On a construction of perfectly normal spaces and its applications to dimension theory

by

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Abstract. We present a method of construction of perfectly normal spaces by a modification of topology of metrizable spaces. We apply this method to obtain some examples in the dimension theory of perfectly normal spaces.

1. Introduction. Let (X, ϱ) be a metrizable space, and let α be an ordinal number. Suppose we have a family $\{X_\beta: \beta < \alpha\}$ of closed subsets of X such that

(1) $$\emptyset = X_0 \subset X_1 \subset \ldots X_\beta \subset \ldots \subset X;$$

(2) $X_\beta = \mathrm{cl}_\varrho(\bigcup_{\gamma<\beta} X_\gamma)$ for every limit ordinal $\beta < \alpha$; and

(3) $$X = \bigcup_{\beta<\alpha} X_\beta.$$

Define a finer topology τ on X by taking the family

$$\{U \cap X_\beta: U \text{ is open in } (X, \varrho) \text{ and } \beta \leqslant \alpha\}$$

as a base for τ. In [13], R. Pol proved that if $\alpha = \omega_1$ and each X_β is separable, then (X, τ) is perfectly normal and collectionwise normal, but need not be subparacompact. In this paper, we give a relatively easy proof that (X, τ) is perfectly normal and collectionwise normal in the general case. We then apply this result to obtain some examples in dimension theory. In particular, we obtain perfectly normal spaces which are locally weakly infinite dimensional (resp., countable dimensional or 0-countable dimensional) but not weakly infinite dimensional (resp., countable dimensional or 0-countable dimensional)(1). We also give an example of a perfectly normal, locally second countable space X with $\mathrm{loc}\dim X = 0$ which is not strongly countable

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(1) See Section 3 for the definitions of these notions.

dimensional. In these constructions we use the methods developed in [8], [9] and [10].

Our terminology follows [2], [3], and [5].

2. In this section, we prove that the space (X, τ) defined in the introduction is perfectly normal and collectionwise normal.

In our proof, the function $\varkappa\colon X \to \alpha$ by $\varkappa(x) = \min\{\beta\colon x \in X_\beta\}$ will be useful. Note that each point $x \in X$ has a local base in (X, τ) consisting of sets of the form $U \cap X_{\varkappa(x)}$, where U is an open set in (X, ϱ) containing x.

Proof that (X, τ) is perfectly normal. Let $\{\mathscr{G}_n\}_{n\in\omega}$ be a development for (X, ϱ) such that for each $n \in \omega$, $\mathscr{G}_{n+1}$ is a star-refinement of $\mathscr{G}_n$, and each element of $\mathscr{G}_{n+1}$ meets only finitely many members of $\mathscr{G}_n$.

Suppose $H \subset X$ is closed in (X, τ). For each $G \in \mathscr{G}_n$, let $\alpha(G) = \sup\{\beta < \alpha\colon G \cap X_\beta \cap H = \emptyset\}$. If $G \cap X_{\alpha(G)} \cap H \neq \emptyset$, note that the cofinality of $\alpha(G)$, denoted $\mathrm{cf}(\alpha(G))$, is countable (since $X_\gamma = \bigcup_{\beta<\gamma} X_\beta$ if $\mathrm{cf}(\gamma) > \omega$). In this case, let $\{\alpha_m(G)\}_{m\in\omega}$ be an increasing sequence of ordinals converging to $\alpha(G)$. On the other hand, if $G \cap X_{\alpha(G)} \cap H = \emptyset$, let $\alpha_m(G) = \alpha(G)$ for each $m \in \omega$.

Define $\mathscr{U}_{n,m} = \{G - X_{\alpha_m(G)}\colon G \in \mathscr{G}_n\}$. Let $U_{n,m} = \bigcup \mathscr{U}_{n,m}$. We wish to show that

$$H = \bigcap_{n,m\in\omega} U_{n,m} = \bigcap_{n,m\in\omega} \mathrm{cl}_\tau U_{n,m}.$$

Let $x \in H$, and fix $n, m \in \omega$. There exists $G \in \mathscr{G}_n$ with $x \in G$. Since $G \cap X_{\alpha_m(G)} \cap H = \emptyset$, $x \notin X_{\alpha_m(G)}$. Thus $x \in G - X_{\alpha_m(G)} \subset U_{n,m}$, and so $H \subset \bigcap_{n,m\in\omega} U_{n,m}$. Now let $x \in \bigcap_{n,m\in\omega} \mathrm{cl}_\tau U_{n,m}$, and suppose $x \notin H$. Then there exists a set V open in (X, ϱ) containing x such that $V \cap X_{\varkappa(x)} \cap H = \emptyset$. Choose $n \in \omega$ such that $\mathrm{st}(\mathrm{st}(x, \mathscr{G}_n), \mathscr{G}_n) \subset V$, and choose $G \in \mathscr{G}_{n+1}$ containing x.

Let $G_0, G_1, \ldots, G_k$ be the elements of $\mathscr{G}_n$ which meet G. For each $i \leqslant k$, we have $G_i \cap X_{\varkappa(x)} \cap H = \emptyset$, and so $\alpha(G_i) \geqslant \varkappa(x)$. If $\alpha(G_i) \gneqq \varkappa(x)$, there exists $m_i \in \omega$ such that $\alpha_{m_i}(G_i) > \varkappa(x)$, and so $X_{\varkappa(x)} \cap (G_i - X_{\alpha_{m_i}(G_i)}) = \emptyset$. If $\alpha(G_i) = \varkappa(x)$, then $\alpha_m(G_i) = \alpha(G_i) = \varkappa(x)$ for all $m \in \omega$. In either case, then, there exists $m_i \in \omega$ such that $X_{\varkappa(x)} \cap (G_i - X_{\alpha_{m_i}(G_i)}) = \emptyset$. Hence if $m \geqslant \max\{m_i\colon i \leqslant k\}$, we have $G \cap X_{\varkappa(x)} \cap U_{n,m} = \emptyset$. Then $x \notin \mathrm{cl}_\tau U_{n,m}$, contradiction. Thus $x \in H$, and the proof is finished.

Proof that (X, τ) is collectionwise normal. Let $\{\mathscr{G}_n\}_{n\in\omega}$ be as above, and let $\mathscr{H}$ be a discrete collection of closed sets in (X, τ). For each $G \in \mathscr{G}_n$, let

$$\gamma(G) = \sup\{\beta < \alpha\colon (\mathrm{cl}_\varrho G) \cap X_\beta \text{ meets at most one element of } \mathscr{H}\}.$$

If $(\mathrm{cl}_\varrho G) \cap X_{\gamma(G)}$ meets more than one element of $\mathscr{H}$, then $\mathrm{cf}(\gamma(G)) = \omega$, and

we choose an increasing sequence $\{\gamma_m(G)\}_{m\in\omega}$ of ordinals converging to $\gamma(G)$. Otherwise let $\gamma_m(G) = \gamma(G)$ for each $m \in \omega$.

Let $\mathscr{V}_{n,m} = \{G \cap X_{\gamma_m(G)}\colon G \in \mathscr{G}_n\}$. Then $\mathscr{V}_{n,m}$ is locally finite, and the closure of each member of $\mathscr{V}_{n,m}$ meets at most one element of $\mathscr{H}$. Also, if $x \in X$, there exists $n \in \omega$ and $x \in G \in \mathscr{G}_n$ such that $(\mathrm{cl}_\varrho G) \cap X_{\varkappa(x)}$ meets at most one element of $\mathscr{H}$. Hence $\gamma(G) \geqslant \varkappa(x)$, and so there exists $m \in \omega$ with $\gamma_m(G) \geqslant \varkappa(x)$. Thus $\bigcup_{n,m\in\omega} \mathscr{V}_{n,m}$ covers X.

Let $\{\mathscr{V}_{n,m}\colon n, m \in \omega\} = \{\mathscr{W}_n\colon n \in \omega\}$. For $x \in H \in \mathscr{H}$, let n_x be the least integer such that $x \in \bigcup \mathscr{W}_{n_x}$. Choose $W_x \in \mathscr{W}_{n_x}$ such that $x \in W_x$.

Let $W(x) = W_x - \mathrm{cl}_\tau\{\bigcup W'\colon W' \in \mathscr{W}_k,\ (\mathrm{cl}_\tau W') \cap H = \emptyset, \text{ and } k \leqslant n_x\}$. Let $U_H = \bigcup_{x\in H} W(x)$. It is routine to verify that $\{U_H\colon H \in \mathscr{H}\}$ is a disjoint collection of open sets with $H \subset U_H$ for each $H \in \mathscr{H}$. Thus (X, τ) is collectionwise normal.

3. Applications to dimension theory. By the dimension we understand the covering dimension dim. A Tychonoff space X is said to be

a) *countable-dimensional* (c.d.) iff $X = \bigcup_{i=1}^{\infty} X_i$, where $\dim X_i < \infty$,

b) 0-*countable-dimensional* (0-c.d.) iff $X = \bigcup_{i=1}^{\infty} X_i$, where $\dim X_i \leqslant 0$,

c) *strongly countable-dimensional* (s.c.d) iff $X = \bigcup_{i=1}^{\infty} F_i$ where F_i is closed in X and $\dim F_i < \infty$.

We say that a family $\{(A_i, B_i)\}_{i=1}^{\infty}$ of pairs of disjoint closed subsets of a space X is *essential* if whenever L_i is a partition between A_i and B_i in X, then $\bigcap_{i=1}^{\infty} L_i \neq \emptyset$. A space X is said to be *strongly infinite dimensional* (s.i.d.) iff there exists an essential family $\{(A_i, B_i)\}_{i=1}^{\infty}$ in X; X is *weakly infinite-dimensional* (w.i.d.) if it is not strongly infinite-dimensional. Note that every hereditarily normal c.d. space is w.i.d. (see [1; Ch. 10, § 5, Th. 21]). We say that a space X is *locally w.i.d.* (resp., c.d., 0-c.d., s.c.d.) if each point $x \in X$ has a closed neighbourhood which is w.i.d. (resp., c.d., 0-c.d., s.c.d.).

We consider an ordinal to be the set of its predecessors, and cardinals to be initial ordinals of a given cardinality. If λ is a cardinal, then let $D(\lambda)$ and λ denote the spaces of all ordinals less than λ with the discrete topology and the order topology, respectively. A set $S \subset \lambda$ *is stationary* if S meets every closed unbounded subset of λ. The space $D(\lambda)^\omega$ is called the Baire space of weight λ and denoted by $B(\lambda)$.

Now suppose that λ is a regular cardinal, (X, ϱ) is a metrizable space of weight λ and $\{X_\beta\colon \beta < \lambda\}$ is a family of subsets of X satisfying conditions (1)-(3) such that

(4) $$w(X_\beta, \varrho|X_\beta) < \lambda \quad \text{for every } \beta < \lambda.$$

Then the new topology τ determined by this family described in the Introduction has the following property

(5) for each set $A \subset X$ the set $\varkappa(\mathrm{cl}_\varrho A \setminus \mathrm{cl}_\tau A)$ is not stationary in λ.

To show this, it suffices to replace ω_1 by λ in the proof of Lemma 2 of [14].

As shown in [11; Example] if λ is a regular cardinal, and ϱ is the usual topology of the Baire space $B(\lambda)$, then family

(6) $$\{B(\beta)\colon \beta < \lambda\}$$

of subsets of $B(\lambda)$ satisfies conditions (1)-(3). In the sequel we will denote by $\tilde{B}(\lambda)$ the set $B(\lambda)$ with the topology τ determined by the family (6). As shown in Section 2, the space $\tilde{B}(\lambda)$ is perfectly normal and collectionwise normal. For any set $S \subset \lambda$ we put $B(S) = \varkappa^{-1}(S)$, where $\varkappa\colon B(\lambda) \to \lambda$ is defined as in Section 2, i.e. $\varkappa(x) = \min\{\beta\colon x \in B(\beta)\}$. We put

$$B_\alpha = B(\alpha) \setminus \bigcup_{\beta<\alpha} B(\beta) \quad \text{for} \quad \alpha < \lambda.$$

LEMMA 1. *Let λ be a regular cardinal and M be a metrizable separable space. If U is an open subspace of the space $\tilde{B}(\lambda) \times M$, then there exists a nonstationary set $K \subset \lambda$ such that the set $U \setminus (B(K) \times M)$ is open in the subspace $(B(\lambda) \setminus B(K)) \times M$ is the space $B(\lambda) \times M$.*

Proof. Let $\{U_i\}_{i=1}^{\infty}$ be a countable base of the space M. For each $i \in N$ let V_i be the maximal open subset of the space $\tilde{B}(\lambda)$ satisfying $V_i \times U_i \subset U$. By (5), there exists a nonstationary set $K_i \subset \lambda$ such that the set $V_i \setminus B(K_i)$ is open in the subspace $B(\lambda) \setminus B(K_i)$ of the Baire space $B(\lambda)$. The set $K = \bigcup_{i=1}^{\infty} K_i$ satisfies the required conditions.

LEMMA 2. *Let B be a perfectly normal space and X be a subspace of the Cartesian product $B \times I^\omega$ of B and the Hilbert cube I^ω such that every open subset of $B \times I^\omega$ which contains X contains a set $\{x_0\} \times I^\omega$ for some $x_0 \in B$. Then X is strongly infinite-dimensional.*

Proof. Let $\{(A_i, B_i)\}_{i=1}^{\infty}$ be an essential family in I^ω. The sets $B \times A_i$ and $B \times B_i$ are disjoint closed subsets of a perfectly normal space $B \times I^\omega$; hence there exist open subsets U_i and V_i of $B \times I^\omega$, containing $B \times A_i$ and $B \times B_i$ respectively such that $\mathrm{cl}\, U_i \cap \mathrm{cl}\, V_i = \emptyset$. We shall show that the family $\{(\mathrm{cl}\, U_i \cap X, \mathrm{cl}\, V_i \cap X)\}_{i=1}^{\infty}$ is essential in X. Suppose that for $i = 1, 2, \ldots$, there exists a partition L_i between the sets $\mathrm{cl}\, U_i \cap X$ and $\mathrm{cl}\, V_i \cap X$ in X such that $\bigcap_{i=1}^{\infty} L_i = \emptyset$. Then by Lemma 1.2.9 of [3], there exists a partition L'_i between the sets $B \times A_i$ and $B \times B_i$ in $B \times I^\omega$ such that $L'_i \cap X \subset L_i$. We have $\bigcap_{i=1}^{\infty} L'_i \cap X = \emptyset$, hence the set $U = (B \times I^\omega) \setminus \bigcap_{i=1}^{\infty} L'_i$ is an open subset of $B \times I^\omega$ that contains X. Take $x_0 \in B$ such that $\{x_0\} \times I^\omega \subset U$. Then for each $i = 1, 2, \ldots$ the set $L''_i = L'_i \cap (\{x_0\} \times I^\omega)$ is a partition between $\{x_0\} \times A_i$ and $\{x_0\} \times B_i$ in the space $\{x_0\} \times I^\omega$ such that $\bigcap_{i=1}^{\infty} L''_i = \emptyset$, which is a contradiction.

EXAMPLE 1. There exists a subspace Y of the Cartesian product $\tilde{B}(c^+) \times I^\omega$, where c^+ is the first cardinal after the continuum c such that

(a) every open subset of $\tilde{B}(c^+) \times I^\omega$ that contains Y contains a set $\{x_0\} \times I^\omega$ for some $x_0 \in B$ and

(b) for each $\alpha < c^+$ the subspace $(B_\alpha \times I^\omega) \cap Y$ has dimension zero.

We apply the construction given in [9]: let $\{S_\alpha\colon \alpha < c\}$ be a decomposition of c^+ into c disjoint stationary sets and let $I^\omega = \{x_\alpha\colon \alpha < c\}$; then we put

$$Y = \bigcup_{\alpha<c} (B(S_\alpha) \times \{x_\alpha\}).$$

It was shown in [9] that the set Y with the subspace topology ϱ' of $B(c^+) \times I^\omega$ is a strongly metrizable space such that for every open subset V of $B(c^+) \times I^\omega$ that contains Y there exists $x_0 \in B(c^+)$ such that $\{x_0\} \times I^\omega \subset V$ (thus (Y, ϱ') is strongly infinite-dimensional by Lemma 2), but each separable subspace of (Y, ϱ') is zero-dimensional.

We will show that the same set Y with the subspace topology of $\tilde{B}(c^+) \times I^\omega$ satisfies the condition (a); the condition (b) is obviously satisfied. Let U be an open subspace of $\tilde{B}(c^+) \times I^\omega$ containing Y. By Lemma 1 there exists a nonstationary subset K of c^+ such that the set $U \setminus (B(K) \times I^\omega)$ is open in the subspace $(B(c^+) \setminus B(K)) \times I^\omega$ of the Cartesian product $B(c^+) \times I^\omega$ of the Baire space $B(c^+)$ and the Hilbert cube. Let U' be an open subset of $B(c^+) \times I^\omega$ such that $U' \setminus (B(K) \times I^\omega) = U \setminus (B(K) \times I^\omega)$. For each $\alpha < c$ the set $U'_\alpha = \{y \in B(c^+)\colon (y, x_\alpha) \in U'\}$ is an open subspace of $B(c^+)$ containing $B(S_\alpha \setminus K)$. Now, the intersection $\bigcap_{\alpha<c} U'_\alpha$ contains a set $B(S)$ for some stationary set S; for the proof see [7; Corollary 3.5] and [13; Section 2.2 and 3.5], compare also [10; Lemma 1.3]. Thus $S \setminus K \neq \emptyset$ and for $x_0 \in B(S \setminus K)$ we have $\{x_0\} \times I^\omega \subset U$, which finishes the proof.

EXAMPLE 2. There exist perfectly normal spaces X_0, X_1 and X_2 such that

a) X_0 is locally w.i.d., but is not w.i.d.,

b) X_1 is locally c.d., but is not c.d.,

c) X_2 is locally 0-c.d., but is not 0-c.d.

Let Y be a space from Example 1. By Lemma 2, Y is strongly infinite-dimensional, hence is not c.d. and is not 0-c.d. Let $\alpha_0 = \min\{\beta\colon Y\cap(B(\beta)\times I^{\omega})$ is strongly infinite-dimensional$\}$. We have $\mathrm{cf}(\alpha_0) > \omega$, because, by a theorem of Levšenko (see [1; Ch. 10, § 5, Theorem 21]), a perfectly normal space which is the union of countably many w.i.d. subspaces is w.i.d. Thus, the space $X_0 = Y\cap(B(\alpha_0)\times I^{\omega})$ is locally w.i.d., but is not w.i.d. Similarly, if we take $\alpha_1 = \min\{\beta\colon Y\cap(B(\beta)\times I^{\omega})$ is not c.d.$\}$ and $\alpha_2 = \min\{\beta\colon Y\cap(B(\beta)\times I^{\omega})$ is not 0-c.d.$\}$, then the spaces $X_1 = Y\cap(B(\alpha_1)\times I^{\omega})$ and $X_2 = Y\cap(B(\alpha_2)\times I^{\omega})$ have the required properties.

In a similar way, we can construct a perfectly normal locally s.c.d. space, which is not s.c.d., but we have a stronger example:

EXAMPLE 3. There exists a perfectly normal, locally second-countable space X with $\operatorname{loc}\dim X = 0$ which is not strongly countable dimensional.

There exists a compact metrizable countable-dimensional space Z which is the union of a family $\{I^i\}_{i\in N}$ of disjoint subsets homeomorphic with i-dimensional Euclidean cubes and of a subset $P = Z\setminus \bigcup_{i=1}^{\infty} I^i$ homeomorphic with the space of irrationals, and such that

(7) each open subset of Z contains infinitely many of the cubes I^i

(see [1; Ch. 10, § 3] or [4; Remark 1.4]). For each $i = 1, 2, \ldots$ and $m = 0, 1, \ldots, i$ let R_i^m be the set of points in I^i exactly m of whose coordinates are rational; we have $I^i = \bigcup_{m=0}^{i} R_i^m$ and $\dim R_i^m = 0$ (see [2; Ex. 7.2.11]). Let us split ω_1 into countably many disjoint stationary sets S_i, $i = 1, 2, \ldots$ and let $S_i = \bigcup_{m=0}^{i} S_i^m$, where S_i^m are also disjoint and stationary sets for $m = 0, 1, \ldots, i$. Let

$$X = \bigcup_{i=1}^{\infty}\bigcup_{m=0}^{i} [B(S_i^m)\times(R_i^m\cup P)] \subset \tilde{B}(\omega_1)\times Z$$

be the subspace of the Cartesian product of the space $\tilde{B}(\omega_1)$ and Z. Then the space X is perfectly normal (see [2; Problem 4.5.16]). For each $\xi < \omega_1$, the subspace

$$(B(\xi)\times Z)\cap X = \bigcup_{i=1}^{\infty}\bigcup_{m=0}^{i} \cup\{B_\alpha\times(R_i^m\cup P)\colon \alpha\leqslant\xi \text{ and } \alpha\in S_i^m\}$$

is a second-countable and zero-dimensional (as the union of countably many closed zero-dimensional sets of the form $B_\alpha\times(R_i^m\cup P)$) open subset of X.

Thus, the space X is locally second-countable and zero-dimensional.

We shall prove now that the space X is not strongly countable-dimensional. Let $Y = \tilde{B}(\omega_1)\times P$ be the subspace of X and let $Y' = B(\omega_1)\times P$

be a metrizable space which is the Cartesian product of the Baire space $B(\omega_1)$ and the space P. We will show that

(8) If $F_1, F_2, \ldots$ is a sequence of closed subsets of Y such that $Y = \bigcup_{i=1}^{\infty} F_i$ then there exist an open subset U of the space Y' and a nonstationary set $K\subset\omega_1$ such that $U\setminus(B(K)\times P)\subset F_{i_0}$ for some $i_0\in N$.

Indeed, let $\bar{F}_i$ be the closure of the set F_i in the space Y'. Since the space Y' is completely metrizable, there exists a set U open in Y' such that $U\subset\bar{F}_{i_0}$ for some $i_0\in N$. By Lemma 1, there exists a nonstationary set $K\subset\omega_1$ such that $\bar{F}_{i_0}\setminus F_{i_0}\subset B(K)\times P$. Then

$$U\setminus(B(K)\times P)\subset\bar{F}_{i_0}\setminus(B(K)\times P)\subset\bar{F}_{i_0}\setminus(\bar{F}_{i_0}\setminus F_{i_0})\subset F_{i_0}.$$

To show that X is not strongly countable-dimensional suppose on the contrary that $X = \bigcup_{i=1}^{\infty} F_i$, where F_i are closed in X and $\dim F_i < \infty$. Then $Y = \bigcup_{i=1}^{\infty}(F_i\cap Y)$ and thus by (8) there exists an open subset U of the space Y' and a nonstationary set $K\subset\omega_1$ such that $U\setminus(B(K)\times P)\subset F_{i_0}$ for some $i_0\in N$. Let us take an open subset V of the space $B(\omega_1)$ and an open subset W of P such that $(V\setminus B(K))\times W\subset U\setminus(B(K)\times P)\subset F_{i_0}$. Since the set P is dense in Z, then there exists a non-void set W' open in Z which is contained in $\mathrm{cl}_Z W$. We have

$$X\cap[(V\setminus B(K))\times W']\subset \mathrm{cl}_X[(V\setminus B(K))\times W]\subset F_{i_0}.$$

By the properties of the space Z there exist $i_1 < i_2 < \ldots$ such that $W'\supset I^{i_k}$ for each $k\in N$. Then $X\cap[(V\setminus B(K))\times I^{i_k}]\subset F_{i_0}$ for each $k\in N$. We shall show that for every $k\in N$

$$\dim(X\cap[(V\setminus B(K))\times I^{i_k}])\geqslant i_k$$

which contradicts the assumption that F_{i_0} is finite dimensional.

The set V contains an open subset $B'(\omega_1)$ of $B(\omega_1)$ of the form

$$B'(\omega_1) = \{x = (\xi_k)\in B(\omega_1)\colon \xi_k = \alpha_k \text{ for } i = 1, 2, \ldots, n_0\}$$

for some $n_0\in N$ and $\alpha_1, \alpha_2, \ldots, \alpha_{n_0}\in D(\omega_1)$.

Let $\alpha_0 = \max\{\alpha_1, \alpha_2, \ldots, \alpha_{n_0}\}$ and denote $L = \{\xi<\omega_1\colon \xi\leqslant\alpha_0\}$. Let $h\colon B'(\omega_1)\to B(\omega_1)$ be a homeomorphism defined by the formula $h((\xi_k)) = (\xi'_k)$, where $\xi'_k = \xi_{k+n_0}$ for $k\in N$. For an arbitrary set $S\subset\omega_1\setminus L$ we have

$B(S)\cap B'(\omega_1)=h^{-1}(B(S))$. Thus, we have

$$\begin{aligned} X\cap[(V\setminus B(K))\times I^{i_k}] &\supset [\bigcup_{m=0}^{i_k} B(S_{i_k}^m)\times R_{i_k}^m]\cap[(B'(\omega_1)\setminus B(K))\times I^{i_k}]\\ &\supset \bigcup_{m=0}^{i_k} [(B(S_{i_k}^m)\setminus K\setminus L)\cap B'(\omega_1))\times R_{i_k}^m]\\ &= \bigcup_{m=0}^{i_k} h^{-1}(B(S_{i_k}^m\setminus K\setminus L))\times R_{i_k}^m\\ &\underset{\text{top}}{=} \bigcup_{m=0}^{i_k} B(S'_m)\times R_{i_k}^m \end{aligned}$$

where the sets $S'_m=S_{i_k}^m\setminus K\setminus L$ for $m=0,1,\ldots,i_k$ are disjoint stationary subsets of ω_1.

Let $X_{i_k}=\bigcup_{m=0}^{i_k} B(S'_m)\times R_{i_k}^m$; as was shown in [11] we have $\dim X_{i_k}\geqslant i_k$, which completes the proof.

By a theorem of Dowker (see [8; Theorem 11.17]), for any hereditarily normal space X with $\operatorname{loc}\dim X=0$ there exists a hereditarily normal space X^* obtained by adding a point to a space X such that $\dim X^*=\operatorname{loc}\dim X$. Thus, we obtain

EXAMPLE 4. There exists a hereditarily normal Lindelöf space X^* with $\dim X^*=0$ containing a perfectly normal and locally second-countable subspace X which is not strongly countable-dimensional.

Remark 1. It is a question whether there is an analogue of the theorem of Dowker for countable-dimensional, 0-c.d. or s.c.d. spaces. In particular, we do not know whether it is possible to add a point to the space X_1 (or X_2) from Example 3 in such a way that the obtained space is hereditarily normal and countable-dimensional (0-countable-dimensional). It is easy to see that we can obtain a hereditarily normal w.i.d. space $X_0^*=X_0\cup\{p\}$ containing X_0. However, there exists a much better example showing that weakly infinite-dimensionality is not monotone: R. Pol [15] constructed a metrizable separable w.i.d. space containing a subspace which is not w.i.d. (this subspace is not, however, open).

Let us note that every closed subspace of a normal w.i.d. (s.c.d.) space is w.i.d. (s.c.d.) and every closed subspace of a hereditarily normal countable-dimensional (0-c.d.) space is c.d. (0-c.d.). We do not know the answer to the question whether every closed subspace of a normal c.d. (0-c.d.) space is c.d. (0-c.d.). However, one can give an example of a Tychonoff space of dimension zero, containing a functionally closed subspace which is not w.i.d. (resp. c.d., 0-c.d., s.c.d.) – see [5]).

Remark 2. Under the assumption of the continuum hypothesis there exists a perfectly normal, locally compact, locally countable, hereditarily separable and first countable space X which is strongly infinite-dimensional (hence, is not c.d.)(2). The one point compactification X^* of the space X is the example of a compact, hereditarily normal and hereditarily separable space of dimension zero, containing a perfectly normal subspace X which is not w.i.d. (hence is not c.d.).

Let us take the Hilbert cube I^ω with the usual topology ϱ and let τ be a finer perfectly normal topology in I^ω obtained by method described in [6; § 1]. We will show that (I^ω,τ) is strongly infinite-dimensional. Let $\{(A_i,B_i)\}_{i=1}^{\infty}$ be an essential family in (I^ω,ϱ) and let U_i and V_i be open subsets of (I^ω,ϱ) containing A_i and B_i respectively such that $\operatorname{cl}U_i\cap\operatorname{cl}V_i=\emptyset$. We will show that the family $\{(\operatorname{cl}_\varrho(U_i),\operatorname{cl}_\varrho(V_i))\}_{i=2}^{\infty}$ is essential in (I^ω,τ). Let L_i be a partition in (I^ω,τ) between the sets $\operatorname{cl}_\varrho U_i$ and $\operatorname{cl}_\varrho V_i$ for $i=2,3,\ldots$ Then there are closed subsets C_i and D_i of (I^ω,τ) such that $\operatorname{cl}_\varrho U_i\subset C_i$, $\operatorname{cl}_\varrho V_i\subset D_i$, $C_i\cup D_i=I^\omega$ and $C_i\cap D_i=L_i$. Therefore the set $L'_i=\operatorname{cl}_\varrho C_i\cap\operatorname{cl}_\varrho D_i$ is a partition between the sets A_i and B_i in (I^ω,ϱ) for each $i=2,3,\ldots$ and hence $\dim(\bigcap_{i=1}^{\infty} L'_i)>0$. Since the topology τ has the property that $|\operatorname{cl}_\varrho A\setminus\operatorname{cl}_\tau A|\leqslant\omega$ for any $A\subset X$, we have $|\bigcap_{i=1}^{\infty} L'_i\setminus\bigcap_{i=1}^{\infty} L_i|\leqslant\omega$ and thus $\bigcap_{i=1}^{\infty} L_i\neq\emptyset$.

Note that the above proof shows that if (X,ϱ) is a metrizable separable strongly infinite-dimensional space, then Kunen's modification (X,τ) of this space is strongly infinite-dimensional.

Remark 3. Let (X,τ) be the modification of a metrizable space (X,ϱ) described in the Introduction, with α a cardinal. Then, under some additional assumptions about the cardinal α and on the dimension of subspaces of (X,ϱ) of weight less than α, the dimensional properties of (X,τ) are closely related to the dimensional properties of the space (X,ϱ).

Namely, we have the following:

PROPOSITION. *Let Y be a subset of X. Then*

a) *if $\alpha=\omega_1$, then*

$$\dim(Y,\tau|Y)\leqslant\operatorname{Ind}(Y,\tau|Y)\leqslant\operatorname{Ind}(Y,\varrho|Y)+1=\dim(Y,\varrho|Y)+1$$

and if $\dim(Y,\varrho)=0$, then $\dim(Y,\tau)=0$,

b) *if every subspace of (Y,ϱ) of weight $<\alpha$ has dimension not greater than m, then*

$$\dim(Y,\tau)\geqslant\dim(Y,\varrho)-m-1,$$

c) *if every subspace of (Y,ϱ) of weight α is weakly infinite-dimensional, then if (Y,ϱ) is strongly infinite-dimensional, then so is (Y,τ).*

(2) This remark was also made by van Douwen.

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Squares of Q sets

by

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Abstract. A Q set is an uncountable separable metric space in which every subset is a G_δ. We show the following statement is consistent with ZFC: There is a Q set of cardinality ω_2 but no square of a space of cardinality ω_2 is a Q set.

A Q set is an uncountable separable metric space in which every subset is a G_δ. The existence of Q sets is consistent with and independent of ZFC. The existence of Q sets is equivalent to several propositions of set theory and topology and is central in a web of interesting implications — see [T], [P], [F]. One concept investigated recently is that of a *strong Q set*, defined to be a Q set all of whose finite powers are Q sets. The main results are:

If X is a strong Q set, then the Pixley–Roy space built from X is a normal nonmetrizable Moore space [PT].

Conversely, if X is a separable metric space whose Pixley–Roy space is a normal nonmetrizable Moore space, then X is a strong Q set [R].

If there is a Q set, then there is a strong Q set of cardinality ω_1, [P].

We complement this last result with

THEOREM. *It is consistent, relative to* ZFC, *that there be a Q set of cardinality ω_2, but no square of a space of cardinality ω_2 is a Q set.*

We sketch the proof of the theorem below. We start with a model, M, of GCH. We define a notion of forcing $\boldsymbol{P}$ which adds a set Y of ω_2 Cohen reals and makes Y into a Q set in the manner of [FM]. Let $Z = \{z_\beta\colon \beta < \omega_2\}$ be a set of reals of cardinality ω_2. We will show that $\Delta = \{(z_\beta, z_{\beta^*})\colon \beta < \beta^* < \omega_2\}$ is not a G_δ in $Z \times Z$. Let $\mathcal{U} = \{U_k\colon k \in \omega\}$ be a family of open sets containing Δ. Using counting arguments (Lemma 3), we find a large subset H of ω_2 such that z_β, $\beta \in H$, are independent over $\mathcal{U}$ (mutually Cohen generic over $\mathcal{U}$, in some sense). Choose $\beta, \beta^* \in H$. There must be $k \in \omega$ and $t \in \boldsymbol{P}$ such

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