Perfect set theorems for $P_{1/2}$
in the universe without choice

by

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Abstract. We work in the theory ZF. We prove the following Theorem 4: if there is a regular
ordinal number $\sigma$ such that there is no function from the continuum onto $\sigma$, then every $P_1$
set either is well-orderable or has a perfect subset in a boolean extension of the universe. Hence we
obtain under the assumption $\sigma_0$ exists and $\sigma_1$ does not exist the following Theorem 5: if there
is a regular ordinal $\sigma$ such that there is no function from the continuum onto $\sigma$, then every $P_1$
set either is well-orderable or has a perfect subset. As one of the corollaries we obtain Theorem 7: if $(x)$
($x^*$ exists) and $\sigma^*$ does not exists, then from every $P_1$ set there is a function onto $\sigma_1$. All these
results follow from a construction, for a given $P_1$ set $A$, of a certain tree, the notion of a tree which
we use being somewhat different from the usual one. $A$ is the projection of that tree. This method
was introduced in [2] and the present paper shows how it can be applied.

In §0 we give a review of the present state of knowledge about perfect subsets of $P_1$ sets, we
prove several easy remarks and give the discussion of our theorems. In §1 we give a construction
of a special tree for a given $P_1$ set. In further sections we prove the theorems.

§0

By the perfect set theorem for a class $\Gamma$ of subsets of $\omega^\omega$ we mean the
following: every set in $\Gamma$ either is countable or has a perfect subset.

Let us first recall the well-known perfect-set theorems. For $P_1$ or $\mathcal{Z}_1$ sets
this is the Mansfield-Solovay theorem. Let us formulate it as follows (see [5],
[8]): if $(x)_x=\omega^{\omega\omega}$ is countable, then every $P_1$ set either is countable or
contains a perfect subset. The proof of this theorem uses the existence of a tree
$T$ for a given $P_1$ set $A$ such that $T \leq \omega^{\omega\omega} \times \omega_1^{\omega\omega}$ and
$A(x) = \langle E, \langle x, f \rangle \rangle$ is a branch of $T$ (see [9]). This characterization of $A$ has the following absoluteness property: if $M \subseteq N$ are inner models and
$T^M$, $T^N$ are defined for $A$ in $M$, $N$ respectively, then $T^M = \Gamma^N \cap (\omega^{\omega\omega} \times (\omega_1^{\omega\omega})^{\omega\omega})$. Hence

(*) if for $x$ there is an $f$ such that $\langle x, f \rangle$ is a branch of $T^M$ for an inner
model $M$, then $A(x)$.

If $T$, $M$ have property (*) we shall say that $T$ has property (*) w.r.t. $M$.

Let us discuss other perfect-set theorems.
If there is exactly one measurable cardinal and $P(\omega) \cap L[\mathcal{M}]$ is countable, then we have the perfect set theorem for $\Pi^0_1$ (see [6]). Again for a $\Pi^0_1$ set $A$ we have a tree $T$ with property $(\ast)$ w.r.t. $L[\mathcal{M}]$. If there is a measurable cardinal and $P(\omega) \cap \text{HOD}$ is countable, then the perfect-set theorem holds for $\Pi^0_1$. Moreover, for every $\Pi^0_1$ set $A$ there is a tree $T$ with property $(\ast)$ w.r.t. $\text{HOD}$. 

Also under the assumption of $\delta_1^1$-determinacy the perfect-set theorem holds for $\Pi^0_1$, even for $\Pi^0_2$ and $\Sigma^1_2$ sets. Again in this case every $\Pi^0_1$ set is a projection of the set of branches of a tree.

There is also another method of finding a perfect subset of a $\Pi^0_1$ set $A$.

Consider the following remarks. If $M$ is a class, $\Pi^0_1(M)$ denotes $\Pi^0_1$ in a parameter from $M$.

**Remark 1.** Let $M$ be an inner model and $P$ a set of forcing conditions in $M$, $P^M(P) \simeq \omega$. Let $A$ be $\Pi^0_1(M)$. Let $\varnothing \in M^x$ be such that $M^x = A(\varnothing)$ and for every $\varnothing \in P$ there are $q_1, q_2 \in \varnothing$ and $m_1, m_2 \in \varnothing$ such that $m_1 \neq m_2$, $q_1 \in \varnothing(\varnothing) = m_1$ and $q_2 \in \varnothing(\varnothing) = m_2$. Then $A$ has a perfect subset.

**Proof.** By the assumption that $P^M(P) \simeq \omega$ we can enumerate all dense subsets of $P$ belonging to $M$ as $D_0, D_1, \ldots$. Let us define the following mapping $\sigma$ from $2^{\omega \times \omega}$ into $P$. Let $\sigma(\varnothing)$ be any condition in $D_0$. If $\sigma(\varnothing)$ is defined, $\sigma(\varnothing) = p$, then let $\sigma(\varnothing(\varnothing))$, $\sigma(\varnothing(\varnothing))$ be such conditions $q_1, q_2 \in \varnothing$ that

1. there are $n, m_1, m_2 \in \varnothing$ such that $q_1 \in \varnothing(\varnothing) = m_1$, $q_2 \in \varnothing(\varnothing) = m_2$,
2. $q_1, q_2$ determines the values of $\varnothing$ at $m_1 + 1$,
3. $q_1 \in D_{m_1 + 1}$.

To find $q_1, q_2$ we first take $q_1, q_2$ satisfying (1) (they exist by the assumptions), next we take $q_1 \leq q_1, q_2 \leq q_2$ so that $q_1, q_2$ satisfy (2) and then we take $q_1, q_2$ so that $q_1 \leq q_1$ and $q_2$ satisfies (3).

By definition, $\sigma(\varnothing) \in D_{m_1}$.

Define an induced mapping $\sigma^*: 2^{\omega \times \omega} \to 2^{\omega}$ as

$$\sigma^*(f)(m) = m \text{ if } \sigma(f_{n+1}) \in \varnothing(\varnothing) = m.$$ 

We shall show that

(i) $\sigma^*: 2^{\omega \times \omega} \to A$,
(ii) $\sigma^*$ is continuous,
(iii) $\sigma^*$ is 1-1.

Consider (i). Notice that $\sigma(f_{n})_{n \in \omega}$ generate a $P, M$-generic filter, $G$. Indeed, this follows by the fact that $\sigma(\varnothing) \in D_{m_1}$. Moreover, $\sigma^*(f) = \varnothing$. Hence $\sigma^*(f) \in A$ because $M^x = A(\varnothing)$ and $A$ is absolute.

Consider (ii). Let $t$ be an initial segment of $\sigma^*(f)$, dom $t = n + 1$. Then by the definition of $\sigma^*$, $\sigma(f_{n+1}) \in \varnothing(\varnothing)$. Hence, for every $t^\prime$, if $f_{n+1} = f_{n+1}^\prime$, then $\sigma(f_{n+1}^\prime) \in \varnothing(\varnothing)$ and thus $\tau \leq \sigma^*(f')$. Hence follows the continuity of $\sigma^*$.

Consider (iii). Let $f \neq f'$. Let $n$ be the first number such that $f(n) \neq f'(n)$. Then, by the definition of $\sigma$,

$$\sigma(f(n+1)) \in \varnothing(\varnothing) = m_1,$$

$$\sigma(f'(n+1)) \in \varnothing(\varnothing) = m_2.$$ 

for different $m_1, m_2$. Hence $\sigma^*(f) \neq \sigma^*(f')$.

By (i), (ii), (iii), the image of $2^{\omega \times \omega}$ under $\sigma^*$ is a perfect subset of $A$.

**Remark 2.** Let $M$ be an inner model and $C$ a complete boolean algebra in $M$, $P^M(P) \simeq \omega$. Let $A$ be $\Pi^0_1(M)$ and assume that there is a $G$ generic over $M$, $C$ such that $A$ has an element in $M[G]$. Then $A$ has a perfect subset.

**Proof.** Let $a$ and $G$ such that $A(a), G$ is generic over $M$ and $\varnothing \in M[G] = \varnothing$. Then, by the absoluteness of $A, M[G] = A(\varnothing)$. Let $\varnothing \in M^x$ be such that $M^x = \varnothing \in \omega^\varnothing$ and $i(\varnothing) = a$. Let $p$ be such that $p \in C \cap G, p \models \varnothing(a)$. We can assume that $p$ is the I of $C$ because otherwise we can restrict $C$ to $p$. Consider the following subalgebra of $C$. $C$ work in $M$. Let $C$ be the complete subalgebra of $C$ generated by the values $i(\varnothing) = \varnothing$ for $n, m \in \varnothing$. We shall show that there is a $p \in C$ such that $C$ restricted to $p$ is atomless. Suppose the converse, i.e., that under every element of $C$ there is an atom of $C$. Then every filter generic over $C$, $M$ is principal. Consider $G$. Let $G' = G \cap C$. Then $G'$ is generic over $C$. $C$ and $G'$ is principal in $C$. Thus $G' \in M$. $G' \in M$. $G' \in M$. $G' \in M$.

So let $p$ be an element of $C$ such that $C$ restricted to $p$ is atomless. Again, we can assume that $p$ is the I of $C$. Now we shall show that $C$ satisfies the assumptions of Remark 1. We can treat $\varnothing$ a generic of $M^x$. Then $M^x = A(a)$ because $M^x = A(\varnothing)$ and $C$ is a complete subalgebra of $C$.

To prove the main assumption of Remark 1, take $p \in C$. By the fact that $C$ is atomless, there are $G'G'$ such that $G'G'$ are different and generic over $M, G \cap G'$. Then there is a generator of $C$ of the form $\varnothing(\varnothing) = \varnothing$ which is in $G'G'$ or in $G'G'$ because otherwise $G', G'$ being equal at the generators, would be equal. Let $\varnothing \in p \in G'G'$. $G'G'$ is equal at the generators, $\varnothing \in p \in G'G'$.

Then $q_1, q_2$ are as required in Remark 1. Thus, by Remark 1, $A$ has a perfect subset. 

In [1], [2] we studied $\Pi^0_1$ sets for which there is a tree $T$ and a family $\mathcal{D}$ of its dense subsets such that $A$ is the collection of the family of $\mathcal{D}$-generic branches of $T$ or a projection of such a collection. If this characterization is absolute in the following sense:

$$\ast \ast \ast$$

If $M$ is an inner model and $T^M, \mathcal{D}^M$ are defined in $M$ for $A$ and $(E'/(a, f'))$ is a $\mathcal{D}^M$-generic branch of $T^M$ then $A(a)$, then we can infer that $A$ is countable or has a perfect subset in $V^C$ where $C$ enumerates with natural numbers the family $\mathcal{D}$ and $P(\omega)$. Indeed, in this case, if $A$ is not countable then the assumptions of Remark 2 are satisfied in $V^C$ with $M = V$. 

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In this paper we shall define an arbitrary $\mathcal{P}_1$ set $A$, a tree $T$ and a collection $\mathcal{S}$ of its dense subsets such that $A$ is a projection of the collection of $\mathcal{S}$-generic branches of $T$ (Theorem 1). However, this representation does not have the absoluteness property (**). Nevertheless, it will help us to prove a certain theorem about perfect subsets in ZF without choice. That theorem is the main result of the paper and states the following (Theorem 4):

If there is a regular ordinal $\kappa$ such that there is no function from the continuum onto $\kappa$, then every $\mathcal{P}_1$ set either is well-orderable and of power less than $\kappa$ or has a perfect subset in a boolean extension of the universe.

Now we will discuss the question how our theorem is related to the present knowledge about perfect subsets of $\mathcal{P}_1$ sets.

Consider again the known perfect-set theorems for a class $\Gamma$. They are of the following form: "If certain assumptions hold, then for every set $A$ in $\Gamma$ there is a tree $T$ of height $\omega$ such that $A$ is a projection of the set of branches of $T$ and $T$ has the property (*) w.r.t. a certain inner model $M$. Then if $P(\omega) \cap M$ is countable, every set in $\Gamma$ either is countable or has a perfect subset".

In fact, we can derive more from a theorem of this type. From the existence of $T$ for a set $A$ we can infer that the sentence $(\exists a) A(a)$ is absolute w.r.t. $M$. Moreover, without the assumption $P(\omega) \cap M = \omega$, we can infer another theorem about perfect sets, namely: every set $A$ in $\Gamma$ either is included in $M$ or has a perfect subset.

Thus we have:

1. Every $\mathcal{P}_1(\{x\})$ set either is included in $L(x)$, and thus is well-orderable, or has a perfect subset;
2. If a measurable cardinal exists, then every $\mathcal{P}_1(\{x\})$ set either is included in $\text{HOD}(x)$, and thus is well-orderable, or has a perfect subset.

In both cases we have a complementary theorem to Theorem 4. Consider the following definitions:

**Definition 1.** Let $M$ be an inner model. We say that $M$ is $\mathcal{P}_1^1$-correct if, for every $\mathcal{P}_1^1(M)$ set $A$, the sentence $(\exists a) A(a)$ is absolute w.r.t. $M$.

Let $M$ be called $\mathcal{P}_1^1$-correct if the above absoluteness holds for $\mathcal{P}_1$ sets.

**Definition 2.** Let $M$ be an inner model. We say that $M$ is generally $\mathcal{P}_1^1$-correct if, for every real $\beta$ such that $\beta$ is generic over the universe $M[\beta]$ of sets relatively constructible from $\beta$ and the class $M$ is $\mathcal{P}_1^1$-correct.

**Definition 3.** Let $K$ be Jensen's core-model. Let $K^M$ be the core-model of an inner model $M$. If $y$ is a real then $K_y$ we mean the relativization of the core-model to $y$, i.e., the union of mice relativized to $y$. Analogously we define $K_y^M$.

If $x$ is a real then $K[x]$, $K_y[x]$, $K^M[x]$, $K_y^M[x]$ denote the class of sets relatively constructible from $x$ and the classes $K$, $K_y$, $K^M$, $K_y^M$ respectively.

Consider the following remark:

**Remark 3.** Let $M \models \text{ZFC}$ be an inner model. Let $\kappa$ be the cardinality of

$P(\omega)$ in $M$. Let $C \subseteq L$ be the usual algebra collapsing $\kappa^{<\kappa}$ onto $\omega$. Assume that, in $V^C$, $M$ is generally $\mathcal{P}_1^1$-correct. Then every $\mathcal{P}_1^1(M)$ set either is included in $M$ or has a perfect subset in $V^C$.

**Proof.** Let $C$ be the subalgebra of $\mathcal{C}$ collapsing $\kappa$ onto $\omega$. Work in $V^C$. Let $f$ be a function enumerating $\kappa$ in type $\omega$, generic over $C$. Note that $f$ is generic over $L$. Then in $M[f]$ there is a function $g$ such that $\forall \omega \lim_{n \to \omega} P_n^M(\omega)$. Let $A$ be $\mathcal{P}_1(M)$. Suppose that $A \notin M$. Then $(\exists a) (A(a) \land \forall n (\alpha \notin g(n)))$. By the $\mathcal{P}_1^1$-correctness of $M[f]$, in $M[f]$

$(\exists a) (A(a) \land \forall n (\alpha \notin g(n)))$.

But then the assumptions of Remark 2 are satisfied in $V^C$. Hence $A$ has a perfect subset.

The following theorem was proved by Jensen in [4]:

If $(x) (x^* \text{ exists})$ and $0^+ \text{ does not exist}$, then $K$ is $\mathcal{P}_1^1$-correct.

The relativized version of this theorem is the following:

If $(x) (x^* \text{ exists})$ and $y^+ \text{ does not exist}$ where $y$ is a real, then $K_y$ is $\mathcal{P}_1^1$-correct.

We have the following remark:

**Remark 4.** If $(x) (x^* \text{ exists})$ and $0^+ \text{ does not exist}$ and $C$ is a boolean algebra, then $V$ is $\mathcal{P}_1^1$-correct in $V^C$.

**Proof.** Let $A$ be $\mathcal{P}_1$. Let $y$ be the parameter of the definition of $A$. Assume that $(\exists a) (A(a) \land \forall n (\alpha \notin g(n)))$ holds in $V^C$. Consider $K_y$. Then $K_y$ in the sense of $V^C$ is the same as $K_y$. By the fact that $K_y$ is $\mathcal{P}_1^1$-correct in $V^C$ we have

$(\exists a) (A(a) \land \forall n (\alpha \notin g(n)))$.

Hence $(\exists a) (A(a))$.

Consider the following conjecture:

$(\exists a) (x^* \text{ exists})$ and $y^+ \text{ does not exist}$ where $y$ is a real, then $K_y$ is $\mathcal{P}_1^1$-correct. Assume $(\exists a) (x^* \text{ exists})$ and $y^+ \text{ does not exist}$.

**Assume $(\exists a) (x^* \text{ exists}).$**

Then we have the following theorem:

**Remark 5.** If $(x) (x^* \text{ exists})$ and $0^+ \text{ does not exist}$, then every $\mathcal{P}_1$ set $A$ either is included in $K_y$, where $y$ is the parameter of the definition of $A$ or has a perfect subset in a boolean extension of the universe.

**Proof.** We apply Remark 3 with $M = K_y$.

**Corollary.** If $(x) (x^* \text{ exists})$ and $0^+ \text{ does not exist}$, then every $\mathcal{P}_1^1$ set is well-orderable or has a perfect subset in a boolean extension of the universe.

Thus, under the assumption $(x) (x^* \text{ exists})$ and $0^+ \text{ does not exist}$ and under the hypothesis $(\exists a) (x^* \text{ exists})$, we have proved a theorem similar to Theorem 4 by other methods than those used in this paper.

By all that we have said:

Theorem 4 is most interesting in the case where $(\exists a) (x^* \text{ does not exist})$ note that if $0^+ \text{ exists}$ then Mansfield's theorem [6] work.
Returning to the assumption $(\exists x)(x^* \text{ exists})$ and $0'$ does not exist observe that in this case the conclusion of Theorem 4 can be stated as follows: every $\Pi^1_1$ set is well-orderable or has a perfect subset (in the universe). Indeed, in this case $V$ is $\Sigma^1_1$ correct in $V^\mathcal{F}$ by Remark 4. But "having a perfect subset" is $\Sigma^1_1$ for a $\Pi^1_1$ set. Thus if a $\Pi^1_1$ set has a perfect subset in $V^\mathcal{F}$, it just has a perfect subset.

Hence we have proved the following (under $(\star \star \star)$):
If $(\exists x)(x^* \text{ exists})$ and $0'$ does not exist, then every $\Pi^1_1$ set is either included in $K$ or has a perfect subset, every $\Pi^1_2 (\{y\})$ set is either included in $K$, or has a perfect subset.

Thus we see that, as in the case of $L$ for $\Pi^1_1$ sets and in the case of $\text{HOD}$ for $\Pi^1_1$ sets under the assumption of the existence of a measurable cardinal, in the case of $K$ as well the property of $\Sigma^1_1$-correctness is connected with the fact that the perfect set theorem for $\Pi^1_1$ of the second form holds w.r.t. $K$ (although we do not know whether there is a tree for a $\Pi^1_1$-set in $K$).

Finally, we observe that if there are arbitrarily large regular numbers then the assumption of Theorem 4 is satisfied. This follows from Remark 6 below.

Notice that if $(\exists x)(x^* \text{ exists})$ and $(\exists x)(x^* \text{ exists})$, then, by the covering lemma w.r.t. the appropriate $L(x)$, there are arbitrarily large regular numbers. Thus if $(\exists x)(x^* \text{ exists})$ and $(\exists x)(x^* \text{ exists})$ then the conclusion of Theorem 4 holds.

Consider

**Remark 6 (ZF).** There is a cardinal $\nu \in On$ such that there is no function from the continuum onto $\nu$.

Proof of the remark. Let us define the following function $f$ from $\mathcal{P}(P(2^\nu))$ into $On$ as follows:

$$f(\mathcal{A}) = \begin{cases} \alpha & \text{if } \mathcal{A} \text{ is well-orderable,} \\ 0 & \text{otherwise.} \end{cases}$$

Let $B = \{ \beta : (E(\mathcal{A}))_{\mathcal{P}(\mathcal{A})}(f(\mathcal{A}) = \beta) \}$. Then by replacement $B$ is a set. Let $\nu$ not belong to $B$. Let us show that $\nu$ is not the continuum. Indeed, suppose that there is a function $g$ from the continuum onto $\nu$. Let $B' = \{ x \in 2^{<\nu} : g(x) = \xi \}$. Let $\mathcal{A}' = (B', g)$. Then $\mathcal{A}'$ is well-orderable and $f(\mathcal{A}') = \nu$. Hence $\nu \in B$. Contradiction.

Let $\kappa$ be regular $\kappa > \nu$. Then $\kappa$ is as required in Theorem 4.

To end this section we consider a few remarks concerning the theory ZF without the axiom of choice. In this theory we can develop a large part of the theory of projective sets. We consider the following projective hierarchy: $\Sigma^1_2$ are sets definable by an arithmetical formula with a parameter. If the class of $\Sigma^1_2$ sets is defined, then $\Pi^1_2$ is the class of complements of $\Sigma^1_2$ sets and $\Pi^1_{n+1}$ is the class of projections of $\Pi^1_n$ sets.

Note that this does not necessarily coincide with the topological hierarchy, for instance there may be borel sets that are not $\Pi^1_1$.

For simple families of sets there are selectors because of the Kondo-Addison theorem.

§1

First we shall introduce auxiliary notions and notation.

Let us recall from [1] what we mean by a tree in a topological space.

Let $(T, \mathcal{F})$ be a topological space in the sense that $\mathcal{F}$ is a basis in a topology in $T$. Any subset $T \subseteq \mathcal{F}$ is called a tree. An $x \in \mathcal{F}$ is called a branch of a tree $T$ if

$$(p)_x(x \in p \Rightarrow (Eq)_{x\subseteq y}(x \in q \subseteq p)).$$

Let $\mathcal{A} \in \mathcal{F}$ be called a tree. Let $T \subseteq \mathcal{F}$ be a family $\mathcal{F}$ of dense sections of $\mathcal{T}$ such that $\mathcal{F} < \mathcal{F}$ and $\mathcal{A}$ is the set of $\mathcal{F}$-generic branches of $\mathcal{T}$.

The reals are identified with elements of $\omega^\omega$ and will be denoted by $x, y, z, \ldots, \alpha, \beta, \ldots$. If a variable of this type runs over another set, we shall indicate this.

If $y \in \mathcal{F}^x$ is a well-ordering as a characteristic function of a set of pairs, then let $\mathcal{F}$ denote its type and $[\gamma]_{\mathcal{F}}$, for $\gamma \in \omega$, the characteristic function of the ordering

$$\{ (m, k) : y((\langle k, n \rangle) = 0) \}.$$
a well-ordering then
\[ \psi(x, y) = (n) \left( F_{\text{wc}}(a) \text{ is a real } \Rightarrow (E y') R(x, F_{\text{wc}}(a), y') \right). \]

Proof. We shall give an outline of the proof. For details the reader is referred to [3] and [10]. Let \( F(w, z, a) \) be the \( \Delta_1^1 \) formula with the property that, for a well-ordering \( z, F(w, z, a) \equiv (w \text{ codes the set } F_{\text{wc}}(a)) \) where coding can be done as in [10]. We have
\[
(n) \left( F_{\text{wc}}(a) \text{ is a real } \Rightarrow (E y') R(x, F_{\text{wc}}(a), y') \right)
\equiv (n) (E z)(E w)[z = [y]_a \& F(w, z, a) \& (w \text{ does not code a real } \vee (E a)(w \text{ codes } u \& (E y') R(x, u, y')))].
\]

Consider the formula "\( w \) does not code a real". Let us indicate how to prove that it is \( \Sigma_1^1 \). We have "\( w \) does not code a real" iff there is an \( n' \) in the collection of almost maximal vertices of \( w \) which is not a code of a pair of integers or there are two codes of pairs in this collection which have the same first coordinate and different second coordinates or there is an \( m \) for which there is no \( n \) such that the code of \( \langle m, n \rangle \) is an almost maximal vertex of \( w \).

Consider the formula "\( w \) codes \( u' \)". We have "\( w \) codes \( u' \)" iff the collection of almost maximal vertices of \( w \) consists of codes of pairs of integers that are in \( u' \). It follows that the formula
\[
(n) (E z)(E w)[z = [y]_a \& F(w, z, a) \& (w \text{ does not code a real } \vee (E a)(w \text{ codes } u \& (E y') R(x, u, y')))]
\]
is equivalent to a \( \Sigma_1^1 \) formula. Let \( \psi \) be this \( \Sigma_1^1 \) formula. ■

FACT 3. If \( \theta(x, y) \) is a \( \Sigma_1^1 \) formula of \( x \) in a parameter \( y \), then there is a tree \( T \subseteq \omega^{\omega} \times \omega^{\omega} \times \omega^{\omega} \) such that
\[
\psi(x, y) \equiv (E z)(\langle x, y, z \rangle \text{ is a branch of } T).
\]

Proof. We have
\[
\neg \theta(x, y) \equiv (z)(E n) Q(z, x, y, n)
\]
where \( Q(x, z, n) \) is recursive in \( y \). Thus \( (E n) Q(z, x, y, n) \) is recursively enumerable in \( y \). But, by the definition of relative recursive enumerability, there is a recursive \( Q' \subseteq \omega^{\omega} \times \omega^{\omega} \times \omega^{\omega} \times \omega^{\omega} \) such that
\[
Q(z, x, y, n) = (E k) Q'(z, x, y, k).
\]
Thus
\[
\theta(x, y) \equiv (E n)(E k) \neg Q'(z, x, y, k).
\]

Let
\[
T = \langle s, t, u \rangle: \text{ dom } s = \text{ dom } t = \text{ dom } u
\]
\[& \& (E s', t', u')(s \leq s', t \leq t', u \leq u'), \text{ dom } s' = \text{ dom } t' = \text{ dom } u' \]
\[& \& Q'(s', t', u', \text{ dom } s').
\]
Then we have
\[
\theta(x, y) = (E z)(\langle x, y, z \rangle \text{ is a branch of } T).
\]

Combining Fact 2 and Fact 3 we obtain

FACT 4. There is a tree \( T \subseteq \omega^{\omega} \times \omega^{\omega} \times \omega^{\omega} \) such that
\[
\psi(x, y) = (E z)(\langle x, y, z \rangle \text{ is a branch of } T).
\]

Consider the formula
\[
\phi(x, y) = (n) (F_{\text{wc}}(a) \text{ is a real } \Rightarrow (E y') R(x, F_{\text{wc}}(a), y')).
\]

We show the following

FACT 5. There is a tree \( T_i \subseteq \omega^{\omega} \times \omega^{\omega} \times \omega^{\omega} \) and a family \( \mathcal{D} \) of its dense subsets such that
\[
\phi(x, y) \equiv (E z)(\langle x, f, y, z \rangle \text{ is a } \mathcal{D}-\text{generic branch of } T_i).
\]

Proof. Let \( \psi(x, y) \) be the \( \Sigma_1^1 \) formula defined in Fact 2 and let \( T \) be the tree such that
\[
\psi(x, y) \equiv (E z)(\langle x, y, z \rangle \text{ is a branch of } T).
\]

Let us define \( T_i \) as follows:
\[
\langle s, t, u \rangle \in T_i
\]
iff
\[
(1) \quad s \in \omega^{\omega}, t \in \omega^{\omega}, u \in \omega^{\omega}
\]
\[& \& \text{dom } s = \text{dom } t = \text{dom } u,
\]
\[
(2) \quad \langle s, t, u \rangle \in T,
\]
\[
(3) \quad u(\langle m, n \rangle) = 0 \Rightarrow t(m) < t(n),
\]
\[
(4) \quad (E z)(\langle x, f, y, z \rangle \text{ is a branch of } T)(\langle s, t, u \rangle \leq \langle x, f, y, z \rangle \& \langle x, y, z \rangle)
\]
\[= f(\langle m, n \rangle)_f < f(n) & f: \omega^{\omega} \to \omega^{\omega}.
\]
i.e. we require that through every element of \( T_i \) there should go a branch of \( T_i \).
Let $\eta \in \xi$. Let

$$D_\eta = \{ \langle s, t, v, \eta \rangle \in T_\eta : \eta \in \text{rng } t \}.$$ 

By (4) $D_\eta$ is dense in $T_\eta$.

$$D_\xi = \{ D_\eta \}_{\eta \in \xi}.$$ 

We must show that $T_\xi$ is as required, i.e.,

$$\varphi(\alpha, \xi) \equiv (E_y, y, z)(\langle \alpha, f, y, z \rangle \text{ is a } D_\xi\text{-generic branch of } T_\eta).$$ 

We have:

$$\varphi(\alpha, \xi) \equiv (E_y, y, z)(y \text{ is a well-ordering of } \omega \text{ in type } \xi \land \psi(\alpha, y)).$$

Thus assume $\varphi(\alpha, \xi)$. Take $y$ such that $y$ is a well-ordering of type $\xi$ and $\psi(\alpha, y)$. Then there is a $z$ such that $\langle \alpha, f, y, z \rangle$ is a branch of $T$. Define $f \in \omega^\omega$ as

$$f(n) = [\gamma].$$

Then

$$f : \omega \rightarrow \xi \text{ and } (m)(n)(f(m) < f(n) \equiv y(\langle m, n \rangle) = 0).$$

Hence $\langle \alpha, f, y, z \rangle$ is a branch of $T_\xi$. It is a $D_\xi$-generic branch because $f$ is onto $\xi$.

Conversely, assume that there are $f, y, z$ such that $\langle \alpha, f, y, z \rangle$ is a $D_\xi$-generic branch of $T_\eta$. Then $\langle \alpha, f, y, z \rangle$ is a branch of $T$ and thus $\psi(\alpha, y)$. Moreover,

$$f : \omega \rightarrow \xi \text{ and } (m)(n)(y(\langle m, n \rangle) = 0 \equiv f(m) < f(n)).$$

Hence $y$ is a well-ordering of type $\xi$. Thus $\varphi(\alpha, \xi)$. 

We can join the last two coordinates of the elements of $T_\xi$. So let us assume that $T_\xi \subseteq \omega^\omega \times \xi^\omega \times \omega^\omega$. We have

**FACT 6.**

$$A(\alpha) = (\exists t, \langle F_s, g \rangle)(F_s(\alpha) \text{ is a real } \Rightarrow (E_y)R(\alpha, F_s(\alpha), y))$$

$$= (\exists t, \langle F_s, y \rangle)(F_s(\alpha) \text{ is a real } \Rightarrow (E_y)R(\alpha, F_s(\alpha), y))$$

$$= (\exists t, \varphi(\alpha, \xi))$$

$$= (\exists t, \varphi(\alpha, \xi)).$$

Still, without choice, we can define the following tree:

$$T \subseteq \omega^\omega \times \bigotimes_{\xi \in \omega_1} (\xi^\omega \times \omega^\omega).$$

Let

$$\langle s, \langle \langle \xi_0, \langle t_{\xi_0}, v_{\xi_0} \rangle \rangle, \ldots, \langle \xi_n, \langle t_{\xi_n}, v_{\xi_n} \rangle \rangle \rangle \rangle \in T.$$
\[\langle \alpha, f \rangle\] is a \(\mathcal{D}_T\)-generic branch of \(T\). Conversely, if there is an \(f\) such that \(\langle \alpha, f \rangle\) is a \(\mathcal{D}_T\)-generic branch of \(T\), then, for every \(\zeta\), \(\langle \alpha, (f)(\zeta) \rangle, (f)(\zeta)_i\) is a \(\mathcal{D}_{T_i}\)-generic branch of \(T_i\). Hence \(A(\alpha)\). \]

\(\S\) 2

In this section we prove a theorem about perfect subsets of \(\cap_1\) sets. Its proof is an illustration of the method used in \(\S\) 3 to prove a stronger theorem.

**Theorem 2.** Let \(A\) be \(\cap_1\) Assume that \(\omega_1\) is regular and there is no function from \(A\) onto \(\omega_1\). Then either \(A\) is countable or \(A\) has a perfect subset in some boolean extension of the universe.

Let us first explain the idea of the proof. Consider the tree \(T\) defined for \(A\) in \(\S\) 1. It would be natural to treat \(T\) as a set of forcing conditions. Then a \(V\)-generic filter over \(T\) would provide a sequence \(\langle a, f \rangle_{\mathcal{D}_T}\) such that \(\langle a, f \rangle\) is a \(\mathcal{D}_T\)-generic branch of \(T\), i.e.

\[(\mathcal{D}_T \cap \eta_0)(F_\eta(a)) \text{ is a real } \Rightarrow (E\gamma)(R(\alpha, F_\eta(a), \gamma'))\]

If the forcing \(T\) does not collapse \(\omega_1\), then in the extended universe we have

\[(\mathcal{D}_T \cap \eta_0)(F_\eta(a)) \text{ is a real } \Rightarrow (E\gamma)(R(\alpha, F_\eta(a), \gamma'))\]

and thus \(A(\alpha)\).

We shall show that, under the assumptions of the theorem, \(T\) is c.c.c. The usual proof shows that c.c.c. together with the regularity of \(\omega_1\) implies ZF (without choice) that \(\omega_1\) is not collapsed by \(T\). Thus \(T\) enables us to add elements of \(A\). We are not able yet to add a perfect set of elements of \(A\), because \(T\) is not necessarily separable (for instance if \(A\) is provably a singleton) and generic filters over \(T\) can then provide \(\alpha\)'s in \(V\). The next idea will be to observe that either \(A\) is countable or there is a c.c.c. atomless separable non-empty subset of \(\mathcal{P}\), splitting at the first coordinate. Then, by standard methods, we show that \(A\) has a perfect subset in \(V^\mathcal{C}\) where \(C\) is a boolean algebra enumerating with natural numbers the family of dense subsets of \(\mathcal{P}\).

We introduce the following definition:

**Definition 4.** Let \(p \in T\), \(p = \langle \alpha, \langle \xi_0, (t_0, v_0) \rangle, \ldots, \langle \xi_n, (t_n, v_n) \rangle \rangle\) and \(\alpha < \omega^n\).

We say that \(\alpha\) goes through \(p\) if there are \(f_0, \theta_0, \ldots, f_n\) such that \(\leq \alpha, t_0, \ldots, v_0, \theta_0, f_0, \theta_0, \ldots, f_n, \theta_n, \alpha = \omega^{\alpha + x}\) and \(\xi_0, f_0, \theta_0, \xi_1, f_1, \theta_1, \ldots, f_n, \theta_n, \alpha = \omega^{\alpha + x}\) is a branch of \(T\). Notice that, by the definition of \(T\), through every element of \(T\) goes an \(\alpha\) in \(A\).

**Fact 7.** \(T\) is c.c.c.

**Proof.** Observe first that if

\[p = \langle \alpha, \langle \xi_0, (t_0, v_0) \rangle, \ldots, \langle \xi_n, (t_n, v_n) \rangle \rangle,\]

\[q = \langle \alpha, \langle \eta_0, (t_0, v_0) \rangle, \ldots, \langle \eta_n, (t_n, v_n) \rangle \rangle\]

and \(\langle t_0, v_0, \ldots, t_n, v_n \rangle, \langle \xi_0, \eta_0, \ldots, \xi_n, \eta_n \rangle\) are compatible in \(\bigotimes (\xi_i \times \omega^{\alpha + x})\) and there is one and the same \(\alpha \in A\) going through \(p\) and through \(q\), then \(p, q\) are compatible in \(T\).

Indeed, let

\[r = \langle s, \langle \zeta_0, \langle \xi_0, (t_0, v_0) \rangle, \ldots, \langle \xi_n, (t_n, v_n) \rangle \rangle, \langle \eta_0, \langle \xi_0, (t_0, v_0) \rangle, \ldots, \langle \eta_n, (t_n, v_n) \rangle \rangle \rangle\]

where

\[\{\zeta_0, \ldots, \zeta_n\} = \{\xi_0, \ldots, \xi_n\} \cup \{\eta_0, \ldots, \eta_n\}\]

and if \(\xi_i = \zeta_i\) then \(t_i = t_i, v_i = v_i\) and if \(\xi_i = \eta_i\) then \(t_i = t_i, v_i = v_i\).

Then the same \(\alpha\) in \(A\) which goes through \(p\) and through \(q\) goes through \(r\), and thus \(r \in T\), \(r \leq p, r \leq q\).

Now suppose that there exists a family \(F \subseteq T\) such that \(F\) consists of pairwise incompatible conditions and is of power \(\omega_1\). By the regularity of \(\omega_1\) we can assume that all conditions in \(F\) have the same \(s\) at the first coordinate and the same length.

Let \(\alpha \in A\), and \(F_\alpha = \{p \in F : \alpha \text{ goes through } p\}\). Then \(F_\alpha\) consists of pairwise incompatible conditions such that through every two of them goes the same \(\alpha\) in \(A\). Thus, by our observation, for every pair of conditions \(p, q \in F_\alpha\), \((p)\) is incompatible with \((q)\) in \(\bigotimes (\xi_i \times \omega^{\alpha + x})\). Hence \(F_\alpha\) is countable because \(\bigotimes (\xi_i \times \omega^{\alpha + x})\) is c.c.c. By the regularity of \(\omega_1\) there is a \(\zeta < \omega_1\) such that \(F_\zeta \subseteq T \cap (\omega^{\alpha + x} \times \bigotimes (\xi_i \times \omega^{\alpha + x}))\).

Let \(\zeta\) be the least such \(\zeta\). So we have defined a function \(A\) into \(\omega_1\). By our assumption that there is no function from \(A\) onto \(\omega_1\) and by the regularity of \(\omega_1\), \(\sup \xi_i < \omega_1\). Let \(\sup \xi_i = \eta_i\). Then, by the fact that \(F = \bigcup_{\alpha \in \omega_1} F_\alpha \subseteq T \cap (\omega^{\alpha + x} \times \bigotimes (\xi_i \times \omega^{\alpha + x}))\), Hence \(F\) is countable, contradiction.

**Definition 5.** Consider a subset \(P\) of \(T\). We say that \(P\) splits at the first coordinate if, for every \(p \in P\), there are \(q_1, q_2\) in \(P\) such that \(q_1 \leq p\) and \((q_1)\) is incompatible with \((q_2)\) in \(\omega^{\alpha + x}\).

**Fact 8.** Either \(A\) is countable or there is a subset \(P\) of \(T\) such that \(P\) is non-empty and c.c.c. and that \(P\) splits at the first coordinate.

**Proof.** We modify Solovay's idea [7]. Let us define

\[T^\alpha = T,\]

\[T^{\alpha + 1} = \{p \in T^\alpha : (Ex_\alpha) (A(\alpha) & \alpha \text{ goes through } p \& \alpha \neq \alpha_1)\}\]

\[&\& (q)_\alpha (\alpha \text{ goes through } q \Rightarrow q \in T^\eta)\}.

For limit \(\lambda\), let \(T^\lambda = \bigcap \{T^\xi : \xi < \lambda\}\). Since \(T\) is well-orderable, the usual
cardinality argument shows that the minimum $\xi$ such that $T^\xi = T^{\xi+1}$ is less
that $\omega_1$. We denote this minimal $\xi$ by $\xi(T)$.
Let $P = T^{\omega_1}$. We have:

$$p \in P \text{ iff } (\exists_{\xi_1, \xi_2})(\xi_1, \xi_2) \in A(\xi_1) \land \xi_1 \neq \xi_2 \land (q)_{\xi_1}(\xi_1 \text{ goes through } q \Rightarrow q \in P)$$

Suppose that $P = \emptyset$. We shall show that $A$ is countable. Let $\alpha$ be such that
$A(\alpha)$. Then there is a $\xi$ such that there is a $p$ in $T^\xi$ such that $q$ goes through $p$ and
$p \in T^{\xi+1}$. Indeed, otherwise $(q)_{\xi_1}(\xi_1 \text{ goes through } q \Rightarrow q \in P)$ and hence
$P \neq \emptyset$.

Let $\xi_1$ be the least such $\xi$ that there is a $p$ in $T^\xi - T^{\xi+1}$ such that $q$ goes through $p$.
Take the least such $p$. Then by definition we have

$$(\ast) \quad (q)_{\xi_1}(\xi_1 \text{ goes through } q \Rightarrow q \in T^\xi)$$

Let us show that $\alpha$ is the only member of $A$ with property $(\ast)$ going through $p$.
Indeed, suppose that $\alpha'$ is another member of $A$ with property $(\ast)$. Then $\alpha, \alpha'$
are such $\xi_1, \xi_2$ as are required to make $p$ belong to $T^{\xi_1 + 1}$. But $p \notin T^{\xi_1 + 1}$. Thus
$\alpha$ is the only member of $A$ going through $p$ with property $(\ast)$. Thus to every $\alpha$
in $A$ corresponds canonically a pair $(\xi_1, p)$ such that $p \in T^\xi$. Hence $A$ has a
well-ordering. Thus, by our assumption that there is no function from $A$ onto $\omega_1$, $A$
is countable.

Thus either $A$ is countable or $P \neq \emptyset$.

Let us show that $P$ has the following properties:

1. If $p \in P$, $\xi \in \omega_1, \eta \in \omega_1, q \in T^\xi$ then there is a $q \leq p$, $q \in P$
such that $\xi \in \text{dom}(q)$ and $\eta \in \text{rg}(q)$.
2. If $p \in P$, there are $q_1, q_2$ in $P$ such that $q_1 \leq q_2$ and $q_1 \leq p$.
3. $P$ is c.e.c.

Let us show (1). Let $p \in P$,

$$p = \langle \epsilon_0, \langle \epsilon_0, \epsilon_0, \epsilon_0 \rangle, \ldots, 
\langle \epsilon_n, \epsilon_n, \epsilon_n \rangle \rangle$$

Then

$$(\ast \ast) \quad (q)_{\xi_1}(\xi_1 \text{ goes through } q \Rightarrow q \in P)$$

Take $\xi_1$. Then there are $f_0, f_1, \ldots, f_0, g_0, g_1, g_2, g_3, g_4, g_5$ such that $\langle \alpha_1, f, g \rangle$
is a branch of $T^\xi$.

Now $q \in P$. Hence $q \in P$ because

$$(q)_{\xi_1}(\xi_1 \text{ goes through } q \Rightarrow q \in P)$$

To show (2) we define, respectively, $q_1, q_2$ for $\alpha_1, \alpha_2$ as above, where $m$ is
such that $\alpha_1 \downarrow = \alpha_2 \downarrow$.

Let us show (3). Let

$$A' = \{ \alpha \in A : (q)_{\xi_1}(\xi_1 \text{ goes through } q \Rightarrow q \in P) \}$$

We have

$$p \in P \Rightarrow p \in T \land (\exists_{\xi_1})(\xi_1 \text{ goes through } p)$$

Then we can repeat the proof of Fact 7 with $A'$ in place of $A$.

**Corollary.** If $G$ is generic over $P$, then $G$ determines a sequence
$\langle x, f, g, \xi \rangle \downarrow_{\omega}^{\alpha}$ such that $A(\alpha)$.

Indeed, $\omega_1^{\langle \xi \rangle} = \omega_1$ by Fact 8.

By Fact 8, $f_0' \in T^{\omega} \land (\alpha, f_0, g_0) \in T_2$. Then by §1 we have $A(\alpha)$.

**Fact 9.** Assume that $A$ is not countable. Let $C$ collapse $\text{Pr}(P)$ onto $\omega$. Then
$A$ has a perfect subset in $V^C$.

**Proof.** Notice that the assumptions of Remark 1 are satisfied in $V^C$ with
$M = V$.

By the corollary, a filter $G$-generic over $P$ canonically determines a sequence
$\langle x, f, g, \xi \rangle \downarrow_{\omega}^{\alpha}$. Hence there is a canonical name $\alpha$, such that $\alpha$
is realized as $\alpha$ in the extended universe.

We have in $V^C$, $\text{Pr}(P)$ countable. Moreover, if $p \in P$ then there are $q_1, q_2 \leq p$
and $n, m_1, m_2$ such that $q_1 \downarrow \models (q(\emptyset) = m_1)$, $q_2 \downarrow \models (q(\emptyset) = m_2)$. Indeed, this
follows from the fact that $P$ splits the first coordinate. Thus, by Remark 1, $A$
is a perfect subset.

Thus we have completed the proof of Theorem 2.

§3

Now we shall prove a stronger theorem by a similar but more complicated method.

**Theorem 3.** Let $\kappa$ be a regular cardinal. Let $A$ be of $\text{Pr}(P)$ and assume that there
is no function from $A$ onto $\alpha$. Then either $A$ is well-orderable and of power less
than $\alpha$ or $A$ has a perfect subset in a boolean extension of the universe.

**Proof.** First consider a pure combinatorial lemma.

**Lemma 1 (ZF).** If $\kappa$ is regular then $\bigotimes_{\xi} \xi^{\kappa} = \kappa$ c.c.c.
Proof. We shall show by induction on \( n \) that if \( F \subseteq \bigotimes \xi^{<\omega} \) is a family of pairwise incompatible functions with \( n \)-element domains then \( F \) is of power less than \( \kappa \).

If \( n = 1 \) then every \( f \in F \) has the same domain, i.e. there is a \( \xi \in \kappa \) such that \( \forall f \in F \Rightarrow \text{dom } f = \{ \xi \}. \) Thus \( F \subseteq \{ \langle \xi, t \rangle : t \in \xi^{<\omega} \} \). Hence \( F = \xi \) and so \( F < \kappa \).

Assume the inductive assumption for \( n \) and let \( F \subseteq \bigotimes \xi^{<\omega} \) consist of pairwise incompatible functions with \((n+1)\)-element domains.

Suppose that \( F = \kappa \). Fix \( g \in F \). Let \( dom g = \{ \xi_0, \ldots, \xi_n \} \). Define \( F_i = \{ f \in F : \xi_i \in \text{dom } f \} \). Then \( F = \bigcup_{i=0}^{n} F_i \). There is an \( i_0 \) such that \( F_{i_0} = \text{dom } g \).

Let \( t \in \xi^{<\omega} \) \( F_{i_0} = \{ f \in F_{i_0} : f(\xi_{i_0}) = t \} \). We have \( F_{i_0} = \bigcup_{i=0}^{n} F_{i_0} \).

Again by the fact that \( \xi^{<\omega} \) is of power less than \( \kappa \) and by the regularity of \( \kappa \), there is a \( t_0 \in \xi^{<\omega} \) such that \( F_{i_0} \) is of power \( < \kappa \). Now define \( F' = \{ f_{(\text{dom } f) \cup \{ \xi_{i_0} \}} : f \in F_{i_0} \} \). If \( f', g' \in F' \) then there are \( f, g \in F \) such that \( f(\xi_{i_0}) = g(\xi_{i_0}) \) and \( f' = f_{(\text{dom } f) \cup \{ \xi_{i_0} \}} \) \( g' = g_{(\text{dom } g) \cup \{ \xi_{i_0} \}} \). As elements of \( F, f, g \) are incompatible.

By the fact that \( f(\xi_{i_0}) = g(\xi_{i_0}) \), \( f', g' \) are incompatible. Hence \( F' \) consists of pairwise incompatible functions. Notice that if \( f' \in F' \) then \( \text{dom } f' = \{ \xi_{i_0} \} \). Thus, by the inductive assumption, \( F' \) is of power less than \( \kappa \). But \( F = F_{i_0} \quad < \kappa \). Contradiction.

Thus we have proved that if \( F \subseteq \bigotimes \xi^{<\omega} \) consists of pairwise incompatible functions of power \( n \) then \( F \) is of power less than \( \kappa \). Hence, by the regularity of \( \kappa \), \( \bigotimes \xi^{<\omega} \) is \( \kappa \)-closed.

Now we carry out a construction similar to the construction of \( T \). Let \( \xi \) be a countable ordinal. Let us recall the definition of \( T_\xi \) from Fact 5.

Let \( \psi(x, y) \) be the \( \Sigma_1^1 \) formula such that whenever \( y \) is a well-ordering then

\[
\psi(x, y) = \langle (\text{dom } F_{\text{Gr}(x)}(x) \text{ is a real}) = (\text{E}y) \text{ R}(x, F_{\text{Gr}(x)}(x), y) \rangle.
\]

Let \( T \subseteq \omega^{<\omega} \times \exists \omega^{<\omega} \times \omega^{<\omega} \) be such that

\[
\psi(x, y) = \langle \text{E}z \rangle (x, y, z) \quad \text{is a branch of } T.
\]

Let \( T_\xi \) be defined as

\[
\langle s, t, u \rangle \in T_\xi \quad \text{iff}
\]

(1) \( s \in \omega^{<\omega}, t \in \xi^{<\omega}, u \in \omega^{<\omega} \),

\[
\text{dom } s = \text{dom } t = \text{dom } u,
\]

(2) \( \langle s, (u_0, (u_1)) \rangle \in T \),

(3) \( (u_0)(m, n) = 0 \Rightarrow (m < t(n)) \),

(4) \( \langle (\text{E}x, f, y) \rangle_{(\text{dom } x \times \omega^{\<\omega})(x) \times \omega^{\<\omega}} (\langle s, t, u \rangle, \xi) \subseteq \langle s, f, y \rangle \)

& \( \langle x, (y)_0(y)_1 \rangle \) is a branch of \( T \& f: \omega^{<\omega} \rightarrow \xi \)

& \( \langle (m, n)(y)_0(y)_1 \rangle = 0 \Rightarrow (m < f(n)) \).

Then, if \( \varphi(x, \xi) \) means

\[
\langle \text{E}y \rangle (F \text{Gr}(x)) = \langle (\text{E}y) \text{ R}(x, F \text{Gr}(x), y) \rangle,
\]

then

\[
\varphi(x, \xi) = \langle (\text{E}f, g) \rangle (x, f, g) \quad \text{is a branch of } T_\xi \& f: \omega^{<\omega} \rightarrow \xi
\]

provided that \( \xi \) is countable.

Later let \( T_\xi(x, t, u) \) be the formula defining \( T_\xi \). Let \( T^M \) be the relativization of this formula to a class \( M \). Consider the following remark:

Remark 7. If \( B_1, B_2 \) are algebras such that \( V^{B_1} \models \xi \) is countable, then \( V^{B_1} \models T_\xi(x, t, u) \) and \( V^{B_2} \models (x, \xi) \).

Proof. Let \( (x, t, u) \) be given. Assume that \( V^{B_1} \models T_\xi(x, t, u) \). Then, in \( V^{B_2} \), \( (x, t, u) \) satisfies conditions (1)-(3) because they are absolute. Let us show that (4). Work in \( V^{B_1 \times B_2} \). Then \( V^{B_1} \subseteq V^{B_2} \). Thus in \( V^{B_2} \), \( (x, t, u) \) satisfies (4); Notice that, in \( V^{B_1 \times B_2} \), (4) is a \( \Sigma_1 \) sentence with a countable parameter \( \xi \). Thus, by Levy's lemma, (4) is satisfied in \( V^{B_2} \) because \( \xi \) is countable in \( V^{B_2} \).

Let \( \xi \in \kappa \). Let \( B_2 \) be the usual algebra collapsing \( \xi \) onto \( \omega \). Let us define \( T \) as follows:

let \( p \in T \) iff

\[
p = \langle \xi, (\langle \xi_0, (t_0 v_1), \ldots, \langle \xi_n, (t_n v_n) \rangle) \rangle
\]

and

(1) \( s \in \omega^{<\omega}, t_0, e_0^{\xi^{<\omega}}, v_0, e_0^{\omega^{<\omega}}, \xi_0, \xi \in \omega, \)

\[
dom s = \text{dom } t_0 = \text{dom } v_0,
\]

(2) \( \langle \text{E}y \rangle (s, (t_0 v_0)) \),

(3) \( (\text{E}y) (s) \)

& \( \langle \text{E}y \rangle (s, t_1 v_1, \ldots, f_0, \theta)(x, t_1, v_1) \subseteq \langle (x, f_0, \theta) \rangle \)

& \( \langle x, (f_0, \theta) \rangle \) is a branch of \( T_\xi \& f: \omega^{<\omega} \rightarrow \xi_0 \).

Let \( \text{dom } p, \langle p \rangle_0, \langle p \rangle_1 \) be defined as before. Let \( p \leq q \) if \( \text{dom } p \supseteq \text{dom } q \) & \( \langle p \rangle_0 \supseteq \langle q \rangle_0 \& \text{dom } q_0 \supseteq \text{dom } p \).

Let \( p \leq q \) if \( \text{dom } p \supseteq \text{dom } q \) & \( \langle p \rangle_0 \supseteq \langle q \rangle_0 \& \text{dom } q_0 \supseteq \text{dom } p \).
DEFINITION 6. Let $p \in T \setminus \emptyset$, $p = \langle s, \langle \xi_0, \langle t_{\xi_0}, g_{\xi_0} \rangle \rangle, \ldots, \langle \xi_n, \langle t_{\xi_n}, g_{\xi_n} \rangle \rangle \rangle$. Let $\alpha \in \omega^\omega$. Let us say that $\alpha$ goes through $p$ if

$$V_{\alpha}^p = (E_{\alpha}, g_{\alpha}, \ldots, f_{\alpha}, g_{\alpha}) \in (\xi, t_{\xi}, g_{\xi}) \subseteq \langle \xi, t_{\xi}, g_{\xi} \rangle.$$ 

Notice that through every element of $T$ goes an $\alpha$ such that $A(\alpha)$.

The facts that show that $T$ is non-empty if $A$ is non-empty.

The idea of including in $T$ those $p$ through which "goes a full branch" $\langle \alpha, f_0, g_0, \ldots, f_n, g_n \rangle$ not actually in the universe but in a homogenous boolean extension of the universe is an adaptation of an idea of Mansfield from [5]. Also the proof of Fact 10 and the use of Remark 7 in the proof of Fact 10 have much in common with Mansfield [5]. However, the derivation of the existence of a perfect subset of $A$ in a certain situation, which is the content of the proofs of Fact 13 and also of Fact 8, Fact 9 of §2 and Remark 1 of §2, have more in common with Solovay's ideas from [7] than with Mansfield [5].

All these technical devices which come from other papers are applied to our very particular set of forcing conditions, which is characteristic only for the present paper.

FACT 10. Let $A \in U^\omega$. Let $\alpha \in A$. Then

$$V_{\alpha}^p = (E_{\alpha}, g_{\alpha}) \in (\xi, t_{\xi}) \subseteq \langle \xi, t_{\xi}, g_{\xi} \rangle$$

where $\langle \xi, t_{\xi}, g_{\xi} \rangle$ is a branch of $T_{\xi}$. Then $\xi \in A(\alpha)$.

Proof. We have $V_{\alpha}^p = A(\alpha)$ because of the absoluteness of $A$. Thus

$$V_{\alpha}^p = (\alpha, \xi) \in A.$$ 

Therefore follows the required conclusion.

FACT 11. Let $p \in T$, dom $p = \langle \xi_0, \ldots, \xi_n \rangle$, $\xi_0 \in \omega$, $n \in \omega$, $\eta \in \xi_0$, dom $p \subseteq \xi_0$. Then there is a $p \preceq p$ such that $\xi \in \text{dom} q$, $\eta \in \text{rg}(q(\xi))$, $n \in \text{dom}(q_0)$, $\eta \in \text{rg}(q(\xi_0))$.

Proof. Let $\alpha$ be such that $A(\alpha)$ and $\alpha$ goes through $p$. Work in $V_{\alpha}^p$.

Assume that

$$p = \langle s, \langle \xi_0, \langle t_{\xi_0}, g_{\xi_0} \rangle \rangle, \ldots, \langle \xi_n, \langle t_{\xi_n}, g_{\xi_n} \rangle \rangle \rangle.$$ 

Let $f_0, g_0, \ldots, f_n, g_n$ be such that $p(\xi) \subseteq \langle f_0, g_0 \rangle \subseteq \langle \xi, f_0, g_0 \rangle$ is a branch of $T_{\xi}$. There are such $f_0, g_0, \ldots, f_n, g_n$ because $\xi \in \text{rg}(\alpha)$. Let $f, g$ be such that $\langle \xi, f, g \rangle$ is a branch of $T_{\xi}$. There are such $f, g$ by Fact 10. Let $m$ be such that $n \in \text{rg}(f_{m})$. Let $q = \langle \eta_{m}, \langle \xi_0, \langle t_{\xi_0}, g_{\xi_0} \rangle \rangle, \ldots, \langle \xi_n, \langle t_{\xi_n}, g_{\xi_n} \rangle \rangle \rangle$. Then in $V_{\eta_{m}}^p$, $q$ satisfies the conditions (1)-(2) of the definition of $T$, and $q$ goes through $p$. By the fact that $q$ is in $V_{\eta_{m}}^p$, an element of $\omega^{\omega} \times \omega^{\omega} \times \omega^{\omega} \times \omega^{\omega}$, there is an element $g$ of $\omega^{\omega} \times \omega^{\omega} \times \omega^{\omega} \times \omega^{\omega}$ such that $\eta_{m} = g(\xi_0) \neq \emptyset$. Thus $\langle \eta_{m}, g(\xi_0) \rangle \neq \emptyset$. By the homogeneity of $\mathcal{B}_{q}$, this value is 1. Thus $q$ satisfies (1), (2) and $q$ goes through $q$. Hence $q \in T$, $q \subseteq q$. Thus $q$ is as required.

FACT 12. $T$ is $\kappa$-c.c.

Proof. We simply repeat the proof of Fact 7 with $\kappa$ in place of $\omega_1$ and the notion of "going through a condition" determined by Definition 6. We use the regularity of $\kappa$ and Lemma 1.

FACT 13. Either $A$ is of power less than $\kappa$ or there is a set $P$ of $T$ such that $P$ is non-empty, $\kappa$-c.c. and splits at the first coordinate.

Proof. We repeat the proof of Fact 8 with $\kappa$ in place of $\omega_1$, and with the notion of "going through a condition" determined by Definition 6.

Then, if $A$ is not of power less than $\kappa$, $P$ has the following properties:

(1) if $P \in P$, dom $p = \langle \xi_0, \ldots, \xi_n \rangle$, $\xi_0 \in \omega$, $n \in \omega$, $\eta \in \xi_0$, dom $p \subseteq \xi$, then there is a $q \preceq p$, $q \in P$ such that

$$\xi \in \text{dom} q, \eta \in \text{rg}(q(\xi)), n \in \text{dom}(q_0), \eta \in \text{rg}(q(\xi_0)).$$ 

(2) if $P \in P$, then there are $q_1, q_2 \in P$ such that $q_1 \preceq q_2$, $q_2 \preceq q_1$, such that $q_1$ is incompatible with $q_2$ and $q_1 \preceq q_2$.

(3) $P$ is $\kappa$-c.c.

We prove (1), (2) as Fact 11, and (3) as Fact 12 with

$$A = \{ \alpha \in A : (q_1, \alpha) \in \text{rg}(q) \}.$$ 

Now let us derive the conclusion of the theorem from Remark 2, and Facts 12, 13.

Assume that $A$ is not of power less than $\kappa$. Let $C$ enumerate all dense subsets of $P$. Work $V_{\kappa}$. Let $G$ be generic over $P$. Define

$$\alpha = \bigcup \{ s : \langle s, \xi \rangle \in G \},$$ 

$$f_{s} = \bigcup \{ t_{s} : (E_{s}, t_{s}) \in \langle s, \langle \xi, t_{s} \rangle \rangle \in G \},$$ 

$$q_{s} = \bigcup \{ t_{s} : (E_{s}, t_{s}) \in \langle s, \langle \xi, t_{s} \rangle \rangle \in G \}.$$ 

Then, by the property (1) of $P$ in Fact 12,

(1) $[\xi : f_{s} \neq \emptyset]$ it is cofinal in $\kappa$,

(2) $f_{s} \neq \emptyset = f_{s}$, $\omega \subseteq \kappa$,

(3) $f_{s} \neq \emptyset = \langle \alpha, f_{s}, g_{s} \rangle$ is a branch of $T_{\kappa}$. 

Indeed, (1)-(2) are immediate. Let us show (3). We have: \( \xi \) is countable in \( V^C \). Hence, by Remark 7, 

\[
V^C \models \langle s, t, u \rangle \in T \iff V^R \models \langle s, t, u \rangle \in T.
\]

But \((n)(V^C \models \langle \langle t, s, f \rangle, g \rangle \in T)\) by the definition of \( T \). Thus 

\[
(n)(V^C \models \langle \langle t, s, f \rangle, g \rangle \in T) \iff \langle \langle t, s, f \rangle, g \rangle \in T).
\]

Hence \( V^C \models \langle \langle s, t, f \rangle, g \rangle \) is a branch of \( T \). By (1), (2), (3), there is a set \( B \) cofinal in \( \kappa \) and such that \((\exists \eta)(\theta)(\varphi(a, \xi)) \), i.e., 

\[
(\exists \eta)(\theta)(\varphi(a, \xi)) \iff (\exists y)(\varphi(a, \xi, y)) \).
\]

Moreover, by the fact that \( P \) is \( \kappa \) c.c.c. and by the regularity of \( \kappa \), \( \kappa = \alpha_1 \). Thus we can infer \( A(\xi) \). Now, as in Fact 9, we show that there is a perfect subset of \( A \). 

As an immediate corollary we obtain the following theorem.

**Theorem 4.** Assume that there is an ordinal \( \kappa \) such that \( \kappa \) is regular and there is no function from the continuum onto \( \kappa \). Then every \( \Pi^1_1 \) set either is well-orderable and of power less that \( \kappa \) or has a perfect subset in a boolean extension of the universe.

On the basis of \( \S 0 \) we obtain

**Theorem 5.** If \( (x) (x^x \text{ exists}) \) and \( 0' \) does not exist and there is a regular cardinal \( \kappa \) such that there is no function from the continuum onto \( \kappa \), then every \( \Pi^1_1 \) is well-orderable and of power less than \( \kappa \) or has a perfect subset, i.e., is of power \( \mathfrak{c} \).

Also we have

**Theorem 6.** Let \( M \) be \( \Pi^0_1 \)-correct in any \( M^C \) and in \( V \). Let \( A \) be \( \Pi^1_1(M) \). Assume there is no function from \( A \) onto \( \alpha_1 \) in \( M \). Then \( A \models M \) and \( A \) is countable in \( M \).

**Proof.** Define \( T \) for \( A \) inside \( M \) as in \( \S 1 \). Assume \( A \subseteq M \). We shall show that, in \( M \), \( P \neq \emptyset \) where \( P \) is in \( M \) the intersection of derivators of \( T \) such as in Fact 8. Suppose that \( P = \emptyset \). Then, by Fact 8, \( A \) is countable in \( M \). Hence in \( M \) there is a function \( f \) from \( \omega \) onto \( A^M \). We have in the world \( (E)(\langle \alpha \rangle \& (n)(\alpha \neq f(n))) \). By the \( \Pi^1_1 \)-correctness of \( M \), there is such an \( \alpha \) in \( M \). Contradiction. Thus \( P \neq \emptyset \) and hence \( A \) is a perfect subset in \( M \). For collapsing algebra \( C \). But, by the \( \Pi^1_1 \)-correctness of \( M \), \( A \) has a perfect subset in \( M \) and thus there is in \( M \) a function from \( \alpha_1 \) onto \( \alpha_1 \). Contradiction. Thus \( P = \emptyset \in M \) and hence \( A \) is countable in \( M \).

**Theorem 7.** Assume that \( (x) (x^x \text{ exists}) \) and \( 0' \) does not exist. Then, for every \( \Pi^1_1 \) set \( A \) either \( A \) is countable or there is a function from \( A \) onto \( \alpha_1 \).

**Proof.** Indeed, under the above assumptions, \( V \) is \( \Pi^1_1 \)-correct in \( V^C \) (see Remark 4) where \( C \) enumerates \( P(\alpha_1) \) with natural numbers. Suppose that there is no function from \( A \) onto \( \alpha_1 \). Then, by Theorem 2, \( A \) is countable or has a perfect subset in \( V^C \). Thus \( A \) is countable or has a perfect subset. But if \( A \) has a perfect subset, then there is a function from \( A \) onto \( \alpha_1 \). So \( A \) is countable.

**Remark 8.** Compare Theorem 8 with the \( \Pi^1_1 \) case. We have: every \( \Pi^1_1 \) set \( A \) is countable or there is a function from \( A \) onto \( \alpha_1 \) (decomposition into constituents or, if \( A \) is borel, the function from a perfect subset of \( A \) onto \( \alpha_1 \)).

**Theorem 8.** If \( A \) is a \( \Pi^1_1 \) set and is consistent that there is a regular ordinal \( \kappa \) such that there is no function from \( A \) onto \( \kappa \) and \( A \) is not well-orderable, then it is consistent that \( A \) has a perfect subset (we identify \( A \) with its fixed \( \Pi^1_1 \) definition).

This is a consistency version of Theorem 4.

**References**

[4] R. B. Jensen, \( \Sigma^1_3 \) absoluteness w.r.t. the core model, unpublished.

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