Indecomposable continua and the fixed point property II

by

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Abstract. It is proved that if \( X \) is a hereditarily unicoherent Hausdorff continuum and if \( F : X \to X \) is a continuum-valued upper semi-continuous mapping, then there is an indecomposable continuum \( Q \subseteq X \) such that \( Q \subseteq F(Q) \).

1. Introduction. The main purpose of this paper is to prove a theorem which shows that for every hereditarily unicoherent Hausdorff continuum \( X \) and for every continuum-valued upper semi-continuous mapping \( F \) from \( X \) into itself there is an indecomposable continuum \( Q \subseteq X \) such that \( Q \subseteq F(Q) \). This theorem is a generalization of the main result of [5] to the nonmetric case and it implies the fixed point theorems contained in [4] and [6].

The theory of irreducible continua used in the proofs of mentioned theorem is investigated here in more general setting and in a such way that it also gives a theory of Whyburn's cycle which are here indecomposable in some sense. This general approach, also by results of [1] and [8], shows that the fixed point theorems are strongly connected with indecomposable continua.

2. Notations and lemmas. Let \( X \) be an arbitrary Hausdorff and let \( U \) be a fixed family of subcontinua of \( X \) which satisfies the following conditions:

(i) \( X \in U \) and \( \{x\} : x \in X \subseteq U \),

(ii) if \( A \) and \( B \) belong to \( U \) and \( A \cap B \neq \emptyset \), then \( A \cup B \in U \),

(iii) if \( A \) and \( B \) belong to \( U \) and \( B \setminus A = M \cup N \) where \( M \) and \( N \) are separated, then \( \{A \cup M, A \cup N\} \subseteq U \),

(iv) if \( A \) and \( B \) belong to \( U \) and \( B \setminus A \) is connected, then \( \text{cl}(B \setminus A) \in U \),

(v) if \( \emptyset \in U \), then \( \bigcap \{A : A \in \mathcal{P}\} \in U \),

(vi) if \( \emptyset \in U \) and \( \mathcal{P} \) is directed by the inclusion \( \subseteq \), then \( \text{cl}(\bigcup \{A : A \in \mathcal{P}\}) \in U \).

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If $A \in U$, then sometimes we say $A$ is a $U$-continuum. It follows from (vi) that for every two disjoint $U$-subcontinua $A$ and $B$ of $X$ there is a unique minimal $U$-continuum $C$ which intersects both $A$ and $B$. Every such $C$ is called $U$-irreducible between $A$ and $B$; we denote it by $AB$. In particular, $ab$ denotes a unique continuum $U$-irreducible between points $a$ and $b$. One can easily check.

(2.1) $AB$ is $U$-irreducible between every point of $AB$ and every point of $AB \cap B$.

(2.2) If $D \in U$, $A \cap D \neq \emptyset$ $\neq B \cap D$, then $AB \cap B$.

(2.3) If $X = ab$, $b \in C \subset U$, then $X \setminus C$ is connected and cl$(X \setminus C) \in U$.

The set of all points of a $U$-continuum $K$ which can be joined with a set $A \subset K$ by a proper $U$-subcontinuum $L$ of $K$ such that $A \subset L \subset U$ is called a $U$-component of $A$ in $K$ and we will denote it by $C(A, K)$. We say a continuum $C \in U$ is $U$-indecomposable if $C$ can not be decomposed into two proper $U$-subcontinua. Put

$I(a, b) = \bigcap \{cl(ab \setminus C) : b \in C \subset a \neq C \subset U\}$.

It follows from (v) and (2.3)

(2.4) $aeI(a, b) \in U$ and $I(a, b) \subset ab$.

Moreover,

(2.5) $ab = cl(ab \setminus C) \cup I(a, b)$.

Indeed, let $\emptyset = \{c \in U : b \in C \subset ab \neq C\}$. Then $ab = C \cup cl(ab \setminus C)$ for $c \in \emptyset$. Therefore,

$ab = cl\{C : C \in \emptyset\} \cup cl(ab \setminus C) = C(ab \setminus C) \cup I(a, b)$.

We have (compare [22], p. 210-211).

(2.6) $I(a, b)$ is a boundary set in $ab$ if and only if $I(a, b) \subset C(ab, ab) \neq \emptyset$.

In fact, suppose $I(a, b) \subset C(ab, ab) \neq \emptyset$. Then there is $c \in U$ such that $b \subset C \subset ab \neq C$ and $C \cap I(a, b) \neq \emptyset$; thus $ab = C \cup I(a, b)$, and since $C$ is a proper subset of $ab$ we conclude $I(a, b)$ is not a boundary set, a contradiction.

Conversely, suppose the interior of $I(a, b)$ is nonempty in $ab$. Then, by (2.3), $D = cl(ab) \setminus I(a, b) \in U$ and $D$ is a proper subset of $ab$ containing $b$. Hence $D \subset C(ab)$. But $D \cap I(a, b) \neq \emptyset$, because $ab = D \cup I(a, b)$. Therefore, $I(a, b) \subset C(ab, ab)$, a contradiction.

(2.7) $I(a, b)$ is not a boundary set in $ab$, then it is a $U$-indecomposable continuum which has at least two $U$-components.

Indeed, it follows from (2.3) that if $D = cl(ab \setminus I(a, b))$, then $b \in D \in U$ and $D$ is a proper subset of $ab$. Suppose $I(a, b) = A \cup B$ where $A$ and $B$ are proper $U$-subcontinua of $I(a, b)$ and $a \in B$. Then $ab = D \cup A \cup B$. If $B \cap D \neq \emptyset$, then $ab = D \cup B$, thus $I(a, b) \subset cl(ab \setminus D) \subset B$, a contradiction. If $B \cap D = \emptyset$, then $A \cap D \neq \emptyset$. Moreover, $b \in A \cup D \in U$ and $A \cap D \neq ab$, thus $I(a, b) \subset cl(ab \setminus (D \cup A)) \subset B$, a contradiction.

(2.8) If $D \in U$ and $C(a, b) \cap bD = \emptyset$, then $D = ab \cup bD$. Moreover, if $D = \{d\}$, then $C(a, b) \cap bD = \emptyset$ and $I(a, b) \subset C(a, b)$.

In fact, since $ab \cup bD$ is continuum intersecting $[a]$ and $D$ which belongs to $U$, we infer $ab = ab \cup bD$ by (2.2). Then $ab = (ab \cap aD) \cup (bD \cap aD)$. This equality and the connectedness of $ab$ imply that there is a point $z \in (ab \cap aD) \cap (bD \cap aD)$. Then $z \in bD$ and $C(a, b) \cap bD = \emptyset$. Therefore $ab = ab$. Since $[a, z] \subset ab$, we conclude, $az = ab$; thus $ab \subset ab$. Consequently $[a, z] \subset C(a, b)$ which implies $ab \cup bD = aD$. Therefore $ab = ab \cup bD$. If $D = \{d\}$, then equalities $C(a, b) \cap bD = \emptyset$ and $ad = ab \cup bD$ imply that $ad$ is a proper $U$-subcontinuum of $ab$ containing $d$ and belonging to $U$; thus $bd = C(ab, ab)$. If $x \in C(b, ab)$, then $xb \neq ab \setminus a$. We conclude that $xb \cup bd$ is a proper $U$-subcontinuum of $ad$ containing $d$, because $xb \cup bd = C(ab, ab)$. Therefore $x \in C(b, ab)$, so $xb \neq ab \setminus a$. Consequently $C(b, ab) \cup bD \subset [d, b)$. Moreover, if $b \in C(U)$ and $ab \neq C \subset ab$, then $I(a, d) \subset cl(ab \setminus C \cup bD) = cl(ab \setminus C)$. This implies $I(a, d) \subset C(a, b)$ which completes the proof of (2.8).

(2.9) If $[K, L] = U$, $K \subset L \cap I(b, a) \neq I(b, a)$ and $K \cap C(ab, ab) = \emptyset$, then $L \cap (C(ab, ab) \neq \emptyset)$.

Suppose $L \cap C(ab, ab) = \emptyset$ for some $C \in U$ such that $aeC \subset ab \neq C$. Then $ab \subset C \subset U$. Therefore $I(b, a) \subset cl(ab \setminus C) \subset L$, a contradiction.

(2.10) If $I(c, d) \subset I(b, a) \neq I(c, d)$ and $I(c, d) \cap C(ab, ab) = \emptyset$, then $cd \cap (a, ab) = \emptyset$.

Since $I(c, d) \subset I(b, a) \neq I(c, d)$, there is a continuum $D \in U$ such that $d \in D \subset cd \neq D$ and $cl(cD) \cap I(b, a) \neq I(b, a)$. Therefore, if $E = cl(cD) \cap D$, then $CE = E \cap D = \emptyset$ by (2.9). We conclude $cd \subset E \cup D$. Suppose that $D \cap C(ab, ab) \neq \emptyset$ and let $F \in C$ be a proper $U$-subcontinuum of $ab$ containing $a$ and intersecting $D$. If $F \cap E \neq \emptyset$, then $ab \subset F \subset E$, thus $I(b, a) \subset cl(ab \setminus F) \subset I(b, a) = E \cap I(b, a) \neq I(b, a)$, a contradiction. Thus $E \cap I(b, a) = \emptyset$. If $E \cap I(b, a) \neq ab$, then $I(b, a) \subset ab \cap E \cup I(b, a) \subset F \cup D$; in particular $E \subset D$, a contradiction. Thus $I(b, a) \subset cl(ab \setminus F \cup D) \subset E$. But $E \cap I(b, a) \neq I(b, a)$ by the construction, a contradiction.

(2.11) If $C$ is a $U$-component of $a$ in $K \in U$ and it is not closed in $K$, then it is dense in $K$. 
In fact, the collection of all proper $U$-subcontinua of $K$ containing $a$ is directed by the inclusion; thus $\text{cl}C \in U$. According to the assumptions $C \neq C$, thereby $C = K$.

(2.12) Different $U$-componants in a $U$-indecomposable subcontinuum $K$ of $X$ are disjoint and if $K \neq L \subset K$ and $L \in U$, then $L$ is a boundary set in $K$.

Remark. Let $X$ be a metric continuum. Applying Kuratowski's proof used in [3] one can prove that point $a \in X$ is a point of $U$-irreducibility of $X$ if and only if there do not exist two proper $U$-subcontinua $P$ and $R$ of $X$ such that $X = P \cup R$ and $a \in P \cap R$. As a corollary we can obtain many theorems known for ordinary irreducible metric continua; for example: $X$ is $U$-indecomposable if and only if every proper $U$-subcontinuum of $X$ is a boundary set; if $X$ is $U$-indecomposable, then the collection of $U$-componants of $K$ is an uncountable collection members of which are pairwise disjoint and bound in $X$; every $U$-indecomposable continuum is $U$-irreducible between some points etc.

A corollary of it is also the fact that the closure of the union of an increasing family of subcontinua $ab$, always $U$-irreducible between $a$ and some point $b$. This last theorem we will prove in the nonmetric case (compare [4], Lemma (ii)).

(2.13) Let a point $a \in X$ be fixed and $\mathcal{P}$ be a nested family of $U$-irreducible subcontinua $ab$ of $X$. If $\mathcal{F} = \text{cl}(\cup \{ab : a \in \mathcal{P}\})$, then there is a point $p$ and a $U$-indecomposable subcontinuum $P$ of $X$ such that $ap \cup P = \mathcal{F}$ and every $U$-componant in $P$ is equal to $P$.

Proof. First note that $\mathcal{F} \cap U$ by (vi). We may assume that $ab \neq \mathcal{F}$ for $ab \in \mathcal{P}$. Moreover

$$ (1) \quad \mathcal{F} \cap ab = \text{connected for } ab \in \mathcal{P}. $$

Suppose that $\mathcal{F} \cap ab = M \cap N$ where $M$ and $N$ are separated. Then $ab \cup M \cap N \cap N = M \cap N$ belong to $U$ by (iii). If $c \in ab \cup M$ and $c \neq b$, then there is $d \in ab \cup M$ such that $cd \in \mathcal{P}$ (otherwise all $d \in \mathcal{P}$ such that $ac \cap cd$ are contained in $ab \cup M \cap N \cap N$ because $\mathcal{P}$ is nested). But then $ac \cap cd$ and $d \in M \cap N$, a contradiction.

Let $C = \bigcap \{\text{cl}(\mathcal{F} \setminus ab) : ab \in \mathcal{P}\}$. It follows from (1) and (iv)-(v) that

$$ (2) \quad C \in U \text{ and } \text{cl}(\mathcal{F} \setminus ab) \in U \text{ for } ab \in \mathcal{P}. $$

Moreover, we obviously have (compare the proof of (2.5)).

$$ (3) \quad \mathcal{F} = \bigcup \{ab : ab \in \mathcal{P}\} \cup C = ab \cup \text{cl}(\mathcal{F} \setminus ab) \text{ for } ab \in \mathcal{F}. $$

If $\text{cl}(\mathcal{F} \setminus ab)$ is $U$-indecomposable for some $ab \in \mathcal{P}$, then, if $\text{cl}(\mathcal{F} \setminus ab)$ has more than one $U$-componant, we can take a point $p \in \text{cl}(\mathcal{F} \setminus ab) \cap \text{cl}(\mathcal{F} \setminus ab)$, we obtain $ap = ab \cup \text{cl}(\mathcal{F} \setminus ab) = \mathcal{F}$, which completes the proof. If $\text{cl}(\mathcal{F} \setminus ab)$ has only one componant, taking $p = b$ and $P = \text{cl}(\mathcal{F} \setminus ab)$ we also find $\mathcal{F} = ap \cup P$. Therefore we may assume that $\text{cl}(\mathcal{F} \setminus ab)$ are $U$-indecomposable for $ab \in \mathcal{P}$. We claim that

$$ (4) \quad ab \cap C = \emptyset \text{ for } ab \in \mathcal{P}. $$

In fact, suppose $ab \cap C \neq \emptyset$ for some $ab \in \mathcal{P}$. Since $\text{cl}(\mathcal{F} \setminus ab)$ is $U$-indecomposable there are proper $U$-subcontinua $Q$ and $P$ of $\text{cl}(\mathcal{F} \setminus ab)$ whose union is $\text{cl}(\mathcal{F} \setminus ab)$. If $ab \cap Q \cap R$ is nonempty, we obtain the same arguments as in the proof of (1), that $\mathcal{F} \cap (ab \cup (Q \cup R))$ is a contradiction, because $\mathcal{F} \cap (ab \cup (Q \cup R)) \subseteq Q \cup R \cup Q \subseteq \mathcal{F} \cap (ab \cup (Q \cup R))$. Assume $ab \cap R = \emptyset$. If for each $ab \in \mathcal{P}$ there is $ac \in \mathcal{P}$ such that $ac \cap ab = \emptyset$ and thus $\mathcal{F} \cap ab = \emptyset$. Therefore there is $ac \in \mathcal{P}$ such that $ac \cap ab = \emptyset$ and $ac \cap ab = \emptyset$, then $ac \in \mathcal{P}$. But then $ac \cup R$ contains all $ac \in \mathcal{P}$. Thus $ac \cup R$ contains all $ac \in \mathcal{P}$. We conclude $C = \text{cl}(\mathcal{F} \setminus ac) = R$. Since $ab \cap R = \emptyset$, we conclude $ab \cap C = \emptyset$, a contradiction, i.e. (4) holds.

Now it suffices to show that $\mathcal{F} = ac \subset C$. Clearly $ac \subset \mathcal{F}$. So it will suffice to show that $ac \subset \mathcal{F}$ for $ac \in \mathcal{F}$. From (4) we conclude that there is $ad \in \mathcal{P}$ such that $ab \cap \text{cl}(\mathcal{F} \setminus ad) = \emptyset$. It follows from (3) that $ac \in \mathcal{P}$ such that $ac \cap ab = \emptyset$ and $z \in \text{cl}(\mathcal{F} \setminus ad)$. Therefore $ac \cap ab = \emptyset$ and $\text{cl}(\mathcal{F} \setminus ad)$. Consequently $ac \subset \mathcal{F}$. This inclusion completes the proof.

(2.14) Let a point $a \in X$ be fixed and $\mathcal{P}$ be a nested family of $U$-irreducible subcontinua $ab$ of $X$ and $P = \bigcup \{ab : ab \in \mathcal{P}\}$. If $ac = \text{cl}P$, then $I(c, a)$ is a boundary set in $ac$ and $ac \neq b \cap \text{cl}(\mathcal{F} \setminus ac)$. Consequently $ac \subset \mathcal{F}$. The inclusion $P \subset C(a, ac)$ is obvious. Let $e \in C(a, ac)$. Then $ac \cap e \neq ac$. Since $I(c, a)$ is a boundary set in $ac$, we infer that $C(a, ac) \cap I(c, a) = \emptyset$ by (2.6), thus there is a proper subcontinuum $L$ of $ac$ such that $ac \cap (ac \cup L) = \emptyset$. Since $\text{cl}(ac \cup L) = ac$, there is $b \in \mathcal{P}$ such that $b \in \text{cl}(ac \cup L)$. The equality $ac \cap (ac \cup L) = ac$ implies that $ac \subset \mathcal{F}$.

3. Fixed point properties. Recall that $F : X \to X$ is upper semi-continuous (u.s.c.) provided each point image $F(x)$ is closed set and whenever $U$ is an open set containing $F(x)$, there exists an open set $V$ containing $x$ such that $F(t) \subset U$ for each $t \in V$. We say the mapping $F$ is $U$-valued for each $K \in U$ we have $F(K) \cap U$. Such a mapping has a fixed point if there is a point $x \in X$ such that $x \in F(x)$. If $F : X \to X$ is $U$-valued and $A \subset X$ then we put

$$ \mathcal{P}(x) = \{ab : b \in F(a) \cap C(a, ab) = \emptyset\}, $$

and

$$ \mathcal{P}(a, x) = \{ab : a \in \mathcal{P}(a) \text{ and } I(a, b) \subset A\}. $$

We have

(3.1) Theorem. If a mapping $F : X \to X$ is $U$-valued and u.s.c., for each nondegenerate $U$-indecomposable subcontinuum $K$ of $X$, the set $K \cap F(K)$ is a proper subcontinuum of $K$ and if $\mathcal{P}(a, x) \neq \emptyset$, then there is a maximal element in
If \( P(M, A) \), then there is a point \( c \) such that \( a = B \) and \( d \in P(M, A) \) for some \( d \in P(M, A) \).

Consider five cases.

(a) \( I(r, s) \cap F(I(r, s)) = \emptyset \) and \( I(r, s) = \emptyset \), not a boundary set in \( rs \). From (2.7) it follows that \( I(r, s) \) is a U-indecomposable continuum which has at least two U-components. If \( c \in F(I(r, s)) \cap C(r, I(r, s)) \) then \( r \subseteq U \cap F(I(r, s)) \) because \( U \) is a U-continuum intersecting both \( I(b, a) \) and \( F(I(r, s)) \). Therefore, \( F(I(r, s)) \) is a U-continuum in \( I(r, s) \) and \( F(I(r, s)) \) is a U-continuum in \( I(r, s) \) with \( r \subseteq U \cap F(I(r, s)) \) and \( s \subseteq U \cap F(I(r, s)) \). From (6.2) and (2.8) we conclude \( aw = ab \cup rw \) and \( I(a, w) = A \). Since \( C(a, aw) = ab \cup C(r, I(r, s)) \) and \( wI(r, s) \) we can infer \( aw \in I(a, A) \); a contradiction to the maximality of \( ab \).

(b) \( I(r, s) \cap F(I(r, s)) \neq \emptyset \) and \( I(r, s) = \emptyset \) is a boundary set in \( rs \). From (2.7) the continuum \( rs \) is U-indecomposable between \( s \) and \( t \) point of \( I(r, s) \).

Let \( R \) be a minimal subcontinuum of \( I(r, s) \) with respect to the property: \( F(R) \cap F(I(r, s)) = \emptyset \). Let \( t \in R \cap F(I(r, s)) \). Since \( F(R) \cap F(I(r, s)) = \emptyset \), we conclude \( I(r, s) \subseteq F(R) \); thus \( rt \subseteq F(R) \).
= R and if S is a proper subcontinuum of rt such that r ∈ S U U then
F(r) ∩ F(r) = ∅. Therefore I(r, s) = F(r, C(r, r)). In particular I(r, r) ⊂ F(r, r), but I(r, s) is a boundary set by (2.7) and the assumptions. It follows from (2.7) that I(r, s) ⊂ C(r, r) = ∅. There is a point w ∈ I(r, s) such that F(r, w) ∩ I(r, s) = ∅. Since wF(w) ∈ F(r, r), we conclude aw ∈ P(a, A); a contradiction.

(c) I(r, s) ∩ F(I(r, s)) = ∅ and I(r, s) = rs. Put Q = F(I(b, a), F(I(r, s)), s). Since Q ⊂ X \ C(r, r) and Q is a U-contiuum intersecting both \{s\} and F(s), we conclude wF(s) ⊂ Q ⊂ X \ C(r, r). As in case (a) obtain as = ab ∪ rs ∈ P(a, A), which contradicts the maximality of ab.

(d) I(r, s) ∩ F(I(r, s)) = ∅, I(r, s) = rs and I(r, s) is not a boundary set in rs. Then there is a point w such that w ∈ I(r, s) \ ab, rw = I(r, s), rs = I(r, s) \ w and ws is U-irreducible between I(r, s) and s. Since wF(s) = ab ∪ rs = w ∈ P(a, A), we have ab = ab ∪ rs ∈ P(a, A). One can easily check that aw = ab ∪ rw ∈ P(a, A), a contradiction.

(e) I(r, s) = F(I(r, s)) = ∅ and I(r, s) is a boundary set in rs. The upper semi-continuity of F and the definition of I(r, s) implies that there is Q ∈ U such that Q ∩ F(r) = ∅, I(r, s) ⊂ C(r, r) and Q(I(r, s), ∅). Since I(r, s) is a boundary set in rs, we infer p ⊂ I(r, s) = ∅ for p ∈ Q. I(r, s). Let rw be a continuum in Q which is U-irreducible between r and p ∩ Q. It is easy to show that aw ∈ P(a, A), which contradicts the maximality of ab. In this way the proof of Theorem (3.1) is complete.

The main result of the paper is the following:

(3.2) Theorem. If a mapping F : X → X is U-valued and u.s.c., and every U-
indecomposable continuum of X has only dense U-composants, then there is a
U-indecomposable continuum K of X such that K ⊂ F(K).

Proof. Consider the following family \( \mathcal{W} \) of subsequences of X: \( Q ∈ \mathcal{W} \)
provided:

1. \( Q ∈ U \);
2. \( Q ∩ F(Q) = ∅ \);
3. if A is a proper U-subcontinuum of Q and ab ∈ P(ab), then Q ∩ A = ∅.

(4) if ab is a maximal element in P(a, A), and (b, a) is a boundary set in ab, then I(b, a) ∈ \( \mathcal{W} \).

Conditions (1) and (2) of the definition of \( \mathcal{W} \) are satisfied by Theorem (3.2).

Now, let K be a proper U-subcontinuum of I(b, a), and c ∈ P(c, K). It follows from (2.10) that C(a, ab) ∩ cd = ∅; then ad = ab ∪ cd ∈ P(a, A) by (2.8). Thus the maximality of ab in P(a, A) implies (3).

Now we claim that

(5) if ab is a maximal element in P(a, A), then there is a continuum \( Q ∈ \mathcal{W} \) such that \( Q ⊂ a_{b} ∩ C(a_{b}, a_{b}) \).

According to (4) and Propositions (2.6) and (2.7) we may assume that \( Q_{b} \) is an U-indecomposable continuum with at least two U-composants where \( Q_{b} \)

= I(b_{a}, a_{b}). By assumptions we obtain \( Q_{b} ∩ F(Q_{b}) \) is a proper subcontinuum of \( Q_{b} \). Since \( a_{b} ∈ P(a_{b}, A_{b}) \) we infer

(5) \( Q_{b} ∩ F(Q_{b}) ∩ C(a_{b}, a_{b}) = ∅ \).

Take a point a_{i} from \( Q_{b} \) \( (Q_{b} ∩ F(Q_{b})) \) which belongs to the component of \( Q_{b} \) containing \( Q_{b} ∩ F(Q_{b}) \) (we can find \( a_{i} \), because U-composants in \( Q_{b} \) are dense by assumptions). Let \( a_{i} ∈ L \) be a continuum U-irreducible between \( a_{i} \) and \( Q_{b} ∩ F(Q_{b}) \). Then \( a_{i} ∈ C(a_{i}, Q_{b}) \) and \( a_{i} ∈ P(a_{i}, a_{i}) \). By Theorem (3.1) there is a maximal element \( a_{i} ∈ P(a_{i}, a_{i}) \). Then \( a_{i} ∈ C(a_{i}, Q_{b}) \) by the maximality of \( a_{i} \), and conditions (2.8) and (2.10). We find \( Q_{b} = I(b_{a}, a_{b}) \). If \( Q_{b} \) is a boundary set in \( a_{i} \), then \( Q_{b} ∈ \mathcal{W} \) by (4). Thus we can assume that \( Q_{b} \) is an indecomposable continuum and as above we find \( Q_{b} = a_{i} \) and \( d_{i} \) etc. Take \( Q = \bigcap_{n=0}^{N} Q_{n} \). We will check that \( Q ∈ \mathcal{W} \). Since \( Q ∈ U \) and \( Q ∩ F(Q) = ∅ \), i.e. (1) and (2) hold. Now let K be a proper U-
subcontinuum of Q and \( cd ∈ P(c, K) \). Then K is a proper subcontinuum of \( Q_{b} \) for each n. Moreover, K ∩ (a_{b}, a_{b}) = ∅ for each n. Thereby, \( a_{b} ∈ P(a_{b}, A_{b}) \) and \( cd ∈ P(a_{b}, A_{b}) \) by (2.8) and (2.10). The maximality of \( a_{b} \) in \( P(a_{b}, A_{b}) \) implies \( cd ∈ I(b_{a}, a_{b}) = Q_{b} \). This means that \( cd ∈ Q_{b} \), i.e. condition (3) of the definition of \( \mathcal{W} \) holds. The proof of (5) is complete.

Now, let Q be a continuum in X which is minimal in \( \mathcal{W} \) and L be a minimal subcontinuum of Q with respect to the properties: \( L ⊂ U \) and \( L ∩ F(L) = ∅ \).

Then \( L = aF(a) \) for some \( a \). If L is degenerate, then \( L ⊂ F(L) \) and Theorem (3.2) is proved. So let L be nondegenerate. If L is U-decomposable, then I(a, F(a)) is a proper U-subcontinuum of Q and \( L ⊂ P(a, A, F(a)) \). If L is U-
indecomposable, then \( L ∩ F(L) \) is a nonempty proper U-subcontinuum of L (otherwise \( L ⊂ F(L) \), and taking \( a_{i} ∈ C(L) ∩ F(L), L \) (L ∩ F(L)) (L has at least two composedants and therefore they are dense in it) and a continuum \( a_{i} \) U-
irreducible between \( a_{i} \) and \( L ∩ F(L) \) we find \( a_{i}∉ P(a_{i}, a_{i}) \) because \( F(a_{i}) \) is a continuum \( a_{i} ∩ L \subset C(a_{i}, a_{i}) \). Therefore in both cases we have a proper U-subcontinuum D of Q such that \( a_{b} ∈ D \). D ⊂ L. M. If \( a_{b} \) is a maximal element in \( P(a, A) \) contained in Q, because \( Q ∈ \mathcal{W} \); but then, there is a proper U-subcontinuum \( Q_{b} \) of Q such that \( Q_{b} ∈ \mathcal{W} \) by (5). It is a contradiction to the minimality of Q in \( \mathcal{W} \). Therefore Q is degenerate and \( Q = F(Q) \) which finishes the proof of Theorem (3.2).

Remark. The main and some parts of the proofs of Theorems (3.1) and (3.2) are almost the same as in [5]. We repeat them only for the completeness of this paper.

4. Universal subcontinua. We say that a subcontinuum K of X is universal if X is connected. The family of all universal continua of a continuum X satisfies conditions concerning U
without (iv) and (vi). But it is easy to see that if $X$ is either locally connected or hereditarily unicoherent then these two conditions also are satisfied. If a continuum $X$ is locally connected, then universal subcontinua of $X$ are exactly these subcontinua of $X$ which are unions of Whyburn's cycles (compare [8]), and a universal continuum which is not the union of two proper universal subcontinua (i.e. indecomposable in the class of universal subcontinua) is some Whyburn's cycle. If $X$ is hereditarily unicoherent, then all its subcontinua are universal and then indecomposable continua satisfy the additional assumption from Theorem (3.2). According to Lemma 3 in [7], p. 161 from Theorem (3.2) we obtain

(4.1) Corollary. If $X$ is a hereditarily unicoherent Hausdorff continuum, $F: X \to X$ is u.s.c. and such that $F(x)$ is a continuum for each $x \in X$, then there is an indecomposable continuum $K$ such that $K = F(K)$.

This result is a generalization of Theorem 1 from [5] to the nonmetric case.

The following problems remain open:

For which collections $U$ is it true that if every $U$-indecomposable subcontinuum of $X$ has a fixed point property then $X$ has it also?

Some answers are given by Corollary (4.1) (compare [5]), some others are contained in [1] and [4].

If $U$ denotes the family of all universal subcontinua of $X$, then the theory of $U$-irreducible continua investigated in previous sections to obtain some fixed point theorems is not so nice. It can be see from the following examples.

(4.2) Example. Let $I_n$ denote a straight line interval lying in the Euclidean plane $E^2$ and joining $a = (0, 1)$ with a point $b_n$ where $b_0 = (0, 0)$ and $b_n = (1/n, 0)$ for $n = 1, 2, ...$. Consider a continuum $X$ which is a union of $I_n$ for $n = 0, 1, 2, ...$ and disjoint lines $C_k$ such that $b_k$ is a beginning of $C_k$ and $C_k$ approximates $I_n$ for $n = 1, 2, ...$. Then $X$ is $U$-indecomposable and $X = \bigcup_{i=0}^\infty ab_i$ where $ab_i$ is a $U$-irreducible continuum in $X$ between $a$ and $b_i$.

(4.3) Example. Let $I_n$ denotes a straight line interval joining points $a_n$ and $b_n$ where $a_n = (0, 0)$, $b_n = (1/n, 0)$ and $a_n = (1/n, 1)$ for $n = 1, 2, ...$. For each $n = 1, 2, ...$ take two disjoint lines $A_n$ and $B_n$ lying in the strip $1/(n+1) < x < 1/n$ such that both approximate $I_{n-1}$, $a_n$ is a beginning of $A_n$ and $b_n$ is a beginning of $B_n$. Put

$$X = \bigcup_{i=1} A_i \cup B_i \cup I_0.$$

Then $X$ is $U$-irreducible between $a_0$ and $a_1$; the point $a_2$ is not a point of $U$-irreducibility of $X$, but $X$ cannot be decomposed into two proper $U$-subcontinua each of which contains $a_2$. 

References


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