aleph_\alpha$. We well-order V and consider the subset V^* of the first λ vertices in this well-ordering:

$$V^* = \{v_\tau | \ \tau < \lambda\}.$$

Since G is connected, for every τ with $0 < \tau < \lambda$ there is a path joining v_τ with v_0. The lengths of these paths are natural numbers, and since λ is regular it follows from the Pigeonhole Principle that there are $\aleph_\alpha$ pairwise different vertices w_τ, $\tau < \lambda$, such that the corresponding paths P_τ joining w_τ with v_0 all have equal length, say n.

Let $w_\tau = b_\tau^0, b_\tau^1, \ldots, b_\tau^n = v_0$ be the vertices of P_τ, listed in their path order. Then the set $\{b_\tau^{n-1} | \tau < \lambda\}$ has cardinality less than $\aleph_\alpha$ because $\deg(v_0) < \aleph_\alpha$. Again according to the Pigeonhole Principle there exists a subset $S_1 \subset \lambda$ with $|S_1| = \aleph_\alpha$ such that all b_τ^{n-1}, $\tau \in S_1$, are equal. Repeating this process we obtain a set $S_2 \subset S_1 \subset \lambda$ with $|S_2| = \aleph_\alpha$ such that all b_τ^{n-2}, $\tau \in S_2$, are equal, and after $n-1$ steps we arrive at a set $S_{n-1} \subset \lambda$ of cardinality $\aleph_\alpha$ such that all b_τ^1, $\tau \in S_{n-1}$, are equal, say $= z$. But since z is adjacent to all w_τ, $\tau \in S_{n-1}$, this contradicts the inequality $\deg(z) < \aleph_\alpha$.

Using the concept of weakly inaccessible numbers we can reformulate our theorem as follows:

THEOREM 2. *Let $D = \{a_\nu | \nu < \lambda\}$ be a set of cardinals, where $\mu < \nu < \lambda$ implies $a_\mu < a_\nu$. Then there exists a connected graph having D as degree set in all cases with the exception of the following: λ is a weakly inaccessible ordinal ω_α and $a_\nu < \aleph_\alpha$ for all $\nu < \lambda$.*

Proof. According to Theorem 1 in the exceptional case λ is a regular initial ordinal ω_α with $\alpha > 0$ and $a_\nu < \aleph_\alpha$ for all $\nu < \lambda$. Then α must be a limit ordinal number. Otherwise there would exist an ordinal β such that $\alpha = \beta+1$, and we would have $a_\nu \leqslant \aleph_\beta$ for $\nu < \lambda$, which gives $|D| \leqslant \aleph_0 + |\beta| \leqslant \aleph_\beta < \aleph_\alpha$ contradicting $|D| = |\lambda| = \aleph_\alpha$. Now a regular initial ordinal ω_α with a limit number α as index is weakly inaccessible. Hence our statement follows from Theorem 1.

References

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