Indecomposable continua and the fixed point property II

by

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Abstract. It is proved that if X is a hereditarily unicoherent Hausdorff continuum and if $F: X \to X$ is a continuum-valued upper semi-continuous mapping, then there is an indecomposable continuum $Q \subseteq X$ such that $Q \subset F(Q)$.

1. Introduction. The main purpose of this paper is to prove a theorem which shows that for every hereditarily unicoherent Hausdorff continuum X and for every continuum-valued upper semi-continuous mapping F from X into itself there is an indecomposable continuum $Q \subset X$ such that $Q \subset F(Q)$. This theorem is a generalization of the main result of [5] to the nonmetric case and it implies the fixed point theorems contained in [4] and [6].

The theory of irreducible continua used in the proofs of mentioned theorem is investigated here in more general setting and in a such way that it also gives a theory of Whyburn's cycle which are here indecomposable in some sense. This general approach, also by results of [1] and [8], shows that the fixed point theorems are strongly connected with indecomposable continua.

2. Notations and lemmas. Let X be an arbitrary Hausdorff and let U be a fixed family of subcontinua of X which satisfies the following conditions:

(i) $X \in U$ and $\{\{x\}: x \in X\} \subset U$,

(ii) if A and B belong to U and $A \cap B \neq \emptyset$, then $A \cup B \in U$,

(iii) if A and B belong to U and $B \setminus A = M \cup N$ where M and N are separated, then $\{A \cup M, A \cup N\} \subset U$,

(iv) if A and B belong to U and $B \setminus A$ is connected, then $\mathrm{cl}(B \setminus A) \in U$,

(v) if $\mathscr{P} \subset U$, then $\bigcap \{A: A \in P\} \in U$,

(vi) if $\mathscr{P} \subset U$ and $\mathscr{P}$ is directed by the inclusion $\subset$, then $\mathrm{cl}(\bigcup \{A: A \in \mathscr{P}\}) \in U$.

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If $A \in U$, then sometimes we say A is a *U-continuum*. It follows from (v) that for every two disjoint U-subcontinua A and B of X there is a unique minimal U-continuum C which intersects both A and B. Every such C is called *U-irreducible between A and B*; we denote it by AB. In particular, ab denotes a unique continuum U-irreducible between points a and b. One can easily check.

(2.1) *AB is U-irreducible between every point of $AB \cap A$ and every point of $AB \cap B$.*

(2.2) *If $D \in U$, $A \cap D \neq \emptyset \neq B \cap D$, then $AB \subset D$.*

(2.3) *If $X = ab$, $b \in C \subset U$, then $X \setminus C$ is connected and $\mathrm{cl}(X \setminus C) \in U$.*

The set of all points of a U-continuum K which can be joined with a set $A \subset K$ by a proper subcontinuum L of K such that $A \subset L \in U$ is called a *U-composant of A in K* and we will denote it by $C(A, K)$. We say a continuum $C \in U$ is *U-indecomposable* if C can not be decomposed into two proper U-subcontinua. Put

$$I(a, b) = \bigcap \{\mathrm{cl}(ab \setminus C)\colon\ b \in C \subset ab \neq C \text{ and } C \in U\}.$$

It follows from (v) and (2.3)

(2.4) $$a \in I(a, b) \in U \quad \text{and} \quad I(a, b) \subset ab.$$

Moreover,

(2.5) $$ab = C(b, ab) \cup I(a, b).$$

Indeed, let $\mathscr{C} = \{c \in U\colon\ b \in C \subset ab \neq C\}$. Then $ab = C \cup \mathrm{cl}(ab \setminus C)$, for $C \in \mathscr{C}$. Therefore,

$$ab = \bigcup \{C\colon\ C \in \mathscr{C}\} \cup \bigcap \{\mathrm{cl}(ab \setminus C)\colon\ C \in \mathscr{C}\} = C(b, ab) \cup I(a, b).$$

We have (compare [2], p. 210–211).

(2.6) *$I(a, b)$ is a boundary set in ab if and only if $I(a, b) \cap C(b, ab) = \emptyset$.*

In fact, suppose $I(a, b) \cap C(a, ab) \neq \emptyset$. Then there is $C \in U$ such that $b \in C \subset ab \neq C$ and $C \cap I(a, b) \neq \emptyset$; thus $ab = C \cup I(a, b)$ and since C is a proper subset of ab we conclude $I(a, b)$ is not a boundary set, a contradiction. Conversely, suppose the interior of $I(a, b)$ is nonempty in ab. Then, by (2.3), $D = \mathrm{cl}(ab \setminus I(a, b)) \in U$ and D is a proper subset of ab containing b. Hence $D \subset C(b, ab)$. But $D \cap I(a, b) \neq \emptyset$, because $ab = D \cup I(a, b)$. Therefore, $I(a, b) \cap C(b, ab) \neq \emptyset$, a contradiction.

(2.7) *If $I(a, b)$ is not a boundary set in ab, then it is a U-indecomposable continuum which has at least two U-composants.*

Indeed, it follows from (2.3) that if $D = \mathrm{cl}(ab \setminus I(a, b))$, then $b \in D \in U$ and D is a proper subset of ab. Suppose $I(a, b) = A \cup B$ where A and B are proper U-subcontinua of $I(a, b)$ and $a \in B$. Then $ab = D \cup A \cup B$. If $B \cap D \neq \emptyset$, then $ab = D \cup B$, thus $I(a, b) \subset \mathrm{cl}(ab \setminus D) \subset B$, a contradiction. If $B \cap D = \emptyset$, then $A \cap D \neq \emptyset$. Moreover, $b \in A \cup D \in U$ and $A \cup D \neq ab$, thus $I(a, b) \subset \mathrm{cl}(ab \setminus (D \cup A)) \subset B$, a contradiction.

(2.8) *If $D \in U$ and $C(a, ab) \cap bD = \emptyset$, then $aD = ab \cup bD$. Moreover, if $D = \{d\}$, then $C(b, ab) \cup bd \subset C(d, ad)$ and $I(a, d) \subset I(a, b)$.*

In fact, since $ab \cup bD$ is continuum intersecting $\{a\}$ and D which belongs to U, we infer $aD \subset ab \cup bD$ by (2.2). Then $aD = (ab \cap aD) \cup (bD \cap aD)$. This equality and the connectedness of aD imply that there is a point $z \in (ab \cap aD) \cap (bD \cap aD)$. Then $z \in ab \setminus C(a, ab)$, because $z \in bD$ and $C(a, ab) \cap bD = \emptyset$. Therefore $az = ab$. Since $\{a, z\} \subset aD$, we conclude, $az \subset aD$; thus $ab \subset aD$. Consequently $\{a, b\} \subset aD$ which implies $ab \cup bD \subset aD$. Therefore $aD = ab \cup bD$. If $D = \{d\}$, then equalities $C(a, ab) \cap bd = \emptyset$ and $ad = ab \cup bd$ imply that bd is a proper subcontinuum of ad containing d and belonging to U; thus $bd \subset C(d, ad)$. If $x \in C(b, ab)$, then $xb \subset ab \setminus \{a\}$. We conclude that $xb \cup bd$ is a proper U-subcontinuum of ad containing d, because $xb \cup bd \subset ad \setminus \{a\}$. Therefore $x \in C(d, ad)$ because $xb \cup bd \in U$. Consequently $C(b, ab) \cup bd \subset D(d, ab)$. Moreover, if $b \in C \in U$ and $ab \neq C \subset ab$, then $I(a, d) \subset \mathrm{cl}(ad \setminus (C \cup bd)) = \mathrm{cl}(ab \setminus C)$. This implies $I(a, d) \subset I(a, b)$ which completes the proof of (2.8).

(2.9) *If $\{K, L\} \subset U$, $K \subset L \cap I(b, a) \neq I(b, a)$ and $K \cap C(a, ab) = \emptyset$, then $L \cap C(a, ab) = \emptyset$.*

Suppose $L \cap C \neq \emptyset$ for some $C \in U$ such that $a \in C \subset ab \neq C$. Then $ab \subset C \cup L$. Therefore $I(b, a) \subset \mathrm{cl}(ab \setminus C) \subset L$, a contradiction.

(2.10) *If $I(c, d) \subset I(b, a) \neq I(c, d)$ and $I(c, d) \cap C(a, ab) = \emptyset$, then $cd \cap C(a, ab) = \emptyset$.*

Since $I(c, d) \subset I(b, a) \neq I(c, d)$, there is a continuum $D \in U$ such that $d \in D \subset cd \neq D$ and $\mathrm{cl}(cd \setminus D) \cap I(b, a) \neq I(b, a)$. Therefore, if $E = \mathrm{cl}(cd \setminus D)$, then $E \cap C(a, ab) = \emptyset$ by (2.9). We conclude $cd = E \cup D$. Suppose that $D \cap C(a, ab) \neq \emptyset$ and let $F \in U$ be a proper subcontinuum of ab containing a and intersecting D. If $F \cap E \neq \emptyset$, then $ab \subset F \cup E$, thus

$$I(b, a) \subset \mathrm{cl}(ab \setminus F) \cap I(b, a) \subset E \cap I(b, a) \neq I(b, a),$$

a contradiction. Thus let $F \cap E = \emptyset$. We have $ab \subset F \cup D \cup E$. The continuum $(F \cup D) \cap ab$ is a proper U-subcontinuum of ab containing a (otherwise, if $(F \cup D) \cap ab = ab$, then $I(c, d) \subset E \cap (F \cup D)$; in particular $c \in D$, a contradiction). Thus $I(b, a) \subset \mathrm{cl}(ab \setminus (F \cup D)) \subset E$. But $E \cap I(b, a) \neq I(b, a)$ by the construction, a contradiction.

(2.11) *If C is a U-composant of a in $K \in U$ and it is not closed in K, then it is dense in K.*

In fact, the collection of all proper U-subcontinua of K containing a is directed by the inclusion; thus $\mathrm{cl}\, C \in U$. According to the assumptions $\mathrm{cl}\, C \neq C$; thereby $\bar{C} = K$.

(2.12) *Different U-composants in a U-indecomposable subcontinuum K of X are disjoint and if $K \neq L \subset K$ and $L \in U$, then L is a boundary set in K.*

Remark. Let X be a metric continuum. Applying Kuratowski's proof used in [3] one can prove that point $a \in X$ is a point of U-irreducibility of X if and only if there do not exist two proper U-subcontinua P and R of X such that $X = P \cup R$ and $a \in P \cap R$. As a corollary we can obtain many theorems known for ordinary irreducible metric continua; for example: X is U-indecomposable if and only if every proper U-subcontinuum of X is a boundary set; if X is U-indecomposable, then the collection of U-composants of K is an uncountable collection members of which are pairwise disjoint and boundary in X; every U-indecomposable continuum is U-irreducible between some points etc.

A corollary of it is also the fact that the closure of the union of an increasing family of subcontinua ab_n is always U-irreducible between a and some point b. This last theorem we will prove in the nonmetric case (compare [4], Lemma (i)).

(2.13) *Let a point $a \in X$ be fixed and $\mathscr{P}$ be a nested family of U-irreducible subcontinua ab of X. If $\mathscr{I} = \mathrm{cl}(\bigcup \{ab\colon ab \in \mathscr{P}\})$, then there is a point p and a U-indecomposable subcontinuum P of X such that $ap \cup P = \mathscr{I}$ and every U-composant in P is equal to P.*

Proof. First note that $\mathscr{I} \in U$ by (vi). We may assume that $ab \neq \mathscr{I}$ for $ab \in \mathscr{P}$. Moreover

(1) $\mathscr{I} \setminus ab$ is connected for $ab \in \mathscr{P}$.

Suppose that $\mathscr{I} \setminus ab = M \cup N$ where M and N are separated. Then $ab \cup M$ and $ab \cup N$ belong to U by (iii). If $c \in ab \cup N$ and $ac \in \mathscr{P}$, then there is $d \in ab \cup M$ such that $ac \subset ad \in \mathscr{P}$ (otherwise all $ad \in \mathscr{P}$ such that $ac \subset ad$ are contained in $ab \cup N$, thus $\mathscr{I} \subset ab \cup N$ because $\mathscr{P}$ is nested). But then $ad \subset ab \cup M$; thus $ac \subset ab \cup M$ for $ac \in \mathscr{P}$, which means $\mathscr{I} \subset ab \cup M$, a contradiction.

Let $C = \bigcap \{\mathrm{cl}(\mathscr{I} \setminus ab)\colon ab \in \mathscr{P}\}$. It follows from (1) and (iv)–(v) that

(2) $C \in U$ and $\mathrm{cl}(\mathscr{I} \setminus ab) \in U$ for $ab \in \mathscr{P}$.

Moreover, we obviously have (compare the proof of (2.5)).

(3) $$\mathscr{I} = \bigcup \{ab\colon ab \in \mathscr{P}\} \cup C = ab \cup \mathrm{cl}(\mathscr{I} \setminus ab) \quad \text{for} \quad ab \in \mathscr{I}.$$

If $\mathrm{cl}(\mathscr{I} \setminus ab)$ is U-indecomposable for some $ab \in \mathscr{P}$, then, if $\mathrm{cl}(\mathscr{I} \setminus ab)$ has more than one U-composant, we can take a point $p \in \mathrm{cl}(\mathscr{I} \setminus ab) \setminus C(ab \cap \mathrm{cl}(\mathscr{I} \setminus ab), \mathrm{cl}(\mathscr{I} \setminus ab))$ we obtain $ap = ab \cup \mathrm{cl}(\mathscr{I} \setminus ab) - \mathscr{I}$, which completes the proof. If $\mathrm{cl}(\mathscr{I} \setminus ab)$ has only one composant, taking $p = b$ and $P = \mathrm{cl}(\mathscr{I} \setminus ab)$

we also find $\mathscr{I} = ap \cup P$. Therefore we may assume that $\mathrm{cl}(\mathscr{I} \setminus ab)$ are U-decomposable for $ab \in \mathscr{P}$. We claim that

(4) $$ab \cap C = \emptyset \quad \text{for} \quad ab \in \mathscr{P}.$$

In fact, suppose $ab \cap C \neq \emptyset$ for some $ab \in \mathscr{P}$. Since $\mathrm{cl}(\mathscr{I} \setminus ab)$ is U-decomposable there are proper U-subcontinua Q and P of $\mathrm{cl}(\mathscr{I} \setminus ab)$ whose union is $\mathrm{cl}(\mathscr{I} \setminus ab)$. If $ab \cap Q \cap R$ is nonempty, we obtain, the same arguments as in the proof of (1), that $\mathscr{I} \setminus (ab \cup (Q \cap R))$ is connected, a contradiction, because $\mathscr{I} \setminus (ab \cup (Q \cap R)) \subset Q \setminus R \cup R \setminus Q$. Assume $ab \cap R = \emptyset$. If for each $ac \in \mathscr{P}$ there is $ad \in \mathscr{P}$ such that $d \in Q$, then $ac \subset ad \subset ab \cup Q$ and thus $\mathscr{I} \subset ab \cup Q$, a contradiction. Therefore there is $ac \in \mathscr{P}$ such that $ab \subset ac$ and if $ac \subset ad \in \mathscr{P}$, then $d \in R$. But then $ac \cup R$ contains all $ad \in \mathscr{P}$. Thus $\mathscr{I} \subset ac \cup R$. We conclude $C \subset \mathrm{cl}(\mathscr{I} \setminus ac) \subset R$. Since $ab \cap R = \emptyset$, we conclude $ab \cap C = \emptyset$, a contradiction, i.e. (4) holds.

Now it suffices to show that $\mathscr{I} = ac$ for $c \in C$. Clearly $ac \subset \mathscr{I}$. So it will suffice to show that $ab \subset ac$ for $ab \in \mathscr{P}$. From (4) we conclude that there is $ad \in \mathscr{P}$ such that $ab \cap \mathrm{cl}(\mathscr{I} \setminus ad) = \emptyset$. It follows from (3) that there is $az \in \mathscr{P}$ such that $ab \subset az$ and $z \in \mathrm{cl}(\mathscr{I} \setminus ad)$. Therefore $ab \subset az \subset ac \cup \mathrm{cl}(\mathscr{I} \setminus ad)$. Consequently $b \in ac$, i.e. $ab \subset ac$. This inclusion completes the proof.

(2.14) *Let a point $a \in X$ be fixed and $\mathscr{P}$ be a nested family of U-irreducible subcontinua ab of X and $P = \bigcup \{ab\colon ab \in \mathscr{P}\}$. If $ac = \mathrm{cl}\, P$, $I(c, a)$ is a boundary set in ac and $ab \neq ac$ for $ab \in \mathscr{P}$ then $C(a, ac) = P$.*

The inclusion $\mathscr{P} \subset C(a, ac)$ is obvious. Let $e \in C(a, ac)$. Then $ae \subset ac \neq ae$. Since $I(c, a)$ is a boundary set in ac, we infer that $C(a, ac) \cap I(c, a) = \emptyset$ by (2.6), thus there is a proper subcontinuum L of ac such that $a \in L \in U$ and $ae \cap \mathrm{cl}(ac \setminus L) = \emptyset$. Since $L \cup \mathrm{cl}(ac \setminus L) = ac$, there is $ab \in \mathscr{P}$ such that $b \in \mathrm{cl}(ac \setminus L)$. The equality $ab \cup \mathrm{cl}(ac \setminus L) = ac$ implies that $e \in ab$, thus $e \in P$.

3. Fixed point properties. Recall that $F\colon X \to X$ is *upper semi-continuous* (u.s.c.) provided each point image $F(x)$ is closed set and whenever U is an open set containing $F(x)$, there exists an open set V containing x such that $F(t) \subset U$ for each $t \in V$. We say the mapping F is *U-valued* if for each $K \in U$ we have $F(K) \in U$. Such a mapping has a *fixed point* if there is a point $x \in X$ such that $x \in F(x)$. If $F\colon X \to X$ is U-valued and $A \in U$ then we put

$$\mathscr{P}(a) = \{ab\colon bF(b) \cap C(a, ab) = \emptyset\}.$$

and

$$\mathscr{P}(a, A) = \{ab\colon ab \in \mathscr{P}(a) \text{ and } I(a, b) \subset A\}.$$

We have

(3.1) Theorem. *If a mapping $F\colon X \to X$ is U-valued and u.s.c., for each nondegenerate U-indecomposable subcontinuum K of X, the set $K \cap F(K)$ is a proper subcontinuum of K and if $\mathscr{P}(a, A) \neq \emptyset$, then there is a maximal element in*

$\mathscr{P}(a, A)$. *Moreover, if* ab *is a maximal element in* $\mathscr{P}(a, A)$, *then* $I(b, a) \cap F(I(b, a)) \neq \emptyset$.

Proof. It follows from (2.8)

(1) If $ab \in \mathscr{P}(a)$, then $aF(b) = ab \cup bF(b)$.

Moreover,

(2) If $\mathscr{P} \subset \mathscr{P}(a)$, $B = \mathrm{cl}(\bigcup\{ab\colon ab \in \mathscr{P}\})$, $ab \neq B$ for $ab \in \mathscr{P}$, then B is U-decomposable.

Indeed, fix $ab_0 \in \mathscr{P}$. Then $aF(b) \subset ab_0 \cup b_0F(b_0) \cup F(B)$ for each $ab \in \mathscr{P}$. Therefore $B \subset ab_0 \cup b_0F(b_0) \cup F(B)$ by (1). It implies the equality $B = (ab_0 \cap B) \cup (b_0F(b_0) \cap B) \cup (F(B) \cap B)$. All these continua in the decomposition of B are proper in B and belong to U, thus (2) holds.

(3) If $\mathscr{P} \subset \mathscr{P}(a)$, $B = \mathrm{cl}(\{ab\colon ab \in \mathscr{P}\})$, $ab \neq B$ for $ab \in \mathscr{P}$ and $\mathscr{P}$ is nested, then there is a point c such that $ac = B$ and $I(c, a)$ is a boundary set in ac.

In fact, it follows from (2.13) that there is a point c and U-indecomposable subcontinuum C of X such that $ac \cup C = B$ and every U-composant in C is equal to C. Suppose $ac \neq B$. We may assume that ac is U-irreducible between a and C and $ac \subset ab \neq ac$ for each $ab \in \mathscr{P}$. Then $C(c, cb) \subset C(a, ab)$, thus $cb \in \mathscr{P}(c)$. Moreover $cb \neq C$, because C has only one composant. From (2) we obtain C is U-decomposable, a contradiction. Therefore $ac = B$. Suppose $I(c, a)$ is not a boundary set in ac; then, by (2.7), it is a U-indecomposable continuum and $ab_0 \cap I(c, a) \neq \emptyset$ for some $ab_0 \in \mathscr{P}$ by (2.6). Let ad be a continuum U-irreducible between a and $I(c, a)$. Then $ad \subset ab_0$, $ad \cap I(c, a)$ is a boundary set in $I(c, a)$ and $ad \cup I(c, a) = ac$. We may assume that $ad \subset ab \neq ad$ for each $ab \in \mathscr{P}$. Then $C(d, db) \subset C(a, ab)$; thus $db \in \mathscr{P}(d)$ for $ab \in \mathscr{P}$. Moreover $db \neq I(c, a)$ because $c \notin db$. From (2) we obtain a contradiction as above. We will now prove

(4) If $\mathscr{P} \subset \mathscr{P}(a, A)$, $ac = \mathrm{cl}(\{ab\colon ab \in \mathscr{P}\})$, $ab \neq ac$ for $ab \in \mathscr{P}$ and $\mathscr{P}$ is nested, then $I(a, c) \subset A$.

It follows from (3) that $I(c, a)$ is a boundary set in ac. We can assume that $I(c, a) \cap A = \emptyset$ (otherwise $ac \subset A$ by (2.6)). Then there is a U-subcontinuum D of ac such that $a \in D \subset ac \neq D$ and $I(c, a) \subset C \subset X \setminus A$ where $C = \mathrm{cl}(ac \setminus D)$. Obviously we have $c \in C \subset C(c, ac)$. We can assume that $b \in C$ if $ab \in \mathscr{P}$. It is clear that $C(b, ab) \subset C(c, ac)$ for $ab \in \mathscr{P}$ because $ac = ab \cup C$ for $ab \in \mathscr{P}$.

Now, if $K \in U$ and $B \in K \subset ab \neq K$, then $K \cup C \in U$ and $c \in K \cup C \subset ac \neq K \cup C$. Therefore

$$I(a, c) \subset \mathrm{cl}(ac \setminus (K \cup C)) = \mathrm{cl}(ab \setminus (K \cup C)) \subset \mathrm{cl}(ab \setminus K);$$

thus $I(a, c) \subset I(a, b)$, which proves (4).

According to (3) and (4) we obtain

(5) if $\mathscr{P} \subset \mathscr{P}(a, A)$, $B = \mathrm{cl}(\{ab\colon ab \in \mathscr{P}\})$, $ab \neq B$ for $ab \in \mathscr{P}$ and $\mathscr{P}$ is nested, then there is a point c such that $ad = B$ and $ad \in \mathscr{P}(a, A)$ for some $d \in I(c, a)$.

Since $ac = ad$, $C(a, ac) = C(a, ad)$ and $I(c, d) = I(d, a)$, it remains to prove that $dF(d) \cap C(a, ac) = \emptyset$ for some $d \in I(c, a)$. If $I(c, a) \cap F(I(c, a)) \neq \emptyset$, then taking $d \in I(c, a)$ such that $F(d) \cap I(c, a) \neq \emptyset$ we obtain $dF(d) \subset I(c, a)$ which gives (5). Further we assume that sets $I(c, a)$ and $F(I(c, a))$ are disjoint. Since F is u.s.c., we can find, by the definition of $I(c, a)$, $P \in U$ such that $a \in P \subset ac \neq P$ and $Q \cap F(Q) = \emptyset$ where $Q = \mathrm{cl}(ac \setminus P)$. We may assume $b \in Q$ for $ab \in \mathscr{P}$. If $QF(Q) \cap C(a, ac) = \emptyset$, then taking $d = c$ we have $cF(c) \subset I(c, a) \cup F(Q) \cup QF(Q)$, thus $cF(c) \cap C(a, ac) = \emptyset$ (because also $F(Q) \cap C(a, ac) = \emptyset$), i.e. (5) holds.

Therefore we can assume that the set $QF(Q) \cap C(a, ac)$ is not empty. According to (2.14) we may assume that $QF(Q) \cap ab \neq \emptyset$ for each $ab \in \mathscr{P}$ (because $\mathscr{P}$ is nested). But $QF(Q) \subset bF(b)$ for each $ab \in \mathscr{P}$. There are $ab, ab^{\nabla} \in \mathscr{P}$ such that $ab \subset C(a, ab^{\nabla})$; thus $b^{\nabla}P(b^{\nabla}) \cap C(a, ab^{\nabla}) \neq \emptyset$, a contradiction.

Condition (5) implies that if $\mathscr{P}(a, A) \neq \emptyset$, then there is a maximal element in $\mathscr{P}(a, A)$. To finish the proof of Theorem (3.1) we shall show that if ab is a maximal element in $\mathscr{P}(a, A)$, then $I(b, a) \cap F(I(b, a)) \neq \emptyset$.

Suppose, on the contrary, sets $I(b, a)$ and $F(I(b. a))$ are disjoint and let rs be a continuum U-irreducible between these sets with $r \in I(b, a)$ and $s \in F(I(b, a))$. Since $ab \in \mathscr{P}(a, A)$, we have

$$C(a, ab) \cap rs = \emptyset. \tag{6}$$

Consider five cases.

(a) $I(r, s) \cap F(I(r, s)) \neq \emptyset$ and $I(r, s)$ is not a boundary set in rs. It follows from (2.7) that $I(a, b)$ is U-indecomposable continuum which has at least two U-composants. If $c \in F(I(r, s)) \cap C(r, I(r, s))$ then $rs \subset rc \cup F(I(r, s))$ because $rc \cup F(I(r, s))$ is a U-continuum intersecting both $I(b, a)$ and $F(I(b, a))$. Therefore $I(r, s) \subset rc \cup F(I(r, s))$; thus $I(r, s) \setminus rc \subset F(I(r, s))$. Consequently $\mathrm{cl}(I(r, s) \setminus rc) = I(r, s)$ because rc is a boundary set in $I(r, s)$ by (2.12). It implies the inclusion $I(r, s) \subset F(I(r, s))$ contrary to the assumptions. Hence $F(I(r, s)) \cap C(r, I(r, s)) = \emptyset$ and take $w \in I(r, s) \cap F(I(r, s))$. Then $I(r, s) \cap F(I(r, s)) \subset C(w, I(r, s))$ and $rw = I(r, s)$. From (6) and (2.8) we conclude $aw = ab \cup rw$ and $I(a, w) \subset A$. Since $C(a, aw) = ab \cup C(r, I(r, s))$ and $wF(w) \subset F(I(r, s))$ we infer $aw \in (a, A)$; a contradiction to the maximality of ab.

(b) $I(r, s) \cap F(I(r, s)) \neq \emptyset$ and $I(r, s)$ is a boundary set in rs. From (2.6) the continuum rs is U-irreducible between s and every point of $I(r, s)$.

Let R be a minimal subcontinuum of $I(r, s)$ with respect to the property: $r \in R \in U$ and $R \cap F(R) \neq \emptyset$. Let $t \in R \cap F^{-1}(R)$. Since $F(rt) \cap I(r, s) \neq \emptyset \neq F(rt) \cap F(I(b, a))$, we conclude $I(r, s) \subset F(rt)$; thus rt

$= R$ and if S is a proper subcontinuum of rt such that $r \in S \in U$ then $F(s) \cap I(r, s) = \emptyset$. Therefore $I(r, s) \subset F(rt \setminus C(r, rt))$. In particular $I(t, r) \subset F(I(t, r))$; thus $I(t, r)$ is a boundary set by (2.7) and the assumptions. It follows from (2.6) that $I(t, r) \cap C(r, rt) = \emptyset$. There is a point $w \in I(t, r)$ such that $F(w) \cap I(t, r) \neq \emptyset$. Since $wF(w) \subset I(t, r)$, we conclude $aw \in \mathscr{P}(a, A)$; a contradiction.

(c) $I(r, s) \cap F(I(r, s)) = \emptyset$ and $I(r, s) = rs$. Put $Q = F(I(b, a)) \cup F(I(r, s))$. Since $Q \subset X \setminus C(r, rs)$ and Q is a U-continuum intersecting both $\{s\}$ and $F(s)$, we conclude $sF(s) \subset Q \subset X \setminus C(r, rs)$. As in case (a) obtain $as = ab \cup rs \in \mathscr{P}(a, A)$, which contradicts the maximality of ab.

(d) $I(r, s) \cap F(I(r, s)) = \emptyset$, $I(r, s) \neq rs$ and $I(r, s)$ is not a boundary set in rs. Then there is a point w such that $w \in I(r, s) \setminus ab$, $rw = I(r, s)$, $rs = I(r, s) \cup ws$ and ws is U-irreducible between $I(r, s)$ and s. Since $ws \cap ab = \emptyset$, we have $ab \neq ab \cup I(r,s)$. One can easily check that $aw = ab \cup rw \in \mathscr{P}(a, A)$, a contradiction.

(e) $I(r, s) \cap F(I(r, s)) = \emptyset$ and $I(r, s)$ is a boundary set in rs. The upper semi-continuity of F and the definition of $I(r, s)$ implies that there is $Q \in U$ such that $Q \cap F(Q) = \emptyset$, $I(r, s) \subset Q \subset rs$ and $Q \setminus I(r, s) \neq \emptyset$. Since $I(r, s)$ is a boundary set in rs, we infer $ps \cap I(r, s) = \emptyset$ for $p \in Q \setminus I(r, s)$. Let rw be a continuum in Q which is U-irreducible between r and $ps \cap Q$. It is easy to show that $aw \in \mathscr{P}(a, A)$, which contradicts the maximality of ab. In this way the proof of Theorem (3.1) is complete.

The main result of the paper is the following:

(3.2) THEOREM. *If a mapping $F\colon X \to X$ is U-valued and u.s.c., and every U-indecomposable subcontinuum of X has only dense U-composants, then there is a U-indecomposable subcontinuum K of X such that $K \subset F(K)$.*

Proof. Consider the following family $\mathscr{W}$ of subcontinua of X: $Q \in \mathscr{W}$ provided

(1) $Q \in U$; (2) $Q \cap F(Q) \neq \emptyset$ and (3) if A is a proper U-subcontinuum of Q and $ab \in \mathscr{P}(ab)$, then $ab \subset Q$. Firstly, we have

(4) if ab is a maximal element in $\mathscr{P}(a, A)$, and $I(b, a)$ is a boundary set in ab, then $I(b, a) \in \mathscr{W}$.

Conditions (1) and (2) of the definition of $\mathscr{W}$ are satisfied by Theorem (3.2). Now, let K be a proper U-subcontinuum of $I(b, a)$ and $cd \in \mathscr{P}(c, K)$. It follows from (2.10) that $C(a, ab) \cap cd = \emptyset$; then $ad = ab \cup cd \in \mathscr{P}(a, A)$ by (2.8). Thus the maximality of ab in $\mathscr{P}(a, A)$ implies (3).

Now we claim that

(5) if $a_0 b_0$ is a maximal element in $\mathscr{P}(a_0, A_0)$, then there is a continuum $Q \in \mathscr{W}$ such that $Q \subset a_0 b_0 \setminus C(a_0, a_0 b_0)$.

According to (4) and Propositions (2.6) and (2.7) we may assume that Q_0 is an U-indecomposable subcontinuum with at least two U-composants where $Q_0 = I(b_0, a_0)$. By assumptions we obtain $Q_0 \cap F(Q_0)$ is a proper subcontinuum of Q_0. Since $a_0 b_0 \in \mathscr{P}(a_0, A_0)$ we infer

$$(Q_0 \cap F(Q_0)) \cap C(a_0, a_0 b_0) = \emptyset.$$

Take a point a_1 from $Q_0 \setminus (Q_0 \cap F(Q_0))$ which belongs to the composant of Q_0 containing $Q_0 \cap F(Q_0)$ (we can find a_1, because U-composants in Q_0 are dense by assumptions). Let $a_1 b_1$ be a continuum U-irreducible between a_1 and $Q_0 \cap F(Q_0)$. Then $a_1 d_1 \subset C(a_1, Q_0)$ and $a_1 d_1 \in \mathscr{P}(a_1, a_1 d_1)$. By Theorem (3.1) there is a maximal element $a_1 b_1$ in $\mathscr{P}(a_1, a_1 d_1)$. Then $a_1 b_1 \subset C(a_1, Q_0)$ by the maximality of $a_0 b_0$ and conditions (2.8) and (2.10). We find $Q_1 = I(b_1, a_1) \subset Q_0$. If Q_1 is a boundary set in $a_1 b_1$, then $Q_1 \in \mathscr{W}$ by (4). Thus we can assume that Q_1 is an indecomposable continuum and as above we find Q_2, a_2 and d_2 etc. Take $Q = \bigcap_{n=0}^{\infty} Q_n$. We will check that $Q \in \mathscr{W}$. Since $Q_n \in U$ and $Q_n \cap F(Q_n) \neq \emptyset$, we infer $Q \in U$ and $Q \cap F(Q) \neq \emptyset$, i.e. (1) and (2) hold. Now let K be a proper U-subcontinuum of Q and $cd \in \mathscr{P}(c, K)$. Then K is a proper subcontinuum of Q_n for each n. Moreover, $K \cap C(a_n, a_n b_n) = \emptyset$ for each n. Thereby, $a_n d = a_n b_n \cup cd \in \mathscr{P}(a_n, a_n d_n)$ by (2.8) and (2.10). The maximality of $a_n b_n$ in $\mathscr{P}(a_n, a_n d_n)$ implies $cd \subset I(b_n, a_n) = Q_n$. This means that $cd \subset Q$, i.e. condition (3) of the definition of $\mathscr{W}$ holds. The proof of (5) is complete.

Now, let Q be a continuum in X which is minimal in $\mathscr{W}$ and L be a minimal subcontinuum of Q with respect to the properties: $L \in U$ and $L \cap F(L) \neq \emptyset$. Then $L = aF(a)$ for some a. If L is degenerate, then $L \subset F(L)$ and Theorem (3.2) is proved. So let L be nondegenerate. If L is U-decomposable, then $I(a, F(a))$ is a proper U-subcontinuum of Q and $L \in \mathscr{P}(a, I(a, F(a)))$. If L is U-decomposable, then $L \cap F(L)$ is a nonempty proper U-subcontinuum of L (otherwise $L \subset F(L)$), and taking $a_1 \in C(L \cap F(L), L) \setminus (L \cap F(L))$ (L has at least two composants, and therefore they are dense in it) and a continuum $a_1 b_1$ U-irreducible between a_1 and $L \cap F(L)$ we find $a_1 b_1 \in \mathscr{P}(a_1, a_1 b_1)$ because $F(b_1) \subset F(L)$ and $a_1 b_1 \cap F(L) \subset a_1 b_1 \setminus C(a_1, a_1 b_1)$. Therefore in both cases we have a proper U-subcontinuum D of Q such that $\mathscr{P}(d, D) \neq \emptyset$. A maximal element in $\mathscr{P}(a, D)$ is contained in Q, because $Q \in \mathscr{W}$; but then, there is a proper U-subcontinuum Q_1 of Q such that $Q_1 \in \mathscr{W}$ by (5). It is a contradiction to the minimality of Q in $\mathscr{W}$. Therefore Q is degenerated and $Q \subset F(Q)$ which finishes the proof of Theorem (3.2).

Remark. The main and some parts of the proofs of Theorems (3.1) and (3.2) are almost the same as in [5]. We repeat them only for the completness of this paper.

4. Universal subcontinua. We say that a subcontinuum K of X is *universal* provided its intersection with every subcontinuum of X is connected. The family of all universal subcontinua of a continuum X satisfies conditions concerning U

without (iv) and (vi). But it is easy to see that if X is either locally connected or hereditarily unicoherent then these two conditions also are satisfied. If a continuum X is locally connected, then universal subcontinua of X are exactly these subcontinua of X which are unions of Whyburn's cycles (compare [8]), and a universal continuum which is not the union of two proper universal subcontinua (i.e. indecomposable in the class of universal subcontinua) is some Whyburn's cycle. If X is hereditarily unicoherent, then all its subcontinua are universal and then indecomposable continua satisfy the additional assumption from Theorem (3.2). According to Lemma 3 in [7], p. 161 from Theorem (3.2) we obtain

(4.1) Corollary. *If X is a hereditarily unicoherent Hausdorff continuum, $F\colon X \to X$ is u.s.c. and such that $F(x)$ is a continuum for each $x \in X$, then there is a indecomposable continuum K such that $K \subset F(K)$.*

This result is a generalization of Theorem 1 from [5] to the nonmetric case.

The following problems remain open:

For which collections U it is true that if every U-indecomposable subcontinuum of X has a fixed point property then X has it also?

Some answers are given by Corollary (4.1) (compare [5]), some others are contained in [1] and [4].

If U denotes the family of all universal subcontinua of X, then the theory of U-irreducible continua investigated in previous sections to obtain some fixed point theorems is not so nice. It can be see from the following examples.

(4.2) Example. Let I_n denote a straight line interval lying in the Euclidean plane E^2 and joining $a = (0, 1)$ with a point b_n where $b_0 = (0, 0)$ and $b_n = (1/n, 0)$ for $n = 1, 2, \ldots$ Consider a continuum X which is a union of I_n for $n = 0, 1, 2, \ldots$ and disjoint lines C_n such that b_n is a beginning of C_n and C_n approximates I_n for $n = 1, 2, \ldots$. Then X is U-indecomposable and $X = \bigcup_{i=0}^{\infty} ab_i$ where ab_i is a U-irreducible continuum in X between a and b_i.

(4.3) Example. Let I_n denotes a straight line interval joining points a_n and b_n where $a_0 = (0, 0)$, $b_0 = (0, 1)$, $a_n = (1/n, 0)$ and $b_n = (1/n, 1)$ for $n = 1, 2, \ldots$ For each $n = 1, 2, \ldots$ take two disjoint lines A_n and B_n lying in the strip $1/(n+1) < x < 1/n$ such that both approximate I_{n+1} a_n is a beginning of A_n and b_n is a beginning of B_n. Put

$$X = \bigcup_{i=1}^{\infty} (A_n \cup B_n \cup I_n) \cup I_0.$$

Then X is U-irreducible between a_0 and a_1; the point a_2 is not a point of U-irreducibility of X, but X can not be decomposed into two proper U-subcontinua each of which contains a_2.

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