Compact discrete flows

by

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Abstract. In this paper we obtain properties for certain homeomorphisms of locally compact Hausdorff spaces. Compact discrete flows are characterized in terms of the bilateral stability properties of the orbits and stability theorems for cycles are formulated.

1. Introduction. The structure of compact continuous flows on dichotomic 2-manifolds and locally compact Hausdorff spaces are analyzed in terms of the bilateral stability properties of the orbits in [6] and [9]. Our aim here is to analyze compact discrete flows in a similar manner. In order to accomplish this task we develop a cycle stability theorem for discrete flows on locally compact Hausdorff phase spaces. Of course, the structural analysis of compact flows applies to locally compact subsets of the periodic set of an arbitrary discrete flow on a locally compact phase space and the cycle stability theorem indicates the structure of the phase space near periodic orbits.

Throughout the remainder of the paper we shall let (X, π) be a given discrete flow on a locally compact Hausdorff phase space X. The periodic set and the critical set are denoted by P and S, respectively. The extension of (X, π) to the one point compactification X^* of X is denoted by (X^*, π^*). The extension of a function F on X is denoted by F^* and the extended critical set is denoted by S^*.

The integers, nonnegative integers, and nonpositive integers are denoted respectively by Z, Z^+, and Z^-. We let M^0 and ∂M denote the respective interior and boundary sets of a set M. For convenience we let $[a, b] = \{t \in Z\colon a \leqslant t \leqslant b\}$.

The *positive orbit* through a point x denoted by $C^+(x)$ is xZ^+ and the *positive orbit closure* is denoted by $K^+(x)$. The *ω-limit set* $\bigcap\{K^+(xt)\colon t \in Z^+\}$ of a point x is denoted by $L^+(x)$. For a point x the *positive prolongation* is

$D^+(x) = \bigcap \{\overline{C^+(V)}:\ V \in \eta(x)\}$ where $\eta(x)$ is the neighborhood filter of x, or equivalently, $D^+(x) = \{y:\ x_i t_i \to y$ for some net $x_i \to x$ and $t_i \in Z^+\}$. The *positive prolongational limit set* of x is $J^+(x) = \bigcap \{D^+(xt):\ t \in Z^+\}$, or equivalently, $J^+(x) = \{y:\ x_i t_i \to y$ for some net $x_i \to x$, $t_i \in Z^+$, and $t_i \to +\infty\}$. A point x is (*positively*) *weakly attracted* to a set M if and only if the net (xt), $t \in Z^+$, is frequently in every neighborhood of M. The set of all such points x is called the *region of* (*positive*) *weak attraction* of M and is denoted by $A_w^+(M)$. If $A_w^+(M)$ is a neighborhood of M, then M is called a (*positive*) *weak attractor*. A set M is called (*positively*) *stable* if for each neighborhood U of M there exists a neighborhood V of M such that $V = C^+(V) \subset U$. Whenever a set M is both positively stable and a positive weak attractor then it is called (*positively*) *asymptotically stable*. The negative and bilateral versions of each concept above is defined and denoted in the obvious manner. For general references to the concepts above the reader is referred to [1], [3], [5], and [7].

A continuous or discrete flow is called *compact* (*closed*) if each of its orbits is compact (closed). We note that the compact orbits in this paper are either periodic or critical [5, 3.09]. A flow is said to be of *characteristic* 0 *on a set* M if $D(x) = K(x)$ for each x in M. A flow is of *characteristic* 0 whenever it is of characteristic 0 on the phase space. The unilateral versions of this notion are defined similarly and carry the appropriate superscript.

2. Cycle stability. The major results of this section depend upon four key lemmas which we present first. The proof of Lemma 1 is almost immediate if the periodic orbit conjecture holds for discrete flows but this is evidently impossible in view of the counterexample given by Sullivan [11] for continuous flows.

LEMMA 1. *A compact discrete flow without critical points on a locally compact phase space is of characteristic* 0.

Proof. Let (X, π) be compact and $X = P$. We proceed by contradiction showing that $D(x) = C(x)$ for each x in P. For some $x \in P$ let $y \in D(x) - C(x)$. Select an open neighborhood V of $C(x)$ with compact closure $\bar{V}$ contained in $X - C(y)$. Let $A = \{z:\ C(z)$ meets V and $X - \bar{V}\}$ and $B = X - A$. We have A open, B closed, and $C(x) \cup C(y) \subset B$. For any point $z \in A$ define $t_z = \min\{t \in Z^+ :\ zt \in V$ but $z[0, t] \not\subset \bar{V}\}$. Evidently, $t_z < +\infty$. Also for a compact set $M \subset \bar{V} \cap A$ define $T_M = \sup\{t_z:\ z \in M\}$. For any z in M we have by the continuity of π an open neighborhood W_z of z such that $t_p \leqslant t_z$ for each $p \in W_z$. There is a finite subcover $\{W_{z_1}, \ldots, W_{z_k}\}$ of M. Thus, $t_z \leqslant \max\{t_{z_i}:\ 1 \leqslant i \leqslant k\}$, and hence, $T_M < +\infty$. Next, let (x_i) be a net in V converging to x and let (r_i) be a net in Z^+ with each r_i less than the period of x_i such that $x_i r_i \to y$. If (r_i) is bounded by some positive T, then y is in $\bar{V}[0, T]$ so that $C(y) \cap \bar{V} \neq \emptyset$. This is impossible so we have (r_i) unbounded. The nets (x_i) and (r_i) can be chosen so that $(x_i r_i)$ is in $X - \bar{V}$. Let $t_i = \max\{t:\ x_i t \in \bar{V}$ and $0 \leqslant t < r_i\}$. Some subnet $(x_{n_i} t_{n_i})$ of $(x_i t_i)$ converges to a point z in $\bar{V}$. Suppose that $z \in A$. Let G be a compact neighborhood of z contained in A and let $M = G \cap \bar{V}$. Then $x_{n_i} r_{n_i} = (x_{n_i} t_{n_i})(r_{n_i} - t_{n_i})$ is ultimately in $M[0, T_M)$. But this means that $C(y) \cap \bar{V} \neq \emptyset$ which is absurd. Thus, $z \in B$ rather than A. If $z \notin C(x)$, then for some $t > 0$ select a neighborhood W of zt with compact closure such that $z \notin \bar{W}$ and $C(y) \cup C(x) \subset X - \bar{W}$. If each compact subneighborhood of W contains a complete orbit, then there would be a net of periodic orbits $(C(y_i))$ in W such that $y_i \to zt$. This means $(y_i(-t))$ converges in W but $y_i(-t) \to z$. Thus, we can select $\bar{W}$ containing no complete orbit. Either $C(z) \subset \bar{V}$ or $C(z) \subset X - V$ with $C(z) \not\subset \partial V$. First, suppose that $C(z) \subset \bar{V}$ and let $V_0 = V - \bar{W}$. Using the notations employed for V we have $z \in A_0$. As before this is not possible. Next, whenever $C(z) \subset X - V$ let V_0 be the interior of $\bar{V}[0, T]$ where T is the fundamental period of z. If for some subnet (t_{n_j}) of (t_{n_i}) we have $r_{n_j} \leqslant t_{n_j} + T$, then $y \in \bar{V}[0, T]$. Hence, eventually $t_{n_i} + T < r_{n_i}$ with $x_{n_i}(t_{n_i} + T) \to zT = z$ and $t_{n_i} + T = \max\{t:\ x_{n_i} t \in \bar{V}_0$ and $0 \leqslant t < r_{n_i}\}$. As before we have $C(z) \subset \bar{V}_0$ which was shown to be impossible. We have now shown that $z \notin \bar{V} - C(x)$. Thus, z must be in the orbit $C(x)$. Let t be the least nonnegative integer such that $xt \neq z$. Following the proceedures above select a neighborhood W of xt with compact closure such that $z \notin \bar{W}$, $C(y) \subset X - \bar{W}$, and $\bar{W}$ contains no complete orbit. By using the reasoning outlined above for $V_0 = V - \bar{W}$ in place of V we can show that $C(y)$ must meet $\bar{V}_0$ which again is impossible. Whence, we conclude that $D(x) = C(x)$. The proof is complete.

LEMMA 2. *Each orbit in a compact discrete flow without critical points on a locally compact phase space is bilaterally stable.*

Proof. Let (X, π) be compact and let $X = P$. Suppose that there is a compact neighborhood V of some periodic orbit $C(x)$ containing no invariant subneighborhood of $C(x)$. Then there are nets (x_i) in V^0 and (t_i) in Z^+ such that $(x_i t_i)$ is in $X - V$. In view of Lemma 1, no subnet of $(x_i t_i)$ can converge in $X - V$. Hence, some subnet of $(x_i t_i)$ converges to ∞ in (X^*, π^*). The argument used in Lemma 1 can be used here letting $y = \infty$ to obtain the contradiction $\infty \notin D^*(x)$. Consequently, $C(x)$ is bilaterally stable in (X^*, π^*) and hence in (X, π).

LEMMA 3. *Let $C(x)$ be a periodic orbit. Suppose that a neighborhood V of $C(x)$ exists which contains no complete orbit except $C(x)$ and suppose that there do not exist points $y, z \notin C(x)$ such that $L^+(y) = L^-(z) = C(x)$. Then $C(x)$ is either asymptotically stable or negatively asymptotically stable.*

Proof. We can select V open with compact closure. Let y be any element of $X - C(x)$ such that $K^+(y) \subset V$. Then $L^+(y)$ is a nonempty invariant subset of V, and hence, $L^+(y) = C(x)$. We have either $K^+(y) \cap (X - V) \neq \emptyset$ or $L^+(y) = C(x)$. Similarly, the negative version holds. Thus, either $L^+(z) \neq C(x)$ for any $z \in X - C(x)$ or $L^+(z) = C(x)$ for some $z \in X - C(x)$ so that either $K^+(z) \cap (X - V) \neq \emptyset$ for every $z \in X - C(x)$ or $K^-(z) \cap (X - V) \neq \emptyset$ for every $z \in X - C(x)$.

First we consider the case $K^+(z) \cap (X - V) \neq \emptyset$ for each $z \in X - C(x)$ and show that $C(x)$ is negatively asymptotically stable. Now $X - V$ is a weak attractor with $A_w^+(X - V) = X - C(x)$. To show that $C(x)$ is negatively stable we consider the extended flow (X^*, π^*). Suppose that $C(x)$ is not negatively stable. Then V contains no negatively invariant subneighborhood of $C(x)$. There is a net (x_i) converging to x and a net (t_i) in Z^- such that $(x_i t_i)$ converges to some point y in $X^* - V$. Thus, $y \in D^-(x)$ and hence $x \in D^+(y)$. Since V contains only one complete orbit $C(x)$, we can select V so that $C(x)$ is the only complete orbit in $\bar{V}$ and in that case we can select the nets so that $(x_i t_i)$ is in $X^* - V$. For each $p \notin C(x)$ define $t_p = \min\{t \geqslant 1\colon pt \in X^* - \bar{V}\}$. Evidently, $t_p < +\infty$. Define $T = \sup\{t_p\colon p \in X^* - V\}$. Every positive orbit starting in $X^* - V$ meets $X^* - \bar{V}$. By the continuity of π there is a neighborhood V_p of $p \in X^* - V$ such that $t_q \leqslant t_p$ for each $q \in V_p$. There is a finite subcover $\{V_{p_1}, \ldots, V_{p_k}\}$ of the compact set $X^* - V$. Thus, $t_p \leqslant \max\{t_{p_1}, \ldots, t_{p_k}\}$, and hence, $T < +\infty$.

Next, since $x \in D^+(y)$ select nets (y_i) in $X^* - V$ converging to y and (t_i) in Z^+ such that $(y_i t_i)$ is in V and $y_i t_i \to x$. Let (r_i) be the net in Z^+ defined by $r_i = \max\{t \in Z\colon y_i t \in X^* - V \text{ and } 0 \leqslant t < t_i\}$ for each i. Since $X^* - V$ is compact we can choose (y_i) and (t_i) so that $(y_i r_i)$ converges to some point p in $X^* - V$. Since $0 < t_i - r_i < t_{y_i r_i}$ we conclude that $y_i t_i = (y_i r_i)(t_i - r_i)$ converges in $(X^* - V)[0, T]$. But this means that $x \in (X^* - V)[0, T]$ contradicting $C(X^* - V) = X^* - C(x)$. Whence, $C(x)$ is negatively stable.

Finally, we show that $C(x)$ is a negative attractor. Let W be a negatively invariant subneighborhood of $C(x)$ in V. Since $C^-(y) \subset V$ for each $y \in W$ we have $L^-(y) = C(x)$. Hence, $C(x) \subset W \subset A_w^-(C(x))$. We have shown that $C(x)$ is negatively asymptotically stable.

If the second case $K^-(z) \cap (X - V) \neq \emptyset$ for each $z \in X - C(x)$ holds, then a similar argument yields $C(x)$ asymptotically stable. The proof is complete.

Lemma 4. *A periodic orbit $C(x)$ is stable (negatively stable, bilaterally stable) provided $D^+(x) = C(x)$ $(D^-(x) = C(x), D(x) = C(x))$.*

Proof. Let $D^+(x) = C(x)$ and suppose that $C(x)$ is not stable. Using the reasoning of the Lemma 3 proof we can find a net $x_i \to x$ and (t_i) in Z^+ such that $(x_i t_i)$ converges to a point y of $X^* - V$ where V is a compact neighborhood of $C(x)$. But this means that $y \in D^{*+}(x)$ which is impossible. Thus, $C(x)$ is stable in X^* and, therefore, in X. The remainder of the proof follows similarly.

The next theorem is an extension of Ura's prolongational stability theorem for continuous flows [12] to discrete flows.

Theorem 5. *In a discrete flow on a locally compact space a compact set M is stable (negatively stable, bilaterally stable) if and only if $D^+(M) = M$ $(D^-(M) = M, D(M) = M)$.*

Proof. Let $D^+(M) = M$ for some compact set M and suppose that M is not stable. Then there is a compact neighborhood V of M containing no positively invariant subneighborhood of M and thus we can find a net (x_i) converging to x in M and a net (t_i) in Z^+ such that $(x_i t_i)$ is in $X - V$. By choosing t_i so that $x(t_i - 1) \in V$ we can select (x_i) and (t_i) so that $x_i t_i$ converges to a point y in $\overline{V1} - V^0$ Hence, we have $y \in D^+(M) \cap (X - V^0)$ which is absurd. Thus, M is stable.

Conversely, if $y \notin M$, then there is a compact invariant neighborhood of M excluding y. Hence, $y \notin D^+(M)$ and $D^+(M) = M$.

The following version of Ura's alternatives for discrete flows follows from Lemma 3.

Theorem 6. *In a discrete flow on a locally compact space a periodic orbit $C(x)$ has one of the following properties:*

(i) *$C(x)$ is asymptotically stable.*

(ii) *$C(x)$ is negatively asymptotically stable.*

(iii) *There exist points $y, z \notin C(x)$ such that $L^+(y) = L^-(z) = C(x)$.*

(iv) *Every neighborhood of $C(x)$ contains a complete orbit distinct from $C(x)$.*

The subsequent theorem is a partial extension of the Cycle Stability Theorem for continuous flows on dichotomic 2-manifolds. Considering the preceeding results the proof is now evident.

Theorem 7. *In a discrete flow on a locally compact space*

(i) *each orbit of P^0 is bilaterally stable,*

(ii) *a periodic boundary orbit $C(x)$ of P is not stable in any sense if there exist $y, z \notin C(x)$ such that $L^+(y) = L^-(z) = C(x)$, and*

(iii) *a periodic orbit $C(x)$ in the boundary of P is stable, negatively stable, or bilaterally stable provided the flow is of characteristic 0^+, 0^-, or 0 on $C(x)$.*

Corollary 7.1. *In a compact discrete flow on a locally compact space each periodic orbit is bilaterally stable.*

Corollary 7.2. *On a locally compact space a compact discrete flow is of characteristic 0 if and only if each critical point is bilaterally stable.*

Proposition 8. *A compact discrete flow on a locally compact space is of characteristic 0 if and only if it is of characteristic 0^+, 0^-, or $0^{\pm}$.*

Proof. A compact discrete flow on a locally compact phase space is of characteristic 0 if and only if its extended flow is of characteristic 0. The proof follows from Proposition 3 of [8].

3. Compact discrete flows. We turn our attention to characterizations of compact discrete flows. Compact components of the critical set of a compact continuous flow on a locally compact phase space are shown to be bilaterally stable in [9]. The following lemma gives our version of that property for discrete flows.

LEMMA 9. *On a locally compact space a discrete flow has a bilaterally stable critical set provided its boundary is compact.*

Proof. Each periodic orbit $C(x)$ is bilaterally stable and $D(C(x)) = C(x)$. Thus, $D(P) = P$ and so $D(S) = S$. By Theorem 5 the critical set is stable.

The subsequent two theorems are extensions of characterization Theorems 5 and 6 of [9].

THEOREM 10. *A discrete flow on a locally compact space is compact if and only if $S^*(\partial S^*)$ is bilaterally stable, each periodic orbit is bilaterally stable, and $L(X) = P \cup S$.*

Proof. If (X, π) is compact, then the three conditions follow from Theorem 7 and Lemma 9.

Conversely, we proceed by first showing that P is open. For any point y in P select a compact invariant neighborhood V of $C(y)$ disjoint from S^*. Then $K(z)$ is compact for any z in V and $\emptyset \neq L(z) \subset P$. If $z \notin P$, select a point q in $L(z)$ and a compact invariant neighborhood V_0 of $C(q)$ excluding z. This leads to the absurd condition $K(z) \cap C(q) = \emptyset$. Hence, P is open.

Finally, if x is in $X - P \cup S$, then $L^+(x) \subset S^*$ since P is open. Clearly this is impossible since a compact invariant neighborhood of S^* excluding x exists. Consequently, (X, π) is compact.

COROLLARY 10.1 *A closed discrete flow on a locally compact space is compact if and only if S^* and each periodic orbit are bilaterally stable.*

THEOREM 11. *A discrete flow on a locally compact space is compact if and only if S^* is bilaterally stable, each periodic orbit is bilaterally stable, and $P \cup S$ is a global attractor.*

Proof. The set $P \cup S$ is a global attractor if and only if $L^+(X) = P \cup S$. The proof follows from Theorem 10.

COROLLARY 11.1. *A discrete flow on a connected locally compact space is compact if and only if S^* is bilaterally stable, each periodic orbit is bilaterally stable, and $P \cup S$ is a closed attractor.*

A distal compact discrete flow need not be of characteristic 0. A discrete flow on a compact disc X consisting of an annular region of periodic orbits surrounding a closed disc of critical points is not of characteristic 0 on the boundary of the critical set. However, in view of Theorems 1 and 4.10 of [2] and [5] the flow is distal.

Similarly, such a flow need not be equicontinuous. For if so then the flow has proximal pairs the diagonal of $X \times X$ [3, 5.12] and has enveloping semigroup $E(X)$ a compact subgroup of X^X [3, 5.3]. That (X, π) is equicontinuous means that $E(X)$ is equicontinuous, and hence, uniformly equicontinuous. The orbit closure relation is a Hausdorff space [3; 4.4, 4.10] implying that (X, π) is of characteristic 0 [10, 2.4]. This contradiction leads us to the nonequicontinuity of (X, π) at least for compact discrete flows not of characteristic 0 on locally compact phase spaces.

Furthermore, such a flow need not be (locally) weakly almost periodic [5; 2.30 (3), 4.17, 4.24] or uniformly almost periodic [3, 4.4].

The following theorem shows that at least a locally compact periodic set enjoys these properties.

THEOREM 12. *Let (X, π) be a compact discrete (continuous) flow without critical points on a locally compact phase space. Then the flow is*

(a) *(locally) weakly almost periodic,*

(b) *distal,*

(c) *equicontinuous, and*

(d) *uniformly almost periodic.*

Moreover, the enveloping semigroup $E(X)$ of (X, π) is a topological group of homeomorphisms.

Proof. Property (a) is evident from [5; 2.31, 4.24] and Lemma 1. The flow induced on the product $X \times X$ by letting $(x, y)t = (xt, yt)$ for $t \in Z$ and $x, y \in X$ is easily seen to be periodic with no critical points. The extended product flow on $X^* \times X^*$ has one critical point (∞, ∞). In view of Lemma 1, both the product and extended product flows are of characteristic 0 and satisfy property (a). Also both product flows are pointwise almost periodic [5, 4.10], and hence, (X, π) and (X^*, π^*) are distal [3, 5.9]. Now X^* is compact, the extended product flow is weakly almost periodic at each point of the diagonal, and (X^*, π^*) is distal so that Z is equicontinuous relative to (X^*, π^*) [4, Lemma 1]. Thus, $E(X)$ is equicontinuous. According to [3; 4.4, 4.5], (X, π) is uniformly almost periodic and $E(X)$ is a group of homeomorphisms. The smallest closed invariant equivalence relation R such that $E(X^*/R)$ is almost periodic is the diagonal. Consequently, $E(X^*/R) \approx E(X^*)$ is a compact topological group [3, 4.19], and hence, $E(X)$ is a topological group. The proof is complete.

COROLLARY 12.1. *Let (X, π) be a compact discrete (continuous) flow on a locally compact space. Let R consist of $S^* \times S^*$ and the diagonal of $X^* \times X^*$. Then the flow induced on X^*/R is of characteristic 0 and each of the properties of the theorem hold.*

Whenever P is a locally compact subset of X the condition that each periodic orbit be bilaterally stable of Theorem 10 can be replaced by any one of the following: the flow on P is (locally) weakly almost periodic, $E(P)$ is equicontinuous, the flow on P is uniformly almost periodic, $E(P)$ is a group of homeomorphisms.

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Topological extensions and subspaces of η_α-sets

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Abstract. The η_α-sets of Hausdorff have large compactifications (of cardinality $\geqslant \exp(\alpha)$; and of cardinality $\geqslant \exp(\exp(2^{<\alpha}))$ in the Stone-Čech case). If Q_α denotes the unique (when it exists) η_α-set of cardinality α, then Q_α can be decomposed (= partitioned) into homeomorphs of any prescribed nonempty subspace; moreover the subspaces of Q_α can be characterized as those which are regular T_1, of cardinality and weight $\leqslant \alpha$, whose topologies are closed under $<\alpha$ intersections.

Let $\langle A, <\rangle$ be a linearly ordered set. If B and C are subsets of A, we use the notation $B < C$ to mean that $b < c$ for all $b \in B$, $c \in C$. If α is an infinite cardinal number, we say that $\langle A, <\rangle$ is an *η_α-set* if whenever $B, C \subseteq A$ have cardinality $<\alpha$ and $B < C$ then there is an element $a \in A$ with $B < \{a\} < C$. Such ordered sets, invented by Hausdorff [8] (see also [5, 6, 7]), are the forerunners and prototypical examples of saturated relational structures in model theory (see [5, 6]). Our interest in the present note centers on topological issues related to η_α-sets, considered as linearly ordered topological spaces (LOTS's) with the open interval topology.

Roughly stated, our results are these: (i) certain Hausdorff extensions of η_α-sets must have cardinality $\geqslant 2^\alpha$, and some (the compact C^*-extensions) must have cardinality $\geqslant \exp(\exp(2^{<\alpha}))$; (ii) the (unique when it exists; i.e., when $\alpha = \alpha^{<\alpha}$) η_α-set Q_α of cardinality α can be decomposed (= partitioned) into homeomorphs of any prescribed nonempty subspace; and (iii) the subspaces of Q_α are precisely the regular T_1 spaces, of cardinality and weight $\leqslant \alpha$, whose topologies are closed under $<\alpha$ intersections.

1. Preliminaries. We follow the convention that ordinal numbers are the sets of their predecessors and that cardinals are initial ordinals. If α is an infinite cardinal, α^+ denotes the cardinal successor of α ($\omega = \{0, 1, 2, \ldots\}$, $\omega_1 = \omega^+$, etc.) If A is a set, $|A|$ denotes the cardinality of A. If B is another set then ${}^BA$ is the set of all functions $f\colon B \to A$. For cardinals α, β, we let $\alpha^\beta = |{}^\beta\alpha|$ and $\alpha^{<\beta} = \operatorname{Sup}\{\alpha^\gamma\colon \gamma < \beta\}$. $\exp(\alpha)$ sometimes denotes 2^α, especially in interactions: $\exp^2(\alpha) = \exp(\exp(\alpha))$, etc. A useful application of König's Lemma is the following.