Perfect set theorems for $\underset{\sim}{\Pi}_2^1$ in the universe without choice

by

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Abstract. We work in the theory ZF. We prove the following Theorem 4: if there is a regular ordinal number $\varkappa$ such that there is no function from the continuum onto $\varkappa$, then every $\underset{\sim}{\Pi}_2^1$ set either is well-orderable or has a perfect subset in a boolean extension of the universe. Hence we obtain under the assumption $(x)(x^{\#}$ exists) and $0^{\dagger}$ does not exist the following Theorem 5: if there is a regular ordinal $\varkappa$ such that there is no function from the continuum onto $\varkappa$, then every $\underset{\sim}{\Pi}_2^1$ set either is well-orderable or has a perfect subset. As one of the corollaries we obtain Theorem 7: if (x) $(x^{\#}$ exists) and $0^{\dagger}$ does not exists, then from every $\underset{\sim}{\Pi}_2^1$ set there is a function onto ω_1. All these results follow from a construction, for a given $\underset{\sim}{\Pi}_2^1$ set A, of a certain tree, the notion of a tree which we use being somewhat different from the usual one. A is the projection of that tree. This method was introduced in [2] and the present paper shows how it can be applied.

In §0 we give a review of the present state of knowledge about perfect subsets of $\underset{\sim}{\Pi}_2^1$ sets, we prove several easy remarks and give the discussion of our theorems. In §1 we give a construction of a special tree for a given $\underset{\sim}{\Pi}_2^1$ set. In further sections we prove the theorems.

§0

By the perfect set theorem for a class Γ of subsets of ω^{ω} we mean the following: every set in Γ either is countable or has a perfect subset.

Let us first recall the well-known perfect-set theorems. For $\underset{\sim}{\Pi}_1^1$ or $\underset{\sim}{\Sigma}_2^1$ sets this is the Mansfield–Solovay theorem. Let us formulate it as follows (see [5], [8]): if $(x)_{\subseteq\omega}(\omega_1^{L(x)}$ is countable), then every $\underset{\sim}{\Pi}_1^1$ set either is countable or contains a perfect subset. The proof of this theorem uses the existence of a tree T for a given $\underset{\sim}{\Pi}_1^1$ set A such that $T \subseteq \omega^{<\omega} \times \omega_1^{<\omega}$ and $A(x) = (Ef)(\langle x, f\rangle$ is a branch of T) (see [9]). This characterization of A has the following absoluteness property: if $M \subseteq N$ are inner models and T^M, T^N are defined for A in M, N respectively, then $T^M = T^N \cap (\omega^{<\omega} \times (\omega_1^M)^{<\omega})$. Hence

(∗) if for x there is an f such that $\langle x, f\rangle$ is a branch of T^M for an inner model M, then $A(x)$.

If T, M have property (∗) we shall say that T *has property* (∗) *w.r.t.* M.

Let us discuss other perfect-set theorems.

If there is exactly one measurable cardinal and $P(\omega) \cap L[\mu]$ is countable, then we have the perfect set theorem for Π^1_2 (see [6]). Again for a Π^1_2 set A we have a tree with property $(*)$ w.r.t. $L[\mu]$.

If there is a measurable cardinal and $P(\omega) \cap \mathrm{HOD}$ is countable, then the perfect-set theorem holds for Π^1_2. Moreover, for every Π^1_2 set A there is a tree T with property $(*)$ w.r.t. HOD.

Also under the assumption of $\underset{\sim}{\Delta}{}^1_2$-determinancy the perfect-set theorem holds for $\underset{\sim}{\Pi}{}^1_2$, even for $\underset{\sim}{\Pi}{}^1_3$ and $\underset{\sim}{\Sigma}{}^1_4$ sets. Again in this case every $\underset{\sim}{\Pi}{}^1_2$ set is a projection of the set of branches of a tree.

There is also another method of finding a perfect subset of a Π^1_2 set A. Consider the following remarks. If M is a class, $\Pi^1_2(M)$ denotes Π^1_2 in a parameter from M.

REMARK 1. *Let M be an inner model and $\boldsymbol{P}$ a set of forcing conditions in M, $P^M(\boldsymbol{P}) \simeq \omega$. Let A be $\Pi^1_2(M)$. Let $\underset{\sim}{\alpha} \in M^{\boldsymbol{P}}$ be such that $M^{\boldsymbol{P}} \models A(\underset{\sim}{\alpha})$ and for every $p \in \boldsymbol{P}$ there are $q_1, q_2 \leqslant p$ and $n, m_1, m_2 \in \omega$ such that $m_1 \neq m_2$, $q_1 \Vdash (\underset{\sim}{\alpha}(\check{n}) = \check{m}_1)$ and $q_2 \Vdash (\underset{\sim}{\alpha}(\check{n}) = \check{m}_2)$. Then A has a perfect subset.*

Proof. By the assumption that $P^M(\boldsymbol{P}) \simeq \omega$ we can enumerate all dense subsets of $\boldsymbol{P}$ belonging to M as $D_0, D_1, \ldots$ Let us define the following mapping σ from $2^{<\omega}$ into $\boldsymbol{P}$. Let $\sigma(\emptyset)$ be any condition in D_0. If $\sigma(s)$ is defined, $\sigma(s) = p$, then let $\sigma(s^\frown\langle 0\rangle)$, $\sigma(s^\frown\langle 1\rangle)$ be such conditions $q_1, q_2 \leqslant p$ that

(1) there are $n_s, m_1, m_2 \in \omega$ such that

$$q_1 \Vdash 2(\underset{\sim}{\alpha}(\check{n}_s) = \check{m}_1), \qquad q_2 \Vdash (\underset{\sim}{\alpha}(\check{n}_s) = \check{m}_2),$$

(2) q_i determines the values of $\underset{\sim}{\alpha}$ at $\operatorname{dom} s+1$,

(3) $q_i \in D_{\operatorname{dom} s+1}$.

To find q_1, q_2 we first take $\tilde{q}_1, \tilde{q}_2$ satisfying (1) (they exist by the assumptions), next we take $q_1' \leqslant \tilde{q}_1$, $q_2' \leqslant \tilde{q}_2$ so that q_1', q_2' satisfy (2) and then we take q_1, q_2 so that $q_i \leqslant q_i'$ and q_i satisfies (3).

By definition, $\sigma(s) \in D_{\operatorname{dom} s}$.

Define an induced mapping $\sigma^*\colon 2^\omega \to \omega^\omega$ as

$$\sigma^*(f)_{(n)} = m \quad \text{iff} \quad \sigma(f{\restriction}_{n+1}) \Vdash (\underset{\sim}{\alpha}(\check{n}) = \check{m}).$$

We shall show that

(i) $\sigma^*\colon 2^\omega \to A$,

(ii) σ^* is continuous,

(iii) σ^* is 1-1.

Consider (i). Notice that $\{\sigma(f{\restriction}_n)\}_{n\in\omega}$ generate a $\boldsymbol{P}$, M-generic filter, G. Indeed, this follows by the fact that $\sigma(s) \in D_{\operatorname{dom} s}$. Moreover, $\sigma^*(f) = i_G(\underset{\sim}{\alpha})$. Hence $\sigma^*(f) \in A$ because $M^{\boldsymbol{P}} \models A(\underset{\sim}{\alpha})$ and A is absolute.

Consider (ii). Let t be an initial segment of $\sigma^*(f)$, $\operatorname{dom} t = n+1$. Then by the definition of σ^*, $\sigma(f{\restriction}_{n+1}) \Vdash (\check{t} \subseteq \underset{\sim}{\alpha})$. Hence, for every f', if $f'{\restriction}_{n+1} = f{\restriction}_{n+1}$ then $\sigma(f'{\restriction}_{n+1}) \Vdash (\check{t} \subseteq \underset{\sim}{\alpha})$ and thus $t \subseteq \sigma^*(f')$. Hence follows the continuity of σ^*.

Consider (iii). Let $f \neq f'$. Let n be the first number such that $f(n) \neq f'(n)$. Then, by the definition of σ,

$$\sigma(f{\restriction}_{n+1}) \Vdash (\underset{\sim}{\alpha}(\check{n}_{f{\restriction}n}) = \check{m}_1), \qquad \sigma(f'{\restriction}_{n+1}) \Vdash (\underset{\sim}{\alpha}(\check{n}_{f{\restriction}n}) = \check{m}_2)$$

for different m_1, m_2. Hence $\sigma^*(f) \neq \sigma^*(f')$.

By (i), (ii), (iii), the image of 2^ω under σ^* is a perfect subset of A. ∎

REMARK 2. *Let M be an inner model and $\boldsymbol{C}$ a complete boolean algebra in M, $P^M(\boldsymbol{C}) \simeq \omega$. Let A be $\Pi^1_2(M)$ and assume that there is a G generic over $\boldsymbol{C}$, M such that A has an element in $M[G]-M$. Then A has a perfect subset.*

Proof. Let α and G be such that $A(\alpha)$, G is generic over $\boldsymbol{C}$, M and $\alpha \in M[G]-M$. Then, by the absoluteness of A, $M[G] \models A(\alpha)$. Let $\underset{\sim}{\alpha} \in M^{\boldsymbol{C}}$ be such that $M^{\boldsymbol{C}} \models \underset{\sim}{\alpha} \in \omega^\omega$ and $i_G(\underset{\sim}{\alpha}) = \alpha$. Let p be such that $p \in \boldsymbol{C} \cap G$, $p \Vdash A(\underset{\sim}{\alpha})$. We can assume that p is the $\boldsymbol{1}$ of $\boldsymbol{C}$ because otherwise we can restrict $\boldsymbol{C}$ to p. Consider the following subalgebra of $\boldsymbol{C}$, $\boldsymbol{C}'$. Work in M. Let $\boldsymbol{C}'$ be the complete subalgebra of $\boldsymbol{C}$ generated by the values $\|\underset{\sim}{\alpha}(\check{n}) = \check{m}\|$ for $n, m \in \omega$. We shall show that there is a $p \in \boldsymbol{C}'$ such that $\boldsymbol{C}'$ restricted to p is atomless. Suppose the converse, i.e. that under every element of $\boldsymbol{C}'$ there is an atom of $\boldsymbol{C}'$. Then every filter generic over $\boldsymbol{C}'$, M is principal. Consider G. Let $G' = G \cap \boldsymbol{C}'$. Then G' is generic over $\boldsymbol{C}'$, M. Hence G' is principal in $\boldsymbol{C}'$. Thus $G' \in M$. But $\alpha = i_G(\underset{\sim}{\alpha})$ belongs to $M[G']$ – having G' we know all the values of α. Hence $\alpha \in M$. Contradiction.

So let p be an element of $\boldsymbol{C}'$ such that $\boldsymbol{C}'$ restricted to p is atomless. Again, we can assume that p is the $\boldsymbol{1}$ of $\boldsymbol{C}'$. Now we shall show that $\boldsymbol{C}'$ satisfies the assumptions of Remark 1. We can treat $\underset{\sim}{\alpha}$ as an element of $M^{\boldsymbol{C}'}$. Then $M^{\boldsymbol{C}'} \models A(\underset{\sim}{\alpha})$ because $M^{\boldsymbol{C}} \models A(\underset{\sim}{\alpha})$ and $\boldsymbol{C}'$ is a complete subalgebra of $\boldsymbol{C}$.

To prove the main assumption of Remark 1, take $p \in \boldsymbol{C}'$. By the fact that $\boldsymbol{C}'$ is atomless, there are G', G'' which are different and generic over $\boldsymbol{C}'$, M, $p \in G' \cap G''$. Then there is a generator of $\boldsymbol{C}'$ of the form $\|\underset{\sim}{\alpha}(\check{n}) = \check{m}\|$ which is in $G'-G''$ or in $G''-G'$ because otherwise G', G'', being equal at the generators, would be equal. Let $q_1 = p \wedge \|\underset{\sim}{\alpha}(\check{n}) = \check{m}\|$, $q_2 = p \wedge \|\underset{\sim}{\alpha}(\check{n}) = \check{m}'\| \leqslant -\|\underset{\sim}{\alpha}(\check{n}) = \check{m}\|$. Then q_1, q_2 are as required in Remark 1. Thus, by Remark 1, A has a perfect subset. ∎

In [1], [2] we studied Π^1_2 sets for which there is a tree T and a family $\mathscr{D}$ of its dense subsets such that A is the collection of the family of $\mathscr{D}$-generic branches of T or is a projection of such a collection. If this characterization is absolute in the following sense:

$(**)$ if M is an inner model and T^M, $\mathscr{D}^M$ are defined in M for A and $(\mathrm{E}f)$ ($\langle \alpha, f\rangle$ is a $\mathscr{D}^M$-generic branch of T^M) then $A(\alpha)$,

then we can infer that A is countable or has a perfect subset in $V^{\boldsymbol{C}}$ where $\boldsymbol{C}$ enumerates with natural numbers the family $\mathscr{D}$ and $P(\omega)$. Indeed, in this case, if A is not countable then the assumptions of Remark 2 are satisfied in $V^{\boldsymbol{C}}$ with $M = V$.

In this paper we shall define an arbitrary $\underset{\sim}{\Pi}^1_2$ set A, a tree T and a collection $\mathscr{D}$ of its dense subsets such that A is a projection of the collection of $\mathscr{D}$-generic branches of T (Theorem 1). However, this representation does not have the absoluteness property ($**$). Nevertheless, it will help us to prove a certain theorem about perfect subsets in ZF without choice. That theorem is the main result of the paper and states the following (Theorem 4):

If there is a regular ordinal $\varkappa$ such that there is no function from the continuum onto $\varkappa$, then every $\underset{\sim}{\Pi}^1_2$ set either is well-orderable and of power less that $\varkappa$ or has a perfect subset in a boolean extension of the universe.

Now we will discuss the question how our theorem is related to the present knowledge about perfect subsets of $\underset{\sim}{\Pi}^1_2$ sets.

Consider again the known perfect-set theorems for a class Γ. They are of the following form: "if certain assumptions hold, then for every set A in Γ there is a tree T of height ω such that A is a projection of the set of branches of T and T has the property ($*$) w.r.t. a certain inner model M. Then if $P(\omega)\cap M$ is countable, every set in Γ either is countable or has a perfect subset".

In fact, we can derive more from a theorem of this type. From the existence of T for a set A we can infer that the sentence $(\mathrm{E}x)A(x)$ is absolute w.r.t. M. Moreover, without the assumption $P(\omega)\cap M\simeq\omega$, we can infer another theorem about perfect sets, namely: every set A in Γ either is included in M or has a perfect subset.

Thus we have:

(1) Every $\Pi^1_1(\{x\})$ set either is included in $L(x)$, and thus is well-orderable, or has a perfect subsets;

(2) If a measurable cardinal exists, then every $\Pi^1_2(\{x\})$ set either is included in $\mathrm{HOD}(x)$, and thus is well-orderable, or has a perfect subset.

In both cases we have a complementary theorem to Theorem 4. Consider the following definitions:

DEFINITION 1. Let M be an inner model. We say that M is $\underset{\sim}{\Sigma}^1_3$-*correct* if, for every $\Pi^1_2(M)$ set A, the sentence $(\mathrm{E}\alpha)A(\alpha)$ is absolute in w.r.t. M.

Let M be called Σ^1_3-correct if the above absoluteness holds for Π^1_2 sets.

DEFINITION 2. Let M be an inner model. We say that M is *generally* $\underset{\sim}{\Sigma}^1_3$-*correct* if, for every real β such that β is generic over L, the universe $M[\beta]$ of sets relatively constructible from β and the class M is $\underset{\sim}{\Sigma}^1_3$-correct.

DEFINITION 3. Let K be Jensen's core-model. Let K^M be the core-model of an inner model M. If y is a real then by K_y we mean the relativization of the core-model to y, i.e. the union of mice relativized to y. Analogously we define K^M_y.

If x is a real then $K[x]$, $K_y[x]$, $K^M[x]$, $K^M_y[x]$ denote the class of sets relatively constructible from x and the classes K, K_y, K^M, K^M_y respectively.

Consider the following remark:

REMARK 3. *Let* $M\models\mathrm{ZFC}$ *be an inner model. Let* $\varkappa$ *be the cardinality of* $P(\omega)$ *in* M. *Let* $\mathbf{C}\in L$ *be the usual algebra collapsing* $\varkappa^{+L}$ *onto* ω. *Assume that, in* $V^{\mathbf{C}}$, M *is generally* $\underset{\sim}{\Sigma}^1_3$-*correct. Then every* $\Pi^1_2(M)$ *set either is included in* M *or has a perfect subset in* $V^{\mathbf{C}}$.

Proof. Let $\mathbf{C}'$ be the subalgebra of $\mathbf{C}$ collapsing $\varkappa$ onto ω. Work in $V^{\mathbf{C}}$. Let f be a function enumerating $\varkappa$ in type ω, generic over $\mathbf{C}'$, M. Note that f is generic over L. Then in $M[f]$ there is a function g such that $g\colon\omega\xrightarrow{\text{onto}}P^M(\omega)$. Let A be $\Pi^1_2(M)$. Suppose that $A\nsubseteq M$. Then $(\mathrm{E}\alpha)\big(A(\alpha)\,\&\,(n)(\alpha\neq g(n))\big)$. By the $\underset{\sim}{\Sigma}^1_3$-correctness of $M[f]$, in $M[f]$

$$(\mathrm{E}\alpha)\big(A(\alpha)\,\&\,(n)(\alpha\neq g(n))\big).$$

But then the assumptions of Remark 2 are satisfied in $V^{\mathbf{C}}$. Hence A has a perfect subset. ∎

The following theorem was proved by Jensen in [4]:

If (x) ($x^{\#}$ exists) and 0^{+} does not exist, then K is $\underset{\sim}{\Sigma}^1_3$-correct.

The relativized version of this theorem is the following:

If (x) ($x^{\#}$ exist) and y^{+} does not exist where y is a real, then K_y is $\underset{\sim}{\Sigma}^1_3$-correct.

We have the following remark:

REMARK 4. *If* (x) ($x^{\#}$ *exists) and* $0^{\dagger}$ *does not exist and* $\mathbf{C}$ *is a boolean algebra, then* V *is* $\underset{\sim}{\Sigma}^1_3$-*correct in* $V^{\mathbf{C}}$.

Proof. Let A be $\underset{\sim}{\Pi}^1_2$. Let y be the parameter of the definition of A. Assume that $(\mathrm{E}\alpha)A(\alpha)$ holds in $V^{\mathbf{C}}$. Consider K_y. Then K_y in the sense of $V^{\mathbf{C}}$ is the same as K_y. By the fact that K_y is $\underset{\sim}{\Sigma}^1_3$-correct in $V^{\mathbf{C}}$ we have

$$K_y\models(\mathrm{E}\alpha)A(\alpha).$$

Hence $(\mathrm{E}\alpha)A(\alpha)$. ∎

Consider the following conjecture:

($***$) If (x) ($x^{\#}$ exists) and $y^{\dagger}$ does not exist where y is a real, then K_y is generally $\underset{\sim}{\Sigma}^1_3$-correct.

Assume ($***$). Then we have the following remark:

REMARK 5. *If* (x) ($x^{\#}$*exists) and* $0^{\dagger}$ *does not exist, then every* $\underset{\sim}{\Pi}^1_2$ *set* A *either is included in* K_y *where* y *is the parameter of the definition of* A *or has a perfect subset in a boolean extension of the universe.*

Proof. We apply Remark 3 with $M=K_y$.

COROLLARY *If* (x) ($x^{\#}$ *exists) and* $0^{\dagger}$ *does not exist, then every* $\underset{\sim}{\Pi}^1_2$ *set is well-orderable or has a perfect subset in a boolean extension of the universe.*

Thus, under the assumption (x) ($x^{\#}$ exists) and $0^{\dagger}$ does not exist and under the hypothesis ($***$), we have proved a theorem similar to Theorem 4 by other methods than those used in this paper.

By all that we have said:

Theorem 4 is most interesting in the case where $(\mathrm{E}x)$ ($x^{\#}$ does not exist) – note that if $0^{\dagger}$ exists then Mansfield's theorems [6] work.

Returning to the assumption (x) ($x^{\#}$ exists) and $0^{\dagger}$ does not exist observe that in this case the conclusion of Theorem 4 can be stated as follows: every Π^1_2 set is well-orderable or has a perfect subset (in the universe). Indeed, in this case V is Σ^1_3 correct in V^C by Remark 4. But "having a perfect subset" is Σ^1_3 for a Π^1_2 set. Thus if a Π^1_2 set has a perfect subset in V^C, it just has a perfect subset.

Hence we have proved the following (under (***)):

If (x) ($x^{\#}$ exists) and $0^{\dagger}$ does not exist, then every Π^1_2 set is either included in K or has a perfect subset, every $\Pi^1_2(\{y\})$ set is either included in K_y or has a perfect subset.

Thus we see that, as in the case of L for Π^1_1 sets and in the case of HOD for Π^1_2 sets under the assumption of the existence of a measurable cardinal, in the case of K as well the property of Σ^1_3-correctness is connected with the fact that the perfect set theorem for Π^1_2 of the second form holds w.r.t. K (although we do not know whether there is a tree for a Π^1_2-set in K).

Finally, we observe that if there are arbitrarily large regular numbers then the assumption of Theorem 4 is satisfied. This follows from Remark 6 below. Notice that if $(\mathrm{E}x)$ ($x^{\#}$ does not exist), then, by the covering lemma w.r.t. the appropriate $L(x)$, there are arbitrarily large regular numbers. Thus if $(\mathrm{E}x)$ ($x^{\#}$ does not exist) then the conclusion of Theorem 4 holds.

Consider

REMARK 6 (ZF). *There is a cardinal $\nu \in On$ such that there is no function from the continuum onto ν.*

Proof of the remark. Let us define the following function f from $\mathscr{P}(P(2^{\omega}))$ into On as follows:

$$f(\mathscr{A}) = \begin{cases} \bar{\mathscr{A}} & \text{if } \mathscr{A} \text{ is well-orderable,} \\ 0 & \text{otherwise.} \end{cases}$$

Let $B = \{\beta\colon (\mathrm{E}\mathscr{A})_{P(P(2^{\omega}))}(f(\mathscr{A}) = \beta)\}$. Then by replacement B is a set. Let ν not belong to B. Let us show that ν is as required. Indeed, suppose that there is a function g from the continuum onto ν. Let $B_{\xi} = \{x \in 2^{\omega}\colon g(x) = \xi\}$. Let $\mathscr{A} = \{B_{\xi}\}_{\xi \in \nu}$. Then $\mathscr{A}$ is well-orderable and $f(\mathscr{A}) = \nu$. Hence $\nu \in B$. Contradiction. ■

Let $\varkappa$ be regular $\varkappa > \nu$. Then $\varkappa$ is as required in Theorem 4.

To end this section consider a few remarks concerning the theory ZF without the axiom of choice. In this theory we can develop a large part of the theory of projective sets. We consider the following projective hierarchy: Σ^1_0 are sets definable by an arithmetical formula with a parameter. If the class of Σ^1_n sets is defined, then Π^1_n is the class of complements of Σ^1_n sets and Σ^1_{n+1} is the class of projections of Π^1_n sets.

Note that this does not necessarily coincide with the topological hierarchy, for instance there may be borel sets that are not Δ^1_1.

For simple families of sets there are selectors because of the Kondo–Addison theorem.

§1

First we shall introduce auxiliary notions and notation.

Let us recall from [1] what we mean by a tree in a topological space.

Let $\langle \mathscr{X}, \mathscr{O} \rangle$ be a topological space in the sense that $\mathscr{O}$ is a basis in a topology in $\mathscr{X}$. Any subset $T \subseteq \mathscr{O}$ is called a *tree*. An $x \in \mathscr{X}$ is called a *branch of a tree* T iff

$$(p)_{\mathscr{O}}(x \in p \to (\mathrm{E}q)_T(x \in q \subseteq p)).$$

Let $A \subseteq \mathscr{X}$ be called $g.\ G_{\delta}$ (see [2]) if there is a tree $T \subseteq \mathscr{O}$ and a family $\mathscr{D}$ of dense sections of T such that: $\bar{\bar{\mathscr{D}}} < \bar{\bar{\mathscr{X}}}$ and A is the set of $\mathscr{D}$-generic branches of T.

The reals are identified with elements of ω^{ω} and will be denoted by $x, y, z, \ldots, \alpha, \beta, \ldots$ If a variable of this type runs over another set, we shall indicate this.

If $y \in 2^{\omega}$ is a well-ordering as a characteristic function of a set of pairs, then let $\bar{y}$ denote its type and $[y]_n$, for $n \in \omega$, the characteristic function of the ordering

$$\{\langle m, k\rangle\colon\ y(\langle k, n\rangle) = 0\}.$$

If α is a real, let $F_{\xi}(\alpha)$ be the ξth set constructible from α in the Gödel ordering.

By a cardinal we mean an initial ordinal. If $\varkappa$ is a cardinal, then $\bigotimes_{\xi \in \varkappa} a_{\xi}$ denotes the weak product, i.e. the set of finite functions $f\colon \varkappa \to \bigcup_{\xi} a_{\xi}$ such that $f(\xi) \in a_{\xi}$.

If A is a set of pairs, then by a projection of A we mean its projection onto the first coordinate.

The symbol $\langle .,. \rangle$ always denotes a pair (of integers or of reals) and $(.)_0$, $(.)_1$ denote the coordinates of a pair.

Now, in the main part of this section we shall carry out in ZF without choice a construction which is the technical basis of the paper.

Assume that we have a Π^1_2 set A. Let us construct a sequence of trees $(T_{\xi})_{\xi \in \omega_1}$ and of families $(\mathscr{D}_{\xi})_{\xi \in \omega_1}$ such that $T_{\xi} \subseteq \omega^{<\omega} \times \xi^{<\omega} \times \omega^{<\omega}$, $\mathscr{D}_{\xi}$ consists of dense subsets of T_{ξ}, and $A(\alpha) \equiv (\xi)\,\omega_1\,(\mathrm{E}f, g)\,(\langle \alpha, f, g\rangle$ is a $\mathscr{D}_{\xi}$-generic branch of $T_{\xi})$.

Let $A(\alpha) = (x)(\mathrm{E}y)\,R(\alpha, x, y)$ where R is Π^0_1. We ignore the parameters of A.

FACT 1. $A(\alpha) \equiv (L[\alpha] \models A(\alpha)) \equiv (\xi)\,\omega_1\,(F_{\xi}(\alpha)$ *is a real* $\Rightarrow (\mathrm{E}y)$ $R(\alpha, F_{\xi}(\alpha), y))$.

This fact follows from the Shoenfield absoluteness lemma [9].

FACT 2. *There is a Σ^1_1 formula ψ ~~(α, y)~~ where y is a parameter such that if y is* *a well-ordering then*

$$\psi(\alpha, y) \equiv (n)\left[F_{\overline{[y]_n}}(\alpha) \text{ is a real } \Rightarrow (\mathrm{E}y')R(\alpha, F_{\overline{[y]_n}}(\alpha), y')\right].$$

Proof. We shall give an outline of the proof. For details the reader is referred to [3] and [10]. Let $F(w, z, \alpha)$ be the Δ^1_1 formula with the property that, for a well-ordering z, $F(w, z, \alpha) \equiv (w$ codes the set $F_{\bar{z}}(\alpha))$ where coding can be done as in [10]. We have

$$(n)\left(F_{\overline{[y]_n}}(\alpha) \text{ is a real } \Rightarrow (\mathrm{E}y')R(\alpha, F_{\overline{[y]_n}}(\alpha), y')\right)$$
$$\equiv (n)(\mathrm{E}z)(\mathrm{E}w)\left[z = [y]_n \,\&\, F(w, z, \alpha)\right.$$
$$\left.\& \ (w \text{ does not code a real } \vee (\mathrm{E}u)(w \text{ codes } u \,\&\, (\mathrm{E}y')R(\alpha, u, y')))\right].$$

Consider the formula "w does not code a real". Let us indicate how to prove that it is Σ^1_1. We have "w does not code a real" iff there is an n' in the collection of almost maximal vertices of w which is not a code of a pair of integers or there are two codes of pairs in this collection which have the same first coordinate and different second coordinates or there is an m for which there is no n such that the code of $\langle m, n\rangle$ is an almost maximal vertice of w.

Consider the formula "w codes u".

We have "w codes u" iff the collection of almost maximal vertices of w consists of codes of pairs of integers that are in u.

It follows that the formula

$$(n)(\mathrm{E}z)(\mathrm{E}w)\Big[z = [y]_n \,\&\, F(w, z, \alpha) \,\&\, \big(w \text{ does not code}$$
$$\text{a real } \vee (\mathrm{E}u)(w \text{ codes } u \,\&\, (\mathrm{E}y')R(\alpha, u, y'))\big)\Big]$$

is equivalent to a Σ^1_1 formula. Let ψ be this Σ^1_1 formula. ■

FACT 3. *If $\theta(\alpha, y)$ is a Σ^1_1 formula of α in a parameter y, then there is a tree $T \subseteq \omega^{<\omega} \times \omega^{<\omega} \times \omega^{<\omega}$ such that*

$$\theta(\alpha, y) \equiv (\mathrm{E}z)(\langle \alpha, y, z\rangle \text{ is a branch of } T).$$

Proof. We have

$$\neg\theta(\alpha, y) \equiv (z)(\mathrm{E}n)Q(\alpha, z, n)$$

where $Q(\alpha, z, n)$ is recursive in y. Thus $(\mathrm{E}n)Q(\alpha, z, n)$ is recursively enumerable in y. But, by the definition of relative recursive enumerability, there is a recursive $Q' \subseteq \omega^{<\omega} \times \omega^{<\omega} \times \omega^{<\omega} \times \omega^{<\omega}$ such that

$$(\mathrm{E}n)Q(\alpha, z, n) \equiv (\mathrm{E}k)Q'(\alpha_{\restriction k}, z_{\restriction k}, y_{\restriction k}, k).$$

Thus

$$\theta(\alpha, y) \equiv (\mathrm{E}z)(k)(\neg Q'(\alpha_{\restriction k}, z_{\restriction k}, y_{\restriction k}, k)).$$

Let

$$T = \{\langle s, t, u\rangle\colon \operatorname{dom} s = \operatorname{dom} t = \operatorname{dom} u$$
$$\&(\mathrm{E}s', t', u')(s \subseteq s', t \subseteq t', u \subseteq u'), \operatorname{dom} s' = \operatorname{dom} t' = \operatorname{dom} u'$$
$$\& Q'(s', t', u', \operatorname{dom} s')\}.$$

Then we have

$$\theta(\alpha, y) = (\mathrm{E}z)(\langle \alpha, y, z\rangle \text{ is a branch of } T). \ ■$$

Combining Fact 2 and Fact 3 we obtain

FACT 4. *There is a tree $T \subseteq \omega^{<\omega} \times \omega^{<\omega} \times \omega^{<\omega}$ such that*

$$\psi(\alpha, y) \equiv (\mathrm{E}z)(\langle \alpha, y, z\rangle \text{ is a branch of } T).$$

Consider the formula

$$\varphi(\alpha, \xi)\colon (\eta)_\xi\big(F_\eta(\alpha) \text{ is a real } \Rightarrow (\mathrm{E}y')R(\alpha, F_\eta(\alpha), y')\big).$$

We show the following

FACT 5. *There is a tree $T_\xi \subseteq \omega^{<\omega} \times \xi^{<\omega} \times 2^{<\omega} \times \omega^{<\omega}$ and a family $\mathscr{D}_\xi$ of its dense subsets such that*

$$\varphi(\alpha, \xi) \equiv (\mathrm{E}f, y, z)(\langle \alpha, f, y, z\rangle \text{ is a } \mathscr{D}_\xi\text{-generic branch of } T_\xi).$$

Proof. Let $\psi(\alpha, y)$ be the Σ^1_1 formula defined in Fact 2 and let T be the tree such that

$$\psi(\alpha, y) \equiv (\mathrm{E}z)(\langle \alpha, y, z\rangle \text{ is a branch of } T).$$

Let us define T_ξ as follows:

$$\langle s, t, v, u\rangle \in T_\xi$$

iff

(1) $\quad s \in \omega^{<\omega},\ t \in \xi^{<\omega},\ v \in 2^{<\omega},\ u \in \omega^{<\omega}$
$$\& \operatorname{dom} s = \operatorname{dom} t = \operatorname{dom} v = \operatorname{dom} u,$$

(2) $\quad \langle s, v, u\rangle \in T,$

(3) $\quad v(\langle m, n\rangle) = 0 \ \equiv\ t(m) < t(n),$

(4) $\quad (\mathrm{E}\alpha, f, y, z)_{\omega^\omega \times \xi^\omega \times 2^\omega \times \omega^\omega}\big(\langle s, t, v, u\rangle \subseteq \langle \alpha, f, y, z\rangle \,\&\, \langle \alpha, y, z\rangle$
$$\text{is a branch of } T \,\&\, (m, n)\big(y(\langle m, n\rangle) = 0$$
$$\equiv f(m) < f(n)\big) \,\&\, f\colon \omega \xrightarrow{\text{onto}} \xi\big),$$

i.e. we require that through every element of T_ξ there should go a branch of T_ξ.

Let $\eta\in\xi$. Let

$$D_\eta=\{\langle s,t,v,u\rangle\in T_\xi:\ \eta\in\operatorname{rg} t\}.$$

By (4) D_η is dense in T_ξ,

$$\mathscr{D}_\xi=\{D_\eta\}_{\eta\in\xi}.$$

We must show that T_ξ is as required, i.e.

$$\varphi(\alpha,\xi)\equiv(\mathrm{E}f,y,z)(\langle\alpha,f,y,z\rangle \text{ is a } \mathscr{D}_\xi\text{-generic branch of } T_\xi).$$

We have:

$$\varphi(\alpha,\xi)\equiv(\mathrm{E}y)\big(y \text{ is a well-ordering of } \omega \text{ in type } \xi\ \&\ \psi(\alpha,y)\big).$$

Thus assume $\varphi(\alpha,\xi)$. Take y such that y is a well-ordering of type ξ and $\psi(\alpha,y)$. Then there is a z such that $\langle\alpha,y,z\rangle$ is a branch of T. Define $f\in\xi^\omega$ as

$$f(n)=\overline{[y]_n}.$$

Then

$$f:\ \omega\xrightarrow{\text{onto}}\xi \text{ and } (m)(n)\big(f(m)<f(n)\equiv y(\langle m,n\rangle)=0\big).$$

Hence $\langle\alpha,f,y,z\rangle$ is a branch of T_ξ. It is a $\mathscr{D}_\xi$-generic branch because f is onto ξ.

Conversely, assume that there are f,y,z such that $\langle\alpha,f,y,z\rangle$ is a $\mathscr{D}_\xi$-generic branch of T_ξ. Then $\langle\alpha,y,z\rangle$ is a branch of T and thus $\psi(\alpha,y)$. Moreover,

$$f:\ \omega\xrightarrow{\text{onto}}\xi \text{ and } (m)(n)\big(y(\langle m,n\rangle)=0\ \equiv\ f(m)<f(n)\big).$$

Hence y is a well-ordering of type ξ. Thus $\varphi(\alpha,\xi)$. ∎

We can join the last two coordinates of the elements of T_ξ. So let us assume that $T_\xi\subseteq\omega^{<\omega}\times\xi^{<\omega}\times\omega^{<\omega}$. We have

FACT 6.

$$\begin{aligned}A(\alpha)&\equiv(\xi)_{\omega_1}\big(F_\xi(\alpha) \text{ is a real} \Rightarrow (\mathrm{E}y)R(\alpha,F_\xi(\alpha),y)\big)\\&\equiv(\xi)_{\omega_1}(\eta)_\xi\big(F_\eta(\alpha) \text{ is a real} \Rightarrow (\mathrm{E}y')R(\alpha,F_\eta(\alpha),y')\big)\\&\equiv(\xi)_{\omega_1}\varphi(\alpha,\xi)\\&\equiv(\xi)_{\omega_1}(\mathrm{E}f,g)(\langle\alpha,f,g\rangle \text{ is a } \mathscr{D}_\xi\text{-generic branch of } T_\xi).\end{aligned}$$

Still, without choice, we can define the following tree:

$$T\subseteq\omega^{<\omega}\times\bigotimes_{\xi\in\omega_1}(\xi^{<\omega}\times\omega^{<\omega}).$$

Let

$$\big\langle s,\{\langle\xi_0,\langle t_{\xi_0},v_{\xi_0}\rangle\rangle,\ldots,\langle\xi_n,\langle t_{\xi_n},v_{\xi_n}\rangle\rangle\}\big\rangle\in T$$

iff

(1) $s\in\omega^{<\omega}$, $t_{\xi_i}\in\xi_i^{<\omega}$, $v_{\xi_i}\in\omega^{<\omega}$, $\operatorname{dom}s=\operatorname{dom}t_{\xi_i}=\operatorname{dom}v_{\xi_i}$,

(2) $\langle s,t_{\xi_i},v_{\xi_i}\rangle\in T_{\xi_i}$,

(3) $(\mathrm{E}\alpha,f_0,g_0,\ldots,f_n,g_n)\big(\langle s,t_{\xi_i},v_{\xi_i}\rangle\subseteq\langle\alpha,f_i,g_i\rangle$ $\&\ \langle\alpha,f_i,g_i\rangle$ is a branch of $T_{\xi_i}\ \&\ f_i:\ \omega\xrightarrow{\text{onto}}\xi_i\ \&\ A(\alpha)\big)$.

Let $p\in T$. We introduce the following notation:

$\xi\in\operatorname{dom}p$ if $p=\big\langle s,\{\langle\xi_0,\langle t_{\xi_0},v_{\xi_0}\rangle\rangle,\ldots,\langle\xi_n,\langle t_{\xi_n},v_{\xi_n}\rangle\rangle\}\big\rangle$

and ξ is a ξ_i for an i

$p(\xi)$ is the pair $\langle t_\xi,v_\xi\rangle$ if $\xi\in\operatorname{dom}p$,

$(p)_0$ is the first coordinate of p, s,

$(p)_1$ is the sequence $\{\langle\xi_0,\langle t_{\xi_0},v_{\xi_0}\rangle\rangle,\ldots,\langle\xi_n,\langle t_{\xi_n},v_{\xi_n}\rangle\rangle\}$.

Let us define the following family of dense subsets of T:

if $\xi\in\omega_2$, $\eta\in\xi$ then

$$D_{\xi,\eta}=\{p\in T:\ \xi\in\operatorname{dom}p,\ \eta\in\operatorname{rg}(p(\xi))_0\}.$$

By (3) $D_{\xi,\eta}$ is dense in T. Let $\mathscr{D}=\{D_{\xi,\eta}\}_{\substack{\xi\in\omega_1\\ \eta\in\xi}}$. Then, using the axiom of choice, we can prove

THEOREM 1. *Let A be Π^1_2. Then there is a g. G_δ subset B of the space $\omega^\omega\times\prod_{\xi\in\omega_1}(\xi^\omega\times\omega^\omega)$ such that A is a projection of B onto the first coordinate.*

Proof.

$$B=\Big\{\langle\alpha,f\rangle:\ \alpha\in\omega^\omega,\ f\in\prod_{\xi\in\omega_1}(\xi^\omega\times\omega^\omega),$$
$$\big(f(\xi)\big)_0\in\xi^\omega\ \&\ \big(f(\xi)\big)_1\in\omega^\omega\ \&\ (n,m)(\xi_0,\ldots,\xi_n)_{\omega_1}$$
$$\Big\langle\alpha_{\restriction m},\Big\{\big\langle\xi_0,\langle(f(\xi_0))_{0\restriction m},(f(\xi_0))_{1\restriction m}\rangle\big\rangle,\ldots$$
$$\ldots,\big\langle\xi_n,\langle(f(\xi_n))_{0\restriction m},(f(\xi_n))_{1\restriction m}\rangle\big\rangle\Big\}\Big\rangle\in T\Big\}.$$

Then, by definition, $\langle\alpha,f\rangle\in B$ iff it is a $\mathscr{D}$-generic branch of T. We show

$$A(\alpha)\equiv(\mathrm{E}f)(\langle\alpha,f\rangle \text{ is a } \mathscr{D}\text{-generic branch of } T).$$

For a proof we first observe that by Fact 6 we have

$$A(\alpha)\equiv(\xi)_{\omega_1}(\mathrm{E}f,g)(\langle\alpha,f,g\rangle \text{ is a } \mathscr{D}_\xi\text{-generic branch of } T_\xi).$$

Assume $A(\alpha)$. For every ξ choose one pair $\langle f_\xi,g_\xi\rangle$ such that $\langle\alpha,f_\xi,g_\xi\rangle$ is a $\mathscr{D}_\xi$-generic branch of T_ξ. Define $f\in\prod_{\xi\in\omega_1}(\xi^\omega\times\omega^\omega)$ as $f(\xi)=\langle f_\xi,g_\xi\rangle$. Then

$\langle\alpha, f\rangle$ is a $\mathscr{D}$-generic branch of T. Conversely, if there is an f such that $\langle\alpha, f\rangle$ is a $\mathscr{D}$-generic branch of T, then, for every ξ, $\langle\alpha, (f(\xi))_0, (f(\xi))_1\rangle$ is a $\mathscr{D}_\xi$-generic branch of T_ξ. Hence $A(\alpha)$. ∎

§2

In this section we prove a theorem about perfect subsets of $\underset{\sim}{\Pi}^1_2$ sets. Its proof is an illustration of the method used in §3 to prove a stronger theorem.

THEOREM 2. *Let A be $\underset{\sim}{\Pi}^1_2$. Assume that ω_1 is regular and there is no function from A onto ω_1. Then either A is countable or A has a perfect subset in some boolean extension of the universe.*

Let us first explain the idea of the proof. Consider the tree T defined for A in §1. It would be natural to treat T as a set of forcing conditions. Then a V-generic filter over T would provide a sequence $\langle\alpha, f_\xi\rangle_{\xi\in\omega_1}$ such that $\langle\alpha, f_\xi\rangle$ is a $\mathscr{D}_\xi$-generic branch of T_ξ, i.e.

$$(\xi)_{\omega_1}(\eta)_\xi\big(F_\eta(\alpha) \text{ is a real} \Rightarrow (\mathrm{E}y')\,R(\alpha, F_\eta(\alpha), y')\big).$$

If the forcing T does not collapse ω_1, then in the extended universe we have

$$(\xi)^{L(\alpha)}_{\omega_1}(\eta)_\xi\big(F_\eta(\alpha) \text{ is a real} \Rightarrow (\mathrm{E}y')\,R(\alpha, F_\eta(\alpha), y')\big) \text{ and thus } A(\alpha).$$

We shall show that, under the assumptions of the theorem, T is c.c.c. The usual proof shows that c.c.c. together with the regularity of ω_1 implies in ZF (without choice) that ω_1 is not collapsed by T. Thus T enables us to add elements of A. We are not able yet to add a perfect set of elements of A, because T is not necessarily separable (for instance if A is provably a singleton) and generic filters over T can then provide α's in V. The next idea will be to observe that either A is countable or there is a c.c.c. atomless separable non-empty subset of T, $\boldsymbol{P}$, splitting at the first coordinate. Then, by standard methods, we show that A has a perfect subset in $V^{\boldsymbol{C}}$ where $\boldsymbol{C}$ is a boolean algebra enumerating with natural numbers the family of dense subsets of $\boldsymbol{P}$.

We introduce the following definition:

DEFINITION 4. Let

$$p\in T,\quad p = \langle s, \{\langle\xi_0, \langle t_{\xi_0}, v_{\xi_0}\rangle\rangle, \ldots, \langle\xi_n, \langle t_{\xi_n}, v_{\xi_n}\rangle\rangle\}\rangle \quad\text{and}\quad \alpha\in\omega^\omega.$$

We say that α *goes through* p if there are $f_{\xi_0}, g_{\xi_0}, \ldots, f_{\xi_n}, g_{\xi_n}$ such that $s\subseteq\alpha$, $t_{\xi_i}\subseteq f_{\xi_i}$, $v_{\xi_i}\subseteq g_{\xi_i}$, $f_{\xi_i}\colon \omega \xrightarrow{\text{onto}} \xi_i$ and $\langle\alpha, f_{\xi_i}, g_{\xi_i}\rangle$ is a branch of T_{ξ_i}. Notice that, by the definition of T, through every element of T goes an α in A.

FACT 7. *T is c.c.c.*

Proof. Observe first that if

$$p = \langle s, \{\langle\xi_0, \langle t_{\xi_0}, v_{\xi_0}\rangle\rangle, \ldots, \langle\xi_n, \langle t_{\xi_n}, v_{\xi_n}\rangle\rangle\}\rangle,$$
$$q = \langle s, \{\langle\eta_0, \langle t_{\eta_0}, v_{\eta_0}\rangle\rangle, \ldots, \langle\eta_n, \langle t_{\eta_n}, v_{\eta_n}\rangle\rangle\}\rangle$$

and $\langle t_{\xi_0}, v_{\xi_0}, \ldots, t_{\xi_n}, v_{\xi_n}\rangle$, $\langle t_{\eta_0}, v_{\eta_0}, \ldots, t_{\eta_n}, v_{\eta_n}\rangle$ are compatible in $\bigotimes_{\xi\in\omega_1}(\xi^{<\omega}\times\omega^{<\omega})$ and there is one and the same $\alpha\in A$ going through p and through q, then p, q are compatible in T.

Indeed, let

$$r = \langle s, \{\langle\zeta_0, \langle\tilde{t}_{\zeta_0}, \tilde{v}_{\zeta_0}\rangle\rangle, \ldots, \langle\zeta_m, \langle\tilde{t}_{\zeta_m}, \tilde{v}_{\zeta_m}\rangle\rangle\}\rangle$$

where

$$\{\zeta_0, \ldots, \zeta_m\} = \{\xi_0, \ldots, \xi_n\}\cup\{\eta_0, \ldots, \eta_n\}$$

and if $\zeta_i = \xi_j$ then $\tilde{t}_{\zeta_i} = t_{\xi_j}$, $\tilde{v}_{\zeta_i} = v_{\xi_j}$ and if $\zeta_i = \eta_k$ then $\tilde{t}_{\zeta_i} = t_{\eta_k}$, $\tilde{v}_{\zeta_i} = v_{\eta_k}$.

Then the same α in A which goes through p and through q goes through r, and thus $r\in T$, $r\leqslant p$, $r\leqslant q$.

Now suppose that there exists a family $F\subseteq T$ such that F consists of pairwise incompatible conditions and is of power ω_1. By the regularity of ω_1 we can assume that all conditions in F have the same s at the first coordinate and the same length.

Let $\alpha\in A$, and $F_\alpha = \{p\in F\colon \alpha \text{ goes through } p\}$. Then F_α consists of pairwise incompatible conditions such that through every two of them goes the same α in A. Thus, by our observation, for every pair of conditions p, q in F_α, $(p)_1$ is incompatible with $(q)_1$ in $\bigotimes_{\xi\in\omega_1}(\xi^{<\omega}\times\omega^{<\omega})$. Hence F_α is countable because $\bigotimes_{\xi\in\omega_1}(\xi^{<\omega}\times\omega^{<\omega})$ is c.c.c. By the regularity of ω_1 there is a $\zeta<\omega_1$ such that $F_\alpha\subseteq T\cap\big(\omega^{<\omega}\times\bigotimes_{\xi<\zeta}(\xi^{<\omega}\times\omega^{<\omega})\big)$.

Let ζ_α be the least such ζ. So we have defined a function from A into ω_1. By our assumption that there is no function from A onto ω_1 and by the regularity of ω_1, $\sup_{\alpha\in A}\zeta_\alpha<\omega_1$. Let $\sup_{\alpha\in A}\zeta_\alpha = \eta$. Then, by the fact that $F = \bigcup_{\alpha\in A}F_\alpha$, $F\subseteq T\cap\big(\omega^{<\omega}\times\bigotimes_{\xi<\eta}(\xi^{<\omega}\times\omega^{<\omega})\big)$. Hence F is countable, contradiction. ∎

DEFINITION 5. Consider a subset $\boldsymbol{P}$ of T. We say that $\boldsymbol{P}$ *splits at the first coordinate* if, for every p in $\boldsymbol{P}$, there are q_1, q_2 in $\boldsymbol{P}$ such that $q_i\leqslant p$ and $(q_1)_0$ is incompatible with $(q_2)_0$ in $\omega^{<\omega}$.

FACT 8. *Either A is countable or there is a subset $\boldsymbol{P}$ of T such that $\boldsymbol{P}$ is non-empty and c.c.c. and that $\boldsymbol{P}$ splits at the first coordinate.*

Proof. We modify Solovay's idea [7]. Let us define

$$T^0 = T,$$
$$T^{\xi+1} = \{p\in T^\xi\colon (\mathrm{E}\alpha_1, \alpha_2)\,(A(\alpha_i)\,\&\,\alpha_i \text{ goes through } p\,\&\,\alpha_1\neq\alpha_2)$$
$$\&\,(q)_T\,(\alpha_i \text{ goes through } q \Rightarrow q\in T^\xi)\}.$$

For limit λ, let $T^\lambda = \bigcap\{T^\xi\colon \xi<\lambda\}$. Since T is well-orderable, the usual cardinality argument shows that the minimum ξ such that $T^{\xi} = T^{\xi+1}$ is less that ω_2. We denote this minimal ξ by $\xi(T)$.

Let $\boldsymbol{P} = T^{\xi(T)}$. We have:

$$p \in \boldsymbol{P} \text{ iff}$$

$$(\mathrm{E}\alpha_1, \alpha_2)\big(A(\alpha_i) \,\&\, \alpha_i \text{ goes through } p \,\&\, \alpha_1 \neq \alpha_2 \\ \&\, (q)_T(\alpha_i \text{ goes through } q \Rightarrow q \in \boldsymbol{P}).$$

Suppose that $\boldsymbol{P} = \emptyset$. We shall show that A is countable. Let α be such that $A(\alpha)$. Then there is a ξ such that there is a p in T^{ξ} such that α goes through p and $p \notin T^{\xi+1}$. Indeed, otherwise $(q)_T(\alpha$ goes through $q \Rightarrow q \in \boldsymbol{P})$ and hence $\boldsymbol{P} \neq \emptyset$.

Let ξ_α be the least such ξ that there is a p in $T^{\xi} - T^{\xi+1}$ such that α goes through p. Take the least such p. Then by defintttion we have

$$(*) \qquad (q)_T(\alpha \text{ goes through } q \Rightarrow q \in T^{\xi_\alpha}).$$

Let us show that α is the only member of A with property $(*)$ going through p. Indeed, suppose that α' is another member of A with property $(*)$. Then α, α' are such α_1, α_2 as are required to make p belong to $T^{\xi_\alpha+1}$. But $p \notin T^{\xi_\alpha+1}$. Thus α is the only member of A going through p with property $(*)$. Thus to every α in A corresponds canonically a pair $\langle \xi_\alpha, p\rangle$ such that $p \in T^{\xi_\alpha}$. Hence A has a well-ordering. Thus, by our assumption that there is no function from A onto ω_1, A is countable.

Thus either A is countable or $\boldsymbol{P} \neq \emptyset$.

Let us show that $\boldsymbol{P}$ has the following properties:

(1) if $p \in \boldsymbol{P}$, $\xi \in \omega_1$, $n \in \omega$, $\eta \in \xi$ then there is a $q \leqslant p$, $q \in \boldsymbol{P}$ such that $\xi \in \operatorname{dom} q \,\&\, \eta \in \operatorname{rg}(q(\xi))_0 \,\&\, n \in \operatorname{dom}(q)_0$,

(2) if $p \in \boldsymbol{P}$, then there are q_1, q_2 in $\boldsymbol{P}$ such that $(q_1)_0$ is incompatible with $(q_2)_0$ and $q_i \leqslant p$,

(3) $\boldsymbol{P}$ is c.c.c.

Let us show (1). Let $p \in \boldsymbol{P}$, $p = \big\langle s, \{\langle \xi_0, \langle t_{\xi_0}, v_{\xi_0}\rangle\rangle, \ldots \ldots, \langle \xi_n, \langle t_{\xi_n}, v_{\xi_n}\rangle\rangle\}\big\rangle$. Then

$$(\mathrm{E}\alpha_1, \alpha_2)\big(A(\alpha_i) \,\&\, \alpha_i \text{ goes through } p \,\&\, \alpha_1 \neq \alpha_2\big) \\ \&\, (q)_T(\alpha_i \text{ goes through } q \Rightarrow q \in \boldsymbol{P}).$$

Take α_1. Then there are $f_0, g_0, \ldots, f_n, g_n$ such that $\langle \alpha_1, f_i, g_i\rangle$ is a branch of T_{ξ_i} and $f_i\colon \omega \xrightarrow{\text{onto}} \xi_i$. Moreover, by the fact that $A(\alpha_1)$, there are f, g such that $f\colon \omega \xrightarrow{\text{onto}} \xi$ and $\langle \alpha_1, f, g\rangle$ is a branch of T_ξ. Let m be such that $m > n$, $\eta \in \operatorname{rg} f \restriction m$. Let

$$q = \big\langle \alpha_{1\restriction m}, \{\langle \xi_0, \langle f_{0\restriction m}, g_{0\restriction m}\rangle\rangle, \ldots, \langle \xi_n, \langle f_{n\restriction m}, g_{n\restriction m}\rangle\rangle, \langle \xi, \langle f_{\restriction m}, g_{\restriction m}\rangle\rangle\}\big\rangle.$$

Then $q \in T$, α_1 goes through q. Hence $q \in \boldsymbol{P}$ because

$$(q)_T(\alpha_1 \text{ goes through } q \Rightarrow q \in \boldsymbol{P}).$$

To show (2) we define, respectively, q_1, q_2 for α_1, α_2 as above, where m is such that $\alpha_{1\restriction m} \neq \alpha_{2\restriction m}$.

Let us show (3). Let

$$A' = \{\alpha \in A\colon (q)_T(\alpha \text{ goes through } q \Rightarrow q \in \boldsymbol{P}\}.$$

We have

$$p \in \boldsymbol{P} \equiv p \in T \,\&\, (\mathrm{E}\alpha)_{A'}(\alpha \text{ goes through } p).$$

Then we can repeat the proof of Fact 7 with A' in place of A. ■

COROLLARY. *If G is generic over $\boldsymbol{P}$, then G determines a sequence $\langle \alpha, f_\xi, g_\xi\rangle_{\xi \in \omega_1^{L(\alpha)}}$ such that $A(\alpha)$.*

Indeed, $\omega_1^{L(\alpha)} = \omega_1$ by Fact 8.

By Fact 8, $f_\xi\colon \omega \xrightarrow{\text{onto}} \xi \,\&\, \langle \alpha, f_\xi, g_\xi\rangle$ is a branch of T_ξ. Then by §1 we have $A(\alpha)$.

FACT 9. *Assume that A is not countable. Let $\boldsymbol{C}$ collapse $\mathscr{P}(\boldsymbol{P})$ onto ω. Then A has a perfect subset in $V^{\boldsymbol{C}}$.*

Proof. Notice that the assumptions of Remark 1 are satisfied in $V^{\boldsymbol{C}}$ with $M = V$.

By the corollary, a filter G-generic over $\boldsymbol{P}$ canonically determines a sequence $\langle \alpha, f_\xi, g_\xi\rangle_{\xi \in \omega_1^{L(\alpha)}}$. Hence there is a canonical name $\underset{\sim}{\alpha}$ such that $\underset{\sim}{\alpha}$ is realized as α in the extended universe.

We have in $V^{\boldsymbol{C}}$, $\mathscr{P}^M(\mathscr{P}) \simeq \omega$. Moreover, if $p \in \boldsymbol{P}$ then there are $q_1, q_2 \leqslant p$ and n, m_1, m_2 such that $q_1 \Vdash (\underset{\sim}{\alpha}(\check{n}) = \check{m}_1)$, $q_2 \Vdash (\underset{\sim}{\alpha}(\check{n}) = \check{m}_2)$. Indeed, this follows from the fact that $\boldsymbol{P}$ splits the first coordinate. Thus, by Remark 1, A has a perfect subset. ■

Thus we have completed the proof of Theorem 2.

§3

Now we shall prove a stronger theorem by a similar but more complicated method.

THEOREM 3. *Let $\varkappa$ be a regular cardinal. Let A be $\underset{\sim}{\Pi}^1_2$ and assume that there is no function from A onto $\varkappa$. Then either A is well-orderable and of power less than $\varkappa$ or A has a perfect subset in a boolean extension of the universe.*

Proof. First consider a purely combinatorial lemma.

LEMMA 1 (ZF). *If $\varkappa$ is regular then $\bigotimes_{\xi<\varkappa} \xi^{<\omega}$ is $\varkappa$.c.c.*

Proof. We shall show by induction on n that if $F \subseteq \bigotimes_{\xi<\varkappa} \xi^{<\omega}$ is a family of pairwise incompatible functions with n-element domains then F is of power less than $\varkappa$.

If $n = 1$ then every f in F has the same domain, i.e. there is a $\xi \in \varkappa$ such that $f \in F \Rightarrow \operatorname{dom} f = \{\xi\}$. Thus $F \subseteq \{\langle \xi, t\rangle : t \in \xi^{<\omega}\}$. Hence $\bar{F} = \bar{\xi}$ and so $\bar{F} < \varkappa$.

Assume the inductive assumption for n and let $F \subseteq \bigotimes_{\xi<\varkappa} \xi^{<\omega}$ consist of pairwise incompatible functions with $(n+1)$-element domains.

Suppose that $\bar{F} = \varkappa$. Fix $g \in F$. Let $\operatorname{dom} g = \{\xi_0, \ldots, \xi_n\}$. Define $F_i = \{f \in F: \xi_i \in \operatorname{dom} f\}$. Then $F = \bigcup_{i \in n+1} F_i$. There is an i_0 such that $\bar{F}_{i_0} = \varkappa$. Let for $t \in \xi_{i_0}^{<\omega}$ $F_{i_0}^t = \{f \in F_{i_0}: f(\xi_{i_0}) = t\}$. We have $F_{i_0} = \bigcup_t F_{i_0}^t$.

Again by the fact that $\xi_{i_0}^{<\omega}$ is of power less than $\varkappa$ and by the regularity of $\varkappa$, there is a $t_0 \in \xi_{i_0}^{<\omega}$ such that $F_{i_0}^{t_0}$ is of power $\varkappa$. Now define $F' = \{f_{\restriction \operatorname{dom} f - \{\xi_{i_0}\}}: f \in F_{i_0}^{t_0}\}$. If $f', g' \in F'$ then there are $f, g \in F$ such that $f(\xi_{i_0}) = g(\xi_{i_0})$ and $f' = f_{\restriction \operatorname{dom} f - \{\xi_{i_0}\}}$, $g' = g_{\restriction \operatorname{dom} g - \{\xi_{i_0}\}}$. As elements of F, f, g are incompatible.

By the fact that $f(\xi_{i_0}) = g(\xi_{i_0})$, f', g' are incompatible. Hence F' consists of pairwise incompatible functions. Notice that if $f' \in F'$ then $\overline{\overline{\operatorname{dom} f'}} = n$. Thus, by the inductive assumption, F' is of power less than $\varkappa$. But $\bar{F}' = \bar{F}_{i_0}^{t_0} = \varkappa$. Contradiction.

Thu we have proved that if $F \subseteq \bigotimes_{\xi<\varkappa} \xi^{<\omega}$ consists of pairwise incompatible functions of power n then F is of power less than $\varkappa$. Hence, by the regularity of $\varkappa$, $\bigotimes_{\xi<\varkappa} \xi^{<\omega}$ is $\varkappa$.c.c. ∎

Now we carry out a construction similar to the construction of T. Let ξ be a countable ordinal. Let us recall the definition of T_ξ from Fact 5.

Let $\psi(\alpha, y)$ be the Σ_1^1 formula such that whenever y is a well-ordering then

$$\psi(\alpha, y) \equiv (n)\big(F_{\overline{[y]_n}}(\alpha) \text{ is a real}\big) \Rightarrow (\mathrm{E}y')\, R\big(\alpha, F_{\overline{[y]_n}}(\alpha), y'\big).$$

Let $T \subseteq \omega^{<\omega} \times 2^{<\omega} \times \omega^{<\omega}$ be such that

$$\psi(\alpha, y) \equiv (\mathrm{E}z)(\langle \alpha, y, z\rangle \text{ is a branch of } T).$$

Let T_ξ be defined as

$$\langle s, t, u\rangle \in T_\xi$$

iff

(1) $s \in \omega^{<\omega}$, $t \in \xi^{<\omega}$, $(u)_0 \in 2^{<\omega}$, $(u)_1 \in \omega^{<\omega}$,

$$\operatorname{dom} s = \operatorname{dom} t = \operatorname{dom} u,$$

(2) $\langle s, (u)_0, (u)_1\rangle \in T$,

(3) $(u)_0(\langle m, n\rangle) = 0 \equiv t(m) < t(n)$,

(4) $(\mathrm{E}\alpha, f, y)_{\omega^\omega \times \xi^\omega \times \omega^\omega}(\langle s, t, u\rangle \subseteq \langle \alpha, f, y\rangle$

$$\& \langle \alpha, (y)_0 (y)_1\rangle \text{ is a branch of } T \,\&\, f: \omega \xrightarrow{\text{onto}} \xi$$

$$\& (m, n)\big((y)_0(\langle m, n\rangle) = 0 \equiv f(m) < f(n)\big).$$

Then, if $\varphi(\alpha, \xi)$ means

$$(\eta)_\xi\big(F_\eta(\alpha) \text{ is a real} \Rightarrow (\mathrm{E}y')\, R(\alpha, F_\eta(\alpha), y')\big),$$

then

$$\varphi(\alpha, \xi) \equiv (\mathrm{E}f, g)(\langle \alpha, f, g\rangle \text{ is a branch of } T_\xi \,\&\, f: \omega \xrightarrow{\text{onto}} \xi)$$

provided that ξ is countable.

Later let $T_\xi(\langle s, t, u\rangle)$ be the formula defining T_ξ. Let T_ξ^M be the relativization of this formula to a class M. Consider the following remark:

REMARK 7. *If $\mathbf{B}_1$, $\mathbf{B}_2$ are algebras such that $V^{\mathbf{B}_i} \models \xi$ is countable, then*

$$V^{\mathbf{B}_1} \models T_\xi(\langle s, t, u\rangle) \equiv V^{\mathbf{B}_2} \models T_\xi(\langle s, t, u\rangle).$$

Proof. Let $\langle s, t, u\rangle$ be given. Assume that $V^{\mathbf{B}_1} \models T_\xi(\langle s, t, u\rangle)$. Then, in $V^{\mathbf{B}_2}$, $\langle s, t, u\rangle$ satisfies conditions (1)-(3) because they are absolute. Let us show that (4). Work in $V^{\mathbf{B}_1 \times \mathbf{B}_2}$. Then $V^{\mathbf{B}_1} \subseteq V^{\mathbf{B}_1 \times \mathbf{B}_2}$ thus in $V^{\mathbf{B}_1 \times \mathbf{B}_2}$, $\langle s, t, u\rangle$ satisfies (4); Notice that, in $V^{\mathbf{B}_1 \times \mathbf{B}_2}$, (4) is a Σ_1 sentence with a countable parameter ξ. Thus, by Levy's lemma, (4) is satisfied in $V^{\mathbf{B}_2}$ because ξ is countable in $V^{\mathbf{B}_2}$. ∎

Let $\xi \in \varkappa$. Let $\mathbf{B}_\xi$ be the usual algebra collapsing ξ onto ω. Let us define T as follows:

let $p \in T$ iff

$$p = \langle s, \{\langle \xi_0, \langle t_{\xi_0} v_{\xi_0}\rangle\rangle, \ldots, \langle \xi_n, \langle t_{\xi_n} v_{\xi_n}\rangle\rangle\}\rangle$$

and

(1) $s \in \omega^{<\omega}$, $t_{\xi_i} \in \xi_i^{<\omega}$, $v_{\xi_i} \in \omega^{<\omega}$, $\xi_i \in \varkappa$,

$$\operatorname{dom} s = \operatorname{dom} t_{\xi_i} = \operatorname{dom} v_{\xi_i},$$

(2) $V^{\mathbf{B}_{\xi_i}} \models T_{\xi_i}(\langle s, t_{\xi_i} v_{\xi_i}\rangle)$,

(3) $(\mathrm{E}\alpha)(A(\alpha)$

$$\& V^{\mathbf{B}_{\xi_i}} \models (\mathrm{E}f_0, g_0 a, \ldots, f_n, g_n)(\langle s, t_{\xi_i}, v_{\xi_i}\rangle) \subseteq \langle \check{\alpha}, f_i, g_i\rangle$$

$$\& \langle \check{\alpha}, f_i, g_i\rangle \text{ is a branch of } T_{\check{\xi}_i} \,\&\, f_i: \omega \xrightarrow{\text{onto}} \check{\xi}_i).$$

Let $\operatorname{dom} p$, $(p)_0$, $(p)_1$, $p(\xi)$ be defined as before. Let $p \leqslant q$ if

$$\operatorname{dom} p \supseteq \operatorname{dom} q \,\&\, (p)_0 \supseteq (q)_0 \,\&\, (\xi)_{\operatorname{dom} q}\, p(\xi) \supseteq q(\xi).$$

DEFINITION 6. Let $p \in T$, $p = \langle s, \{\langle \xi_0, \langle t_{\xi_0}, v_{\xi_0}\rangle\rangle, \ldots, \langle \xi_n, \langle t_{\xi_n}, v_{\xi_n}\rangle\rangle\}\rangle$. Let $\alpha \in \omega^\omega$. Let us say that α goes through p if

$$V^{\boldsymbol{B}_{\xi_i}} \models (\mathrm{E} f_0, g_0, \ldots, f_n, g_n)(\langle s, \widetilde{t_{\xi_i}, v_{\xi_i}}\rangle \subseteq \langle \check{\alpha}, f_i, g_i\rangle$$
$$\& \langle \check{\alpha}, f_i, g_i\rangle \text{ is a branch of } T_{\xi_i} \& f_i\colon \omega \xrightarrow{\text{onto}} \xi_i).$$

Notice that through every element of T goes an α such that $A(\alpha)$.

The next facts show that T is non-empty if A is non-empty.

The idea of including in T those p through which "goes a full branch" $\langle \alpha, f_0, g_0, \ldots, f_n, g_n\rangle$ not actually in the universe but in a homogeneous boolean extension of the universe is an adaptation of an idea of Mansfield from [5]. Also the proof of Fact 11 and the use of Remark 7 in the proof of Fact 10 have much in common with Mansfield [5]. However, the derivation of the existence of a perfect subset of A in a certain situation, which is the content of the proofs of Fact 13 and also of Fact 8, Fact 9 of §2 and Remark 1 of §0, have more in common with Solovay's ideas from [7] than with Mansfield [5].

All these technical devices which come from other papers are applied to our very particular set of forcing conditions, which is characteristic only for the present paper.

FACT 10. *Let $A \in \underset{\sim}{\Pi}^1_2$. Let $\alpha \in A$. Then*

$$V^{\boldsymbol{B}_\xi} \models (\mathrm{E} f, g)(f\colon \omega \xrightarrow{\text{onto}} \check{\xi} \;\&\; \langle \check{\alpha}, f, g\rangle \text{ is a branch of } T_{\check{\xi}}).$$

Proof. We have $V^{\boldsymbol{B}_\xi} \models A(\check{\alpha})$ because of the absoluteness of A. Thus

$$V^{\boldsymbol{B}_\xi} \models (\varphi(\check{\alpha}, \check{\xi}) \;\&\; \check{\xi} \simeq \omega).$$

Hence follows the required conclusion.

FACT 11. *Let $p \in T$, $\operatorname{dom} p = \{\xi_0, \ldots, \xi_k\}$, $\xi \in \varkappa$, $n \in \omega$, $\eta \in \xi$, $\eta_k \in \xi_k$, $\operatorname{dom} p \subseteq \xi$. Then there is a $q \leqslant p$ such that $\xi \in \operatorname{dom} q$, $\eta \in \operatorname{rg}(q(\xi))_0$, $n \in \operatorname{dom}(q)_0$, $\eta_k \in \operatorname{rg}(q(\xi_k))_0$.*

Proof. Let α be such that $A(\alpha)$ and α goes through p. Work in $V^{\boldsymbol{B}_\xi}$. Assume that

$$p = \langle s, \{\langle \xi_0, \langle t_{\xi_0}, v_{\xi_0}\rangle\rangle, \ldots, \langle \xi_n, \langle t_{\xi_n}, v_{\xi_n}\rangle\rangle\}\rangle.$$

Let $f_0, g_0, \ldots, f_n, g_n$ be such that $p(\xi_i) \subseteq \langle f_i, g_i\rangle \;\&\; \langle \check{\alpha}, f_i, g_i\rangle$ is a branch of $T_{\check{\xi}_i} \;\&\; f_i\colon \omega \xrightarrow{\text{onto}} \check{\xi}_i$. There are such $f_0, g_0, \ldots, f_n, g_n$ because $V^{\boldsymbol{B}_{\xi_i}} \subseteq V^{\boldsymbol{B}_\xi}$. Let f, g be such that $\langle \check{\alpha}, f, g\rangle$ is a branch of $T_{\check{\xi}}$ and $f\colon \omega \xrightarrow{\text{onto}} \check{\xi}$. There are such f, g by Fact 10. Let m be such that $\eta \in \operatorname{rg} f_{\restriction m}$, $m > n$, $\eta_k \in \operatorname{rg} f_{k \restriction m}$. Let $q = \langle \check{\alpha}_{\restriction m}, \{\langle \check{\xi}_0, \langle f_{0\restriction m}, g_{0\restriction m}\rangle\rangle, \ldots, \langle \check{\xi}_n, \langle f_{n\restriction m}, g_{n\restriction m}\rangle\rangle, \langle \check{\xi}, \langle f_{\restriction m}, g_{\restriction m}\rangle\rangle\}\rangle$.

Then in $V^{\boldsymbol{B}_\xi}$, q satisfies the conditions (1)-(2) of the definition of T, and $\check{\alpha}$ goes through q. By the fact that q is in $V^{\boldsymbol{B}_\xi}$ an element of $\omega^{<\omega} \times \bigotimes_{\zeta \leqslant \xi} (\zeta^{<\omega} \times \omega^{<\omega})$, there is an element q of $\omega^{<\omega} \times \bigotimes_{\zeta < \xi} (\zeta^{<\omega} \times \omega^{<\omega})$ such that $\|q = \check{q}\|_{\boldsymbol{B}_\xi} \neq \emptyset$. Thus $\|\check{q}$ satisfies (1), (2) and $\check{\alpha}$ goes through $\check{q}\|_{\boldsymbol{B}_\xi} \neq \emptyset$.

By the homogeneity of $\boldsymbol{B}_\xi$, this value is $\mathbf{1}$. Thus q satisfies (1), (2) and α goes through q. Hence $q \in T$, $q \leqslant p$. Thus q is as required. ∎

FACT 12. *T is $\varkappa$.c.c.*

Proof. We simply repeat the proof of Fact 7 with $\varkappa$ in place of ω_1 and the notion of "going through a condition" determined by Definition 2. We use the regularity of $\varkappa$ and Lemma 1.

FACT 13. *Either A is of power less than $\varkappa$ or there is a subset $\boldsymbol{P}$ of T such that $\boldsymbol{P}$ is non-empty, $\varkappa$.c.c. and splits at the first coordinate.*

Proof. We repeat the proof of Fact 8 with $\varkappa$ in place of ω_1 and with the notion of "going through a condition" determined by Definition 6.

Then, if A is not of power less than $\varkappa$, $\boldsymbol{P}$ has the following properties:

(1) if $p \in \boldsymbol{P}$, $\operatorname{dom} p = \{\xi_0, \ldots, \xi_n\}$, $\xi \in \varkappa$, $n \in \omega$, $\eta \in \xi$, $\eta_i \in \xi_i$, $\operatorname{dom} p \subseteq \xi$, then there is a $q \leqslant p$, $q \in \boldsymbol{P}$ such that

$$\xi \in \operatorname{dom} q, \quad \eta \in \operatorname{rg}(q(\xi))_0, \quad n \in \operatorname{dom}(q)_0, \quad \eta_i \in \operatorname{rg}(q(\xi_i))_0.$$

(2) if $p \in \boldsymbol{P}$, then there are q_1, q_2 in $\boldsymbol{P}$ such that $(q_1)_0$ is incompatible with $(q_2)_0$ and $q_i \leqslant p$.

(3) $\boldsymbol{P}$ is $\varkappa$.c.c.

We prove (1), (2) as Fact 11, and (3) as Fact 12 with

$$A' = \{\alpha \in A\colon (q)_T(\alpha \text{ goes through } q \Rightarrow q \in \boldsymbol{P}) \text{ in place of } A\}.$$

Now let us derive the conclusion of the theorem from Remark 2, and Facts 12, 13.

Assume that A is not of power less than $\varkappa$. Let $\boldsymbol{C}$ enumerate all dense subsets of $\boldsymbol{P}$. Work $V^{\boldsymbol{C}}$. Let G be generic over $\boldsymbol{P}$. Define

$$\alpha = \bigcup \{s\colon \langle s\rangle \in G\},$$
$$f_\xi = \bigcup \Big\{t_\xi\colon (\mathrm{E} s, v_\xi)\Big(\langle s, \{\langle \xi, \langle t_\xi, v_\xi\rangle\rangle\}\rangle \in G\Big)\Big\},$$
$$g_\xi = \bigcup \Big\{v_\xi\colon (\mathrm{E} s, t_\xi)\Big(\langle s, \{\langle \xi, \langle t_\xi, v_\xi\rangle\rangle\}\rangle \in G\Big)\Big\}.$$

Then, by the property (1) of $\boldsymbol{P}$ in Fact 12,

(1) $\{\xi\colon f_\xi \neq \emptyset\}$ is cofinal in $\varkappa$,

(2) $f_\xi \neq \emptyset \Rightarrow f_\xi\colon \omega \xrightarrow{\text{onto}} \xi$,

(3) $f_\xi \neq \emptyset \Rightarrow \langle \alpha, f_\xi, g_\xi\rangle$ is a branch of T_ξ.

Indeed, (1)-(2) are immediate. Let us show (3). We have: ξ is countable in V^C. Hence, by Remark 7,

$$V^C \models \langle s, \widetilde{t, u} \rangle \in T_\xi \equiv V^{B_\xi} \models \langle s, \widetilde{t, u} \rangle \in T_\xi.$$

But $(n)\big(V^{B_\xi} \models (\langle \overbrace{\alpha_{\restriction n}, f_{\xi \restriction n}}, g_{\xi \restriction n} \rangle \in T_\xi)\big)$ by the definition of T. Thus

$$(n)\big(V^C \models (\langle \alpha_{\restriction n}, f_{\xi \restriction n}, g_{\xi \restriction n} \rangle \in T_\xi)\big).$$

Hence $V^C \models \langle \alpha, f_\xi, g_\xi \rangle$ is a branch of T_ξ. By (1), (2), (3), there is a set B cofinal in $\varkappa$ and such that $(\xi)_B \varphi(\alpha, \xi)$, i.e.

$$(\xi)_B (\eta)_\xi (F_\eta(\alpha) \text{ is a real } \Rightarrow (Ey')R(\alpha, F_\eta(\alpha), y')).$$

Moreover, by the fact that P is $\varkappa$.c.c. and by the regularity of $\varkappa$, $\varkappa = \omega_1^{L(\alpha)}$. Thus we can infer $A(\alpha)$. Now, as in Fact 9, we show that there is a perfect subset of A. ∎

As an immediate corollary we obtain the following theorem.

THEOREM 4. *Assume that there is an ordinal $\varkappa$ such that $\varkappa$ is regular and there is no function from the continuum onto $\varkappa$. Then every $\underset{\sim}{\Pi}_2^1$ set either is well orderable and of power less that $\varkappa$ or has a perfect subset in a boolean extension of the universe.*

On the basis of §0 we obtain

THEOREM 5. *If (x) ($x^\#$ exists) and $0^\dagger$ does not exist and there is a regular cardinal $\varkappa$ such that there is no function from the continuum onto $\varkappa$, then every $\underset{\sim}{\Pi}_2^1$ is well-orderable and of power less than $\varkappa$ or has a perfect subset, i.e. is of power $\mathfrak{C}$.*

Also we have

THEOREM 6. *Let M be $\underset{\sim}{\Sigma}_3^1$-correct in any M^C and in V. Let A be $\Pi_2^1(M)$. Assume there is no function from A onto ω_1 in M. Then $A \subseteq M$ and A is countable in M.*

Proof. Define T for A inside M as in §1. Assume $A \nsubseteq M$. We shall show that, in M, $P \neq \emptyset$ where P is in M the intersection of derivators of T such as in Fact 8. Suppose that $P = \emptyset$. Then, by Fact 8, A is countable in M. Hence in M there is a function f from ω onto A^M. We have in the world $(E\alpha)\big(A(\alpha) \,\&\, (n)(\alpha \neq f(n))\big)$. By the $\underset{\sim}{\Sigma}_3^1$-correctness of M, there is such an α in M. Contradiction. Thus $P \neq \emptyset$ in M and hence A has a perfect subset in M^C for a collapsing algebra C. But, by the $\underset{\sim}{\Sigma}_3^1$-correctness of M, A has a perfect subset in M and thus there is in M a function from A onto ω_1. Contradiction. Thus $P = \emptyset$ in M and hence A is countable in M. ∎

THEOREM 7. *Assume that (x) ($x^\#$ exists) and $0^\dagger$ does not exist. Then, for every $\underset{\sim}{\Pi}_2^1$ set A either A is countable or there is a function from A onto ω_1.*

Proof. Indeed, under the above assumptions, V is $\underset{\sim}{\Sigma}_3^1$-correct in V^C (see Remark 4) where C enumerates $\mathscr{P}(\omega_1)$ with natural numbers. Suppose that there is no function from A onto ω_1. Then, by Theorem 2, A is countable or has a perfect subset in V^C. Thus A is countable or has a perfect subset. But if A has a perfect subset, then there is a function from A onto ω_1. So A is countable. ∎

REMARK 8. *Compare Theorem 8 with the $\underset{\sim}{\Pi}_1^1$ case. We have: every $\underset{\sim}{\Pi}_1^1$ set A is countable or there is a function from A onto ω_1 (decomposition into constituents or, if A is borel, the function from a perfect subset of A onto ω_1).*

THEOREM 8. *If A is a Π_2^1 set and it is consistent that there is a regular ordinal $\varkappa$ such that there is no function from A onto $\varkappa$ and A is not well-orderable, then it is consistent that A has a perfect subset (we identify A with its fixed Π_2^1 definition).*

This is a consistency version of Theorem 4.

References

[1] Z. Adamowicz, *A generalization of the Schoenfield theorem on sets*, to appear in Fund. Math.

[2] — *Continuous relations and generalized G_δ sets*, to appear.

[3] J. W. Addison, *Some consequences of the axiom of constructibility*, Fund. Math. 46 (1959), pp. 337–357.

[4] R. B. Jensen, *Σ_3^1 absoluteness w.r.t. the core model*, unpublished.

[5] R. Mansfield, *Perfect subsets of definable sets of real numbers*, Pacific J. Math. 35 (1970), pp. 451–457.

[6] — *A Souslin operation for Π_2^1*, Israel J. Math. 9 (1971), pp. 367–379.

[7] R. Solovay, *A model of set theory in which every set of reals is Lebesgue measurable*, Ann. of Math. 92 (1970), pp. 1–57.

[8] — *Cardinality of Σ_2^1 Sets of Reals*, Foundations of Mathematics, Springer, Berlin 1969.

[9] J. R. Shoenfield, *The Problem of Predicativity*, Essays on the Foundations of Mathematics, Magnes Press, Hebrew University, Jerusalem 1961.

[10] P. Zbierski, *Models for higher order arithmetics*, Bull. Acad. Polon. Sci. 19 (1971), pp. 557–562.

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