Nowhere monotone functions and a problem of K. M. Garg

by

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Abstract. In the paper "Properties of connected functions in terms of their levels" (Fund. Math. 97 (1977)) K. M. Garg asks if a continuous real valued function f defined on a locally connected seperable, complete metric space X (or on R^n) is nowhere monotone, does there exist a residual set of points $x \in X$ such that x is a limit point of the level $f^{-1}(f(x))$ along every simple arc in X that terminates with x? The purpose of this paper is to show that the answer to this question is negative if $X = R^2$.

1. Introduction. Let X and Y be two topological spaces and f a function mapping X into Y, then for every $y \in Y$, the set $f^{-1}(y) = \{x: f(x) = y\}$ is called a *level* set (or fiber) of f. The function f is said to be *monotone* if $f^{-1}(C)$ is connected for every connected subset C of Y (see Kuratowski [8], p. 131]), and f is *nowhere monotone* if f is monotone on no open subset of X, (see [2] and [4]). The function f is said to be *connected* if $f(C)$ is connected for every connected subset C of X. The study of monotone functions and of nowhere monotone functions has a considerable literature and the interested reader is referred to the bibliography at the end of [4] for a few of the appropriate works. In particular, in [2] and [4] Garg investigated nowhere monotone functions by considering properties of their level sets and in [4] he proves the following result.

THEOREM G. *Suppose that X is Hausdorff, second countable, and locally connected, and that f is connected and real valued. If f is also nowhere monotone, then there is a residual set of points x in X such that x is a limit point of the level $f^{-1}(f(x))$.*

Subsequently he asks ([4] p. 34, Problem 5.10): if a continuous real valued function f defined on a locally connected, separable, complete metric space X (or on R^n) is nowhere monotone, does there exist a residual set of points x in X such that x is a limit point of the level $f^{-1}(f(x))$ along every simple arc in X that has x as an endpoint?

The answer is known to be affirmative if $X = R^1$ (see [1], Theorem 2). In a private communication to Garg, Grande has shown that the completeness hypothesis is a necessary one (see [4] p. 36, Added in proof).Then in [5]

Grande constructed an appropriate metric space and subsequently showed that the general answer to Garg's question is in the negative. The purpose of this note is to show that the answer is negative, even for the plane, R^2.

THEOREM. *There exists a continuous, nowhere monotone real valued function f defined on R^2 such that for every point $x \in R^2$ there is an arc terminating at x along which x is not a limit point of $f^{-1}(f(x))$.*

2. The example. Before beginning the construction of the desired function f we make a preliminary construction which will be used inductively later on.

Let S be a square region having vertices v_1, v_2, v_3, and v_4 and let $\varepsilon > 0$ be given. Suppose a real valued function, f, has been defined on the boundary of S such that f has only two relative extrema on the boundary of S, and f is linear (but not constant) on each edge. We first define a process (referred to as process (*)) by which f can be extended to the boundaries of certain subsquares of S. Subsequently, we use process (*) in an inductive manner to extend f continuously to a dense set of horizontal and vertical lines, and finally we show that the resulting function has a continuous extension to the entire plane. First, however, process (*). To simplify notation and without losing essential generality, we assume the vertices of S' to be $v_1(0, 0)$, $v_2(1, 0)$, $v_3(0, 1)$, and $v_4(1, 1)$, that $m = \min\{f(v_i)\} = f(v_1)$, and let $M = \max\{f(v_i)\}$. The set of vertices for the sixteen nonoverlapping subsquares of S' on whose boundaries we wish to extend f is $\{(\frac{1}{4}m, \frac{1}{4}n)\colon m, n = 0, 1, \ldots, 4\}$. To insure that f is linear on the edges of these subsquares, we need only specify f at each of the vertices and extend linearly on edges. Since f is already defined on sixteen of these vertices (those on the boundary of S) we will be finished upon defining f on the remaining nine vertices. To this end, let $A = \{f(\frac{1}{4}m, \frac{1}{4}n)\colon \text{either } m \text{ or } n = 0 \text{ or } 4\}$ and let $[a, b]$ be an interval in $[\frac{1}{8}(3M+5m), \frac{1}{8}(5M+3m)]$ which misses A. Define f at the remaining nine vertices as follows:

1. $f(\frac{1}{4}, \frac{1}{4}) = a$,	4. $f(\frac{1}{2}, \frac{3}{4}) = \frac{1}{8}(a+7b)$,	7. $f(\frac{3}{4}, \frac{1}{4}) = \frac{1}{4}(3a+b)$,
2. $f(\frac{1}{4}, \frac{1}{2}) = \frac{1}{8}(5a+3b)$,	5. $f(\frac{3}{4}, \frac{3}{4}) = b$,	8. $f(\frac{3}{4}, \frac{1}{2}) = \frac{1}{8}(3a+5b)$,
3. $f(\frac{1}{4}, \frac{3}{4}) = \frac{1}{4}(a+3b)$,	6. $f(\frac{1}{2}, \frac{1}{4}) = \frac{1}{8}(7a+b)$,	9. $f(\frac{1}{2}, \frac{1}{2}) = M+\varepsilon$.

Now, as $[a, b]$ misses A it follows that if S' is one of the sixteen aforementioned subsquares of S, then f takes on distinct values at the vertices of S'. Further, as a exceeds $\frac{1}{4}(3m+M)$ it follows that $f(\frac{1}{4}, \frac{1}{4})$ is the maximum of f on the boundary of the subsquare whose vertices are $\{(\frac{1}{4}m, \frac{1}{4}n)\colon m, n = 0 \text{ or } 1\}$. This, and the facts that f is linear on edges and has only two relative extrema on the boundary of S allow us to make the following observation. In this observation B denotes the union of the boundaries of the sixteen subsquares.

OBSERVATION. *If $z \in B \setminus \{(0, 0)\}$ there is an arc $\alpha\colon [0, 1] \to B$ such that $\alpha(0) = (0, 0)$, $\alpha(1) = z$ and if $s < t$ then*

$$f(\alpha(s)) < f(\alpha(t)).$$

Now that this extension process is defined, we are ready to construct the desired function. The construction is accomplished by first inductively defining f on a grid G consisting of those horizontal and vertical lines whose coordinate intercepts are of the form $[2^k m + 2^{n-k}(m+1)]/2^n$ where m and n are integral and $k = 0, 1, 2, \ldots, n$. We then show that this f has a continuous extension to the entire plane. The induction proceeds as follows:

Let $G_0 = \{(x, y)\colon x \text{ or } y \text{ is integral}\}$ and for $(x, y) \in G_0$, define $f(x, y) = x+2y$. The grid G_0 divides the plane into enumerably many square regions of unit edge length and f has been defined in such a manner that it is linear on edges and it takes on distinct values at the vertices of any particular square.

Suppose now that at the nth stage of construction the plane has been divided into enumerably many nonoverlapping square regions of edge length $1/4^{n-1}$. Suppose further that for the boundary of any particular such square that f has been defined so that it is linear on edges, and has only two relative extrema. Divide each of these square regions into sixteen nonoverlapping congruent subsquares, and extend the definition of f on the boundaries using process (*) with $\varepsilon \equiv \varepsilon_{n+1} = 1/2^{n+3}$.

This completes the inductive portion of the construction, and we must now verify that f can be continuously extended to the entire plane.

Let $p \in R^2$ and let $\{p_n\colon n = 1, 2, \ldots\}$ converge to p with each p_n on the grid G defined earlier. In order to show that f has a continuous extension we must show that the sequence $\{f(p_n);\ n = 1, 2, \ldots\}$ is Cauchy. At each of the inductive steps described earlier, the plane is divided into enumerably many nonoverlapping square regions. If B is the boundary of one such square region of the nth stage, we let $d(B) = \max(f/B) - \min(f/B)$ and define $d_n = \sup\{d(B)\colon B \text{ is the boundary of a region of the } n\text{th stage}\}$. Note that $d_0 = 3$ and in general

$$d_{n+1} \leqslant 5d_n/8 + 2^{-n}. \tag{1}$$

Also, if S is a region of the nth stage, and $T \subset S$ is a region of the $(n+1)$st stage, then

$$f(T) \cap f(S) \neq \emptyset. \tag{2}$$

It follows from (1) and (2) that $\{f(p_n)\}$ is Cauchy and consequently that f has a continuous extension to the entire plane. Without ambiguity we let f also denote this continuous extension.

Nowhere monotonicity

Let Δ denote disc in the plane. For some n there is a square region S_n of the nth stage such that $S_n \subset \Delta$. If M and m respectively denote the maximum and minimum of f on the boundary of S_n, then there is a subsquare T of S_n which contains the center of S_n and whose entire boundary maps into the internal $[\frac{1}{8}(5m+3M), \frac{1}{8}(3m+5M)]$. However, $f^{-1}(M)$ intersects both the interior of T (because f is continuous and the center of S_n maps to $M+\varepsilon_n$) and the boundary of S_n. It follows that $f^{-1}(M)$ is disconnected by the boundary of T.

Arc accessibility

Suppose that $p \in G$. Then there is a smallest index $n(p)$ such that p is on the boundary, B, of a grid square of the $n(p)$th stage of construction. Any segment, α, which lies in B and terminates at p will have the property that $f^{-1}(f(p)) \cap \alpha = \{p\}$ because f is linear on S.

Now, let $p \in R^2 - G$ and let $S_n(p)$ be the unique square region of the nth stage which contains p. Let q_n be that point on the boundary of $S_n(p)$ where the restriction of f to the boundary of $S_n(p)$ attains a minimum. Note that q_n is necessarily a vertex of $S_n(p)$, and that $f(q_n) \leqslant f(q_{n+1})$. There is a piecewise linear arc α_n such that $\alpha_n(0) = q_n$, $\alpha_n(1) = q_{n+1}$ and $f(\alpha_n(s)) \leqslant f(\alpha_n(t))$ for $0 \leqslant s < t \leqslant 1$. Finally note that since $p \notin G$ it follows that the sequence $\{f(q_n)\colon n = 1, 2, \ldots\}$ does not terminate. Hence, if $\alpha = \bigcup \alpha_n$ then $\alpha \cup \{p\}$ is an arc at p and $f(\alpha) \cap f(p) = \emptyset$ because $f(q_n) < f(p)$ for every n, and this completes the proof.

It should, perhpas, be noted that the hypothesis that $X = R^2$ could easily be replaced by $X = R^m$ $(m \geqslant 2)$.

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