Analysis on a class of Banach algebras with applications to harmonic analysis on locally compact groups and semigroups

by

Anthony To-Ming Lau* (Edmonton, Alta.)

Abstract. In this paper we introduce and study a class of Banach algebras called F-algebras which includes the group algebra and the Fourier algebra of a locally compact group. It also includes the measure algebra of a locally compact semigroup.

1. Introduction. Let G be a locally compact group with a fixed left Haar measure λ. Let $L_p(G)$ $(1 \leqslant p < \infty)$ denote the Banach space of measurable functions f on G such that $|f|^p$ is integrable. Then $L_1(G)$ is a Banach algebra with norm $\|f\|_1 = \int |f| \, d\lambda$ and product defined by

$$(f * g)(x) = \int_G f(y) g(y^{-1} x) \, d\lambda(y); \quad f, g \in L_1(G).$$

The dual of $L_1(G)$ is the commutative W^*-algebra $L_\infty(G)$ consisting of all essentially bounded measurable functions on G as defined in [19, p. 141] with pointwise multiplication.

Associating with the locally compact group G is another Banach algebra, $A(G)$, which can be defined as follows: $A(G)$ consists of all continuous functions f on G of the form $\bar{k} * \check{h}$, where $k, h \in L_2(G)$, $\bar{k}(x) = \overline{k(x)}$, and $\check{h}(x) = h(x^{-1})$. Then $A(G)$ is contained in $C_0(G)$, the bounded continuous complex-valued functions on G vanishing at infinity (see [19], p. 295). Let $\mathrm{VN}(G)$ denote the von-Neuman algebra generated by the left regular representation of G, i.e. the closure of the operators $\{\varrho(f); f \in L_1(G)\}$ on $L_2(G)$, where $(f)(h) = f * h$ for each $h \in L_2(G)$, in $\mathscr{B}(L_2(G))$, the algebra of bounded linear operators from $L_2(G)$ onto $L_2(G)$, in the weak operator topology. Then

* This research is supported by NRC Grant A-7679.

each $\varphi = \bar{k} * \check{h}$ in $A(G)$ can be regarded as a linear functional on $VN(G)$ defined by

$$\varphi(T) = \langle Th, k\rangle \quad \text{for each } T \in VN(G).$$

P. Eymard [13, p. 210 and p. 218] proved that each ultraweakly continuous linear functional on $VN(G)$ is of this form. Furthermore $A(G)$ with pointwise multiplication and norm

$$\|\varphi\| = \{|\varphi(x)|;\ x \in VN(G) \text{ and } \|x\| \leqslant 1\}$$

is a commutative Banach algebra called the Fourier algebra of G. When G is commutative, then $A(G) = L_1(\hat{G})$, where $\hat{G}$ is the dual group of G (see [13, p. 209]).

The two Banach algebras $L_1(G)$ and $A(G)$ are important topological algebraic objects in the study of harmonic analysis on locally compact groups. They have been shown to have deep relation with the structure of the underlying group G (see Wendel [40] and Walter [39]). They also share a crucial common property: each of them is the predual of a W^*-algebra and the identity of the W^*-algebra is in the spectrum of the Banach algebra.

In this paper, we shall introduce and study a class of Banach algebras, called F-algebras, that includes the algebra $L_1(G)$ and $A(G)$ of a locally compact group G. Roughly speaking, an F-algebra is a Banach algebra A which is the predual of a W^*-algebra M (not necessarily unique!) and the identity of M is in the spectrum of A. The class of F-algebras includes the Fourier Stieltjes algebra $B(G)$ of a locally compact group G and the measure algebra $M(S)$ of a locally compact semigroup S. It also includes the class of convolution measure algebras studied by Taylor [36], [37], [38]; the class of L-algebras (for which the identity of the dual algebra is in the spectrum of the L-algebra) considered by McKilligan and White [28]; predual algebra of a Holf von Neuman algebra [35] and the measure algebra of a locally compact hypergroup [11] or semi-convos [21]. Our approach provides a unified treatment in the study of various aspects of the specific Banach algebras mentioned above.

Part of this research is conducted while the author was visiting the Department of Mathematics, University of California. It is our pleasure to thank Professor M. Rieffel and other members of the department for their kind hospitality during our visit. Many thanks are also due to Professor T. Palmer for several stimulating discussions on some part of this paper.

2. Preliminaries and some notations. Let E be a linear space, and φ be a linear functional on E, then the value of φ at an element x in E will be written as $\varphi(x)$ or $\langle\varphi, x\rangle$. If F is a subspace of the algebraic dual of E, then $\sigma(E, F)$ will denote the weakest locally convex topology on E such that each of the functionals in F is continuous.

If K is a subset of a normed linear space E, the closure of K will be denoted by $\bar{K}$ when the closure is taken with respect to the norm topology, or by $\bar{K}^\tau$ when the closure is taken with respect to topology τ on E different from the norm topology. The continuous dual of E will be denoted by E^*.

If M is a W^*-algebra, then M_* will denote its unique predual. The topology $\sigma(M, M_*)$ on M will be referred to as the ultraweak, or simply the σ-topology.

Let A be a Banach algebra. By the *reversed algebra* of A, denoted by A^c, we shall mean the Banach algebra with the same underlying Banach space as A and the multiplication reversed. The *spectrum* of A will be denoted by $\sigma(A)$.

If A is a normed algebra, then for each $\varphi \in A$, and $x \in A^*$, define the elements $\varphi \cdot x$ and $x \cdot \varphi$ in A^* by

$$\langle x\cdot\varphi, \gamma\rangle = \langle x, \varphi\cdot\gamma\rangle \quad \text{and} \quad \langle\varphi\cdot x, \gamma\rangle = \langle x, \gamma\cdot\varphi\rangle$$

for each $\gamma \in A$. We say that a subspace X of A^* is *topologically left* (resp. *right*) *invariant* if $X\cdot\varphi \subseteq X$ (resp. $\varphi\cdot X \subseteq X$) for each $\varphi \in A$; X is *topologically invariant* if it is both left and right topologically invariant.

If X is a topologically left invariant subspace of A^* and $m \in X^*$, we define an operator m_L from X into A^* by

$$\langle m_L(x), \varphi\rangle = \langle x\cdot\varphi, m\rangle \quad \text{for each } \varphi \in A.$$

We say that X is *topologically left introverted* if $m_L(X) \subseteq X$ for each m in X^*. Similarly, we can define topologically right introverted subspaces of A^*. A subspace X of A^* is *topologically introverted* if it is both left and right topologically introverted.

In [1], Arens defined a product on the second conjugate space A^{**} by

$$\langle m \odot n, x\rangle = \langle m, n_L(x)\rangle \quad \text{for each } m, n \in A^{**},\ x \in A^*.$$

Then A^{**} with resp. to this product becomes a Banach algebra. If X is a topologically left invariant and left introverted subspace of A^*, then the Arens product on X^* makes sense and renders X^* into a Banach algebra.

3. F-algebras. By an *F-algebra* we shall mean a pair (A, M) such that A is a complex Banach algebra and M is a W^*-algebra such that $A = M_*$, the predual of M, and the identity of M is a multiplicative linear functional on A. If there is no confusion, we shall simply say that A is an F-algebra and we shall identify A^* with M. The identity of A^* will be denoted by e. Also $P(A)$ will denote the cone of all positive functionals in A and $P_1(A)$ will denote the set of all φ in $P(A)$ such that $\varphi(e) = 1$.

Note that the W^*-algebra M of an F-algebra (A, M) need not be unique. In fact, let M be a W^*-algebra such that the reversed algebra M^c is not W^*-isomorphic to M (see [6]). Define on $A = M_* = (M^c)_*$ the multiplication $\varphi\cdot\psi = \varphi(e)\psi$ for any $\varphi, \psi \in A$. Then both (A, M) and (A, M^c) are F-algebras even though M and M^c have the same set of positive functionals. However we have the following:

PROPOSITION 3.1. *If (A, M_1) and (A, M_2) are F-algebras such that M_1 and M_2 have the same set of positive functionals. Then there exists central projections z_i in M_i, $i = 1, 2$ such that $M_1 z_1$ is W^*-isomorphic to $M_2 z_2$ and $M_1(e_1 - z_1)$ is W^*-isomorphic to the reversed algebra of $M_2(e_2 - z_2)$, where e_i is the identity of M_i. In particular, if M_1 is commutative, then M_1 and M_2 are W^*-isomorphic.*

Proof. By assumption there exists a linear isometry U from $(M_1)_*$ onto $(M_2)_*$ such that $U(\varphi)$ is a positive linear functional in $(M_2)_*$ if and only if φ is a positive linear functional in $(M_1)_*$. Hence U^* is a linear isometry from M_2 onto M_1 such that $U^*(e_2) = e_1$. The assertion now follows from Theorems 7 and 8 in [24].

The next two propositions show how new F-algebras can be formed from certain C^*-subalgebras of A^* of a given F-algebra A. Since their proofs are rather straight forward, we omit the details.

PROPOSITION 3.2 *Let A be an F-algebra. Let R be a C^*-subalgebra of A^* containing the identity of A^*. If R is topologically invariant and topologically left introverted, then (R^*, R^{**}) is an F-algebra, where R^{**} is the eveloping W^*-algebra of R.*

Let A be an F-algebra, and R be a C^*-subalgebra of A^* which is topologically invariant. Let $I_R = \bigcap_{x \in R} \{\varphi \in A;\ \varphi(x) = 0\}$. Then I_R is a closed two-sided ideal in A. Let A/I_R be the quotient algebra.

PROPOSITION 3.3 *Let A be an F-algebra, and let R be a topologically invariant C^*-subalgebra of A^*. Then there exists a linear isometry from A/I_R onto $(\bar{R}^\sigma)_*$. In particular, if R contains the identity of A^*, then $(A/I_R, \bar{R}^\sigma)$ is an F-algebra.*

Let A_1 be a normed algebra over the complex, let A_2 be an F-algebra and let e_2 be the identity of $(A_2)^*$. We define the *direct sum* of A_1 and A_2, denoted by $A_1 \oplus A_2$, to be the algebra over the complex consisting of all ordered pairs (φ_1, φ_2), $\varphi_1 \in A_1$ and $\varphi_2 \in A_2$ with coordinatewise addition and scalar multiplication, and product of two elements $\varphi = (\varphi_1, \varphi_2)$, $\psi = (\psi_1, \psi_2)$ defined by

$$\varphi\cdot\psi = (\varphi_1\cdot\psi_1 + \varphi_2(e_2)\psi_1 + \psi_2(e_2)\varphi_1,\ \varphi_2\cdot\psi_2).$$

Then $A_1 \oplus A_2$ with norm $\|(\varphi_1, \varphi_2)\| = \|\varphi_1\| + \|\varphi_2\|$ becomes a normed algebra. Note that associativity of multiplication on $A_1 \oplus A_2$ depends heavily on the fact that e_2 is in the spectrum of A_2. Also, when $A_2 = \mathbf{C}$, then $A_1 \oplus A_2$ is the usual unitization of the normed algebra A_1.

As well known, the dual of $A_1 \oplus A_2$ can be identified with $(A_1)^* \times (A_2)^*$ with norm $\|(x_1, x_2)\| = \max\{\|x_1\|, \|x_2\|\}$, $x_i \in (A_i)^*$. In fact, if $(x_1, x_2) \in (A_1)^* \times (A_2)^*$ and $(\varphi_1, \varphi_2) \in A_1 \oplus A_2$, then

$$\langle (x_1, x_2), (\varphi_1, \varphi_2)\rangle = \varphi_1(x_1) + \varphi_2(x_2).$$

PROPOSITION 3.4. *Let A_1 be a normed algebra and A_2 be any F-algebra. Then*

(a) A_2 has a (right, left, two-sided) identity if and only if $A_1 \oplus A_2$ has a (right, left, two-sided) identity.

(b) A_2 has a (right, left, two-sided) bounded approximate identity if and only if $A_1 \oplus A_2$ has a (right, left, two-sided) bounded approximate identity.

Proof. We shall prove (b). The proof of (a) is similar. Let $\{\varphi_{2_\alpha}\}$ be a bounded right approximate identity for A_2. Then $\lim_\alpha \varphi_{2_\alpha}(e_2) = 1$. Hence if $(\psi_1, \psi_2) \in A_1 \oplus A_2$, then

$$\|(\psi_1, \psi_2)\cdot(0, \varphi_{2_\alpha}) - (\psi_1, \psi_2)\| = \|(\varphi_{2_\alpha}(e_2) - 1)\psi_1\| + \|\psi_2\cdot\psi_{2_\alpha} - \psi_2\|$$

which converges to zero. Consequently, $\{(0, \varphi_{2_\alpha}\}$ is a bounded right approximate identity for $A_1 \oplus A_2$.

Conversely, if $(\varphi_{1_\alpha}, \varphi_{2_\alpha})$ is a bounded right approximate identity for A_2, and $\psi_2 \in P_1(A_2)$, then

$$\|(0, \psi_2)(\varphi_{1_\alpha}, \varphi_{2_\alpha}) - (0, \psi_2)\| = \|\varphi_{1_\alpha}\| + \|\psi_2\cdot\varphi_{2_\alpha} - \psi_2\|$$

which converges to zero. Hence $\|\psi_2\cdot\varphi_{2_\alpha} - \psi_2\|$ also converges to zero. Consequently $\{\varphi_{2_\alpha}\}$ is a bounded right approximate identity for A_2. The other cases can be proved similarly.

PROPOSITION 3.5. *If A_1 is a normed algebra and A_2 is any F-algebra, then the Banach algebra $(A_1)^{**} \oplus (A_2)^{**}$ is isometric and algebra isomorphic to the second conjugate algebra $(A_1 \oplus A_2)^{**}$.*

Proof. Note that $(A_2)^{**}$ is an F-algebra by Proposition 3.2. Define a linear map $J\colon (A_1)^{**} \oplus (A_2)^{**} \to (A_1 \oplus A_2)^{**}$ by

$$\langle J(m_1, m_2), (x_1, x_2)\rangle = m_1(x_1) + m_2(x_2)$$

for each $m_i \in (A_i)^{**}$, $x_i \in (A_i)^*$, $i = 1, 2$. Then J is a linear isometry from $(A_1)^{**} \oplus (A_2)^{**}$ onto $(A_1 \oplus A_2)^{**}$. Furthermore, a routine calculation shows that J is even an algebra homomorphism.

If $X_1, \ldots, X_n$ are topological spaces, let $\tilde{X}_i$ denote the n-tuple $(0, \ldots, 0, x_i, 1, \ldots, 1)$ with $x_i \in X_i$ appearing in the ith coordinate. Equip $\tilde{X}_i$ with the topology τ_i induced from X_i. By the *direct sum* of spaces $X_1, \ldots, X_n$, denoted by $X_1 \oplus \ldots \oplus X_n$ we shall mean the set $X = \bigcup\{\tilde{X}_i;\ i = 1, \ldots, n\}$ with the topology τ on X consisting of all subsets 0 of X such that $0 \cap \tilde{X}_i \in \tau_i$ for each $i = 1, \ldots, n$.

Let A_1 be a normed algebra, and $A_2, \ldots, A_n$ be F-algebras. We define inductively the direct sum $A_1 \oplus \ldots \oplus A_n$ by:

$$A_1 \oplus \ldots \oplus A_n = (A_1 \oplus \ldots \oplus A_{n-1}) \oplus A_n.$$

It is easy to see in this case that if $\varphi = (\varphi_1, \ldots, \varphi_n)$, and $\psi = (\psi_1, \ldots, \psi_n)$ are elements in $A_1 \oplus \ldots \oplus A_n$, and $\varphi \cdot \psi = (\gamma_1, \ldots, \gamma_n)$, then

$$\gamma_k = \varphi_k \cdot \psi_k + \left[\sum_{i=k+1}^{n} \psi_i(e_i) \right] \varphi_k + \left[\sum_{i=k+1}^{n} \varphi_i(e_i) \right] \psi_k$$

where e_i is the identity of $(A_i)^*$. Also $\sigma(A_1 \oplus \ldots \oplus A_n)$ consists of all elements in $(A_1)^* \times \ldots \times (A_n)^*$ of the form $(0, \ldots, 0, x_j, e_{j+1}, \ldots, e_n)$ where $x_j \in \sigma(A_j)$.

Proposition 3.6 *Let* (A_i, M_i), $i = 1, \ldots, n$, *be* F-*algebras. Let* $A = A_1 \oplus \ldots \oplus A_n$, *and* $M = M_1 \times \ldots \times M_n$. *Then* (A, M) *is also an* F-*algebra. Furthermore* $\sigma(A)$ *is homeomorphic to* $\sigma(A_1) \oplus \ldots \oplus \sigma(A_n)$.

Proof. For $n = 2$, define a map h from

$$\sigma(A_1) \oplus \sigma(A_2) = \{(x_1, 1);\ x_1 \in \sigma(A_1)\} \cup \{(0, x_2);\ x_2 \in \sigma(A_2)\}$$

into $(A_1 \oplus A_2)^*$ by

$$h(x_1, 1) = (x_1, e_2) \quad \text{and} \quad h(0, x_2) = (0, x_2)$$

for each $x_1 \in \sigma(A_1)$ and $x_2 \in \sigma(A_2)$. Then h is a homeomorphism from $\sigma(A_1) \oplus \sigma(A_2)$ onto $\sigma(A_1 \oplus A_2)$. The general case follows by induction.

4. Left amenable F-algebras. Let A be a Banach algebra. By a left *Banach A-module* we shall mean a Banach space X equipped with a bounded bilinear map from $A \times X \to X$, denoted by $(\varphi, x) \to \varphi \cdot x$, $\varphi \in A$, $x \in X$, such that $\varphi_1 \cdot (\varphi_2 \cdot x) = (\varphi_1 \cdot \varphi_2) \cdot x$ for all $\varphi_1, \varphi_2 \in A$, $x \in X$. A right Banach A-module is defined similarly. A *two-sided Banach A-module* is a left and right A-module such that

$$(\varphi_1 \cdot x) \cdot \varphi_2 = \varphi_1 \cdot (x \cdot \varphi_2) \quad \text{for all } \varphi_1, \varphi_2 \in A,\ x \in X.$$

If X is a two sided Banach A-module, then X^* becomes a two-sided Banach A-module with

$$\langle \varphi \cdot f, x \rangle = \langle f, x \cdot \varphi \rangle \quad \text{and} \quad \langle f \cdot \varphi, x \rangle = \langle f, \varphi \cdot x \rangle$$

for all $f \in X^*$, $\varphi \in A$.

Let X be a two-sided Banach A-module. A *derivation* from A into X^* is a linear map $D\colon A \to X^*$ such that

$$D(\varphi \cdot \psi) = D(\varphi) \cdot \psi + \varphi \cdot D(\psi)$$

for all $\varphi, \psi \in A$. Clearly if $f \in X^*$, then the map $D_f\colon A \to X^*$ defined by

$$D_f(\varphi) = \varphi \cdot f - f \cdot \varphi$$

is a bounded derivation. Any derivation of this form is called an *inner derivation.*

A Banach algebra A is *amenable* if for any two-sided Banach A-module X, any bounded derivation from A into X^* is an inner derivation. B. Johnson proved in [22, Theorem 2.5] that a locally compact group G is amenable if and only if the algebra $L_1(G)$ is amenable. The class of amenable Banach algebras has been studied extensively by Johnson in [22], [23] and Bunce in [3], [4] and [5].

However, Johnson's theorem [22, Theorem 2.5] is no longer valid for semigroups. In fact, the semigroup S of positive integers with addition is amenable, but the Banach algebra $l_1(S)$ as defined in [7] or [18] is not amenable [2, p. 244].

Let A be an F-algebra. A *topological left invariant mean* (abbreviated as TLIM) on A^* is an element m in $P_1(A^{**})$ such that

$$m(x \cdot \varphi) = m(x) \quad \text{for each } \varphi \in P_1(A),\ x \in A^*.$$

The set of TLIM on A^* will be denoted by TLIM(A^*). The notion of TLIM has been considered by many authors for various special cases of A (see for example [8], [20], [33], [42]).

We say that an F algebra A is *left amenable* if for any two-sided Banach A-module X such that $\varphi \cdot x = \varphi(e) x$ for all $\varphi \in A$, $x \in X$, every bounded derivation from A into X^* is inner.

Theorem 4.1. *Let A be an F-algebra. Then A^* has a* TLIM *if and only if A is left amenable.*

Proof. Let m be a TLIM on A^* and let X be a two-sided Banach A-module with $\varphi \cdot x = \varphi(e) x$ for all $\varphi \in A$, $x \in X$. Let D be a bounded derivation from A into X^*. Let L be the restriction of D^* to X, and let $f = L^*(m)$. We shall show that $D = D_{(-f)}$.

Indeed, if $x \in X$, $\varphi \in P_1(A)$ and $\psi \in A$, then

$$\begin{aligned}\langle L(x \cdot \varphi), \psi \rangle &= \langle x \cdot \varphi, D(\psi) \rangle = \langle x, \varphi \cdot D(\psi) \rangle = \langle x, D(\varphi \cdot \psi) - D(\varphi) \cdot \psi \rangle \\ &= \langle L(x) \cdot \varphi, \psi \rangle - \psi(e) \langle D(\varphi), x \rangle\end{aligned}$$

So $L(x \cdot \varphi) = L(x) \cdot \varphi - \langle D(\varphi), x \rangle \cdot e$. Hence

$$\begin{aligned}\langle \varphi \cdot f, x \rangle &= \langle f, x \cdot \varphi \rangle = \langle m, L(x \cdot \varphi) \rangle = \langle m, L(x) \cdot \varphi \rangle - \langle D(\varphi), x \rangle \langle m, e \rangle \\ &= \langle m, L(x) \rangle - \langle D(\varphi), x \rangle = \langle f - D(\varphi), x \rangle.\end{aligned}$$

So $\varphi \cdot f = f - D(\varphi)$ and $f \cdot \varphi = f$. Consequently $D_{-f}(\varphi) = D(\varphi)$ for any $\varphi \in P_1(A)$. Since $P_1(A)$ spans A, it follows that $D_{(-f)} = D$.

Conversely if A is left amenable, then an argument similar to [2, Proposition 4, p. 238] shows that there exists a non-zero $n \in A^{**}$ such that $n(x \cdot \varphi) = \varphi(e) n(x)$ for each $\varphi \in A$, $x \in A^*$. Hence $\varphi \odot n = n$ and $\varphi \odot n^* = n^*$ for all $\varphi \in P_1(A)$. So we may assume that n is self adjoint. Write $n = n^+ - n^-$, the orthogonal decomposition of n. If $\varphi \in P_1(A)$, then $\varphi \odot n = \varphi \odot n^+ - \varphi \odot n^-$. Since $\varphi \odot n^+$, $\varphi \odot n^-$ are positive and

$$\|\varphi \odot n^+\| + \|\varphi \odot n^-\| = (\varphi \odot n^+)(e) + (\varphi \odot n^-)(e) = n^+(e) + n^-(e) = \|n\|$$

it follows that $\varphi \odot n^+ = n^+$ and $\varphi \odot n^- = n^-$ [34, Theorem 1.14.3]. Consequently if $n^+ \neq 0$ (say) and $m = n^+/n^+(e)$, then m is a TLIM on A^*.

The following is an analogue of Johnson's theorem [22, Theorem 2.5]:

COROLLARY 4.2. *A semigroup S is left amenable if and only if the Banach algebra $l_1(S)$ is left amenable.*

COROLLARY 4.3. *A locally compact group G is amenable if and only if the measure algebra $M(G)$ is left amenable.*

Proof. This follows from Theorem 4.1 and proof of Lemma 5.1 in [43].

EXAMPLES.

(1) Any commutative F-algebras are left (and right) amenable. In particular, the algebras $A(G)$ and $B(G)$ of any locally compact group and the measure algebra $M(S)$ of any commutative locally compact semigroup S are left amenable.

In fact, if A is commutative, we consider the commutative semigroup $\mathscr{T} = \{T_\varphi;\ \varphi \in P_1(A)\}$ of affine continuous maps from the compact convex set $(P_1(A^{**}),\ \text{weak*})$ into itself defined by $T_\varphi(m) = \varphi \odot m$ for each $\varphi \in P_1(A)$, $m \in P_1(A^{**})$. Since A is commutative, $\mathscr{T}$ is also commutative. Hence by the Markov–Kakutani fixed point theorem [9, p. 456], $P_1(A^{**})$ contains a common fixed point m for $\mathscr{T}$. Clearly $m \in \text{TLIM}(A^*)$.

(2) Let M be a W^*-algebra and $A = M_*$. Then A is:

(i) left amenable if multiplication on A is defined by $\varphi \cdot \psi = \varphi(e)\psi$;

(ii) right amenable if multiplication on A is defined by $\varphi \cdot \psi = \psi(e)\varphi$;

(iii) both left and right amenable (but not amenable) if multiplication on A is defined by $\varphi \cdot \psi = \psi(e)\varphi(e)\theta$, $\theta \in P_1(A)$ is fixed.

PROPOSITION 4.4 *Let A be an F-algebra. Then A is left amenable if and only if A^{**} is left amenable.*

Proof. Let $m \in \text{TLIM}(A^*)$. If $n \in P_1(A^{**})$, choose a net $\varphi_\alpha \in P_1(A)$ converging to n in the weak*-topology. Then for each $x \in A^*$, we have

$$m(x) = \lim_\alpha m(x \cdot \varphi_\alpha) = \lim_\alpha \varphi_\alpha(m_L(x)) = n(m_L(x)) = n \odot m(x).$$

hence $m \in \text{TLIM}(A^{***})$. Conversely if $\gamma \in \text{TLIM}(A^{***})$, then γ restricted to A^* can easily be seen to be a TLIM on A^*.

PROPOSITION 4.5. *Let A_1 and A_2 be F-algebras. Then $A_1 \oplus A_2$ is left amenable if and only if A_1 is left amenable. In particular $C \oplus A$ is left amenable for any F-algebra A.*

Proof. If $m_1 \in \text{TLIM}(A_1^*)$ and $(\varphi_1, \varphi_2) \in P_1(A_1 \oplus A_2)$, then by Proposition 3.5, we have

$$(\varphi_1, \varphi_2) \odot (m_1, 0) = (\varphi_1 \odot m_1 + \varphi_2(e_2) m_1, 0)$$
$$= (\varphi_1(e_1) m_1 + \varphi_2(e_2) m_1, 0) = (m_1, 0).$$

Hence $(m_1, 0) \in \text{TLIM}((A_1 \oplus A_2)^*)$. Conversely, if (m_1, m_2) is a TLIM on $(A_1 \oplus A_2)^*$, let $\varphi_1 \in P_1(A_1)$, then $(\varphi_1, 0) \in P_1(A_1 \oplus A_2)$, and hence

$$(m_1, m_2) = (\varphi_1, 0) \odot (m_1, m_2) = (\varphi_1 \odot m_1 + m_2(e_2)\varphi_1, 0)$$

by Proposition 3.5. Consequently $m_2 = 0$ and $m_1 = \varphi_1 \odot m_1$. So $m_1 \in \text{TLIM}(A_1^*)$.

THEOREM 4.6. *Let A be any F-algebra. Then the following are equivalent:*

(a) *A is left amenable.*

(b) *There exists a net $\varphi_\alpha \in P_1(A)$ such that $\|\varphi \cdot \varphi_\alpha - \varphi_\alpha\| \to 0$ for each $\varphi \in P_1(A)$.*

(c) *For each $x \in A^*$, the set $\overline{K(x)}^\sigma$ contains λe for some complex number λ, where $K(x) = \{\varphi \cdot x;\ \varphi \in P_1(A)\}$.*

In this case $\lambda e \in \overline{K(x)}^\sigma$ is and only if there exists a TLIM on A^* such that $m(x) = \lambda$.

Proof. The equivalence of (a) and (b) can be proved by an argument similar to that of Namioka's elegant proof of Day's theorem [30, Theorem 2.2].

If (a) holds and m is a TLIM on A^*, let $\{\psi_\alpha\}$ be a net in $P_1(A)$ converging to m in the weak* topology. Then for each $x \in A^*$, the net $\{\psi_\alpha \cdot x\}$ converges to $m(x)e$ in the σ-topology. Hence (c) holds.

That (c) implies (a) follows easily from [26, Theorem 2.1](¹) by considering the semigroup $S = \{T_\varphi;\ \varphi \in P_1(A)\}$ of σ-continuous operators on A^* defined by $T_\varphi(x) = \varphi \cdot x$, $\varphi \in P_1(A)$, $x \in A^*$.

For the algebra $l_1(S)$ of a discrete semigroup S, the equivalence of (a) and (b) was established by Day [7], and the equivalence of (a) and (c) and the last statement is due to Mitchell [29]. Our theorem implies [41, Theorem 5.4 with $X = L_\infty(G)$] and [42, Theorem 3.1 (1)⇔(2)⇔(3)] of Wong. Condition (c) has also been considered by Dunkl–Ramirez for the Fourier algebra of a locally compact group in [10, Theorem 2].

Given an F-algebra A, let $I_0(A) = \{\varphi \in A;\ \varphi(e) = 0\}$. Then $I_0(A)$ is a closed two-sided ideal in A.

THEOREM 4.7. *Let A be an F-algebra. Then the followings are equivalent:*

(a) *A is left amenable.*

(¹) In [26] "=" in condition (1) should be replaced by "≤".

(b) *There exists a net* $\varphi_\alpha \in P_1(A)$ *such that* $\lim_\alpha \|\psi\cdot\varphi_\alpha\| = |\psi(e)|$ *for each* $\psi \in A$.

(c) *For each* $\psi \in I_0(A)$ *and* $\varepsilon > 0$, *there exists* $\varphi \in P_1(A)$ *such that* $\|\psi\cdot\varphi\| < \varepsilon$.

Proof. (a)$\Rightarrow$(b). If A is left amenable, then there exist a net $\{\varphi_\alpha\} \in P_1(A)$ such that $\|\varphi\cdot\varphi_\alpha - \varphi_\alpha\| \to 0$ for each $\varphi \in P_1(A)$ (Theorem 4.6). Let $\psi \in A$ and write $\psi = \sum_{i=1}^n \lambda_i\varphi_i$, where $\varphi_i \in P_1(A)$. Then $|\psi(e)| = |\sum_{i=1}^n \lambda_i|$. Given $\varepsilon > 0$, choose α_0 such that if $\alpha \geqslant \alpha_0$, then $\|\varphi_i\cdot\varphi_\alpha - \varphi_\alpha\| < \varepsilon/n|\lambda_i|$ (of course we may assume that $\lambda_i \neq 0$). Then

$$\|\psi\cdot\varphi_\alpha\| \leqslant \|\sum_{i=1}^n \lambda_i\varphi_i\varphi_\alpha - \sum_{i=1}^n \lambda_i\varphi_\alpha\| + \|\sum_{i=1}^n \lambda_i\varphi_\alpha\|$$
$$\leqslant \sum_{i=1}^n |\lambda_i|\,\|\varphi_i\varphi_\alpha - \varphi_\alpha\| + |\sum_{i=1}^n \lambda_i| \leqslant \varepsilon + |\psi(e)|$$

for all $\alpha \geqslant \alpha_0$. On the other hand

$$|\psi(e)| = |(\psi\cdot\varphi_\alpha)(e)| \leqslant \|\psi\cdot\varphi_\alpha\|$$

for all α. Hence

$$\big||\psi(e)| - \|\psi\cdot\varphi_\alpha\|\big| = \|\psi\cdot\varphi_\alpha\| - |\psi(e)| < \varepsilon$$

for all $\alpha \geqslant \alpha_0$.

(b)$\Rightarrow$(c) is clear.

(c)$\Rightarrow$(a). As in the proof of [17, Theorem 3.7.3] let $\eta_0 \in P_1(A)$ be fixed. Given $\varepsilon > 0$, and $\sigma = \{\varphi_1, \ldots, \varphi_k\}$ a finite subset of $P_1(A)$, let $\psi_1 = \varphi_1\cdot\eta_0 - \eta_0$. Then $\psi_1 \in I_0(A)$. Hence we may find $\eta_1 \in P_1(A)$ such that $\|\psi_1\cdot\eta_1\| < \varepsilon$. Now $\psi_2 = \varphi_2\cdot\eta_0\cdot\eta_1 - \eta_0\cdot\eta_1$ is in $I_0(A)$. So we may find $\eta_2 \in P_1(A)$ such that $\|\psi_2\cdot\eta_2\| < \varepsilon$. Inductively we may find $\eta_i \in P_1(A)$ such that $\|\psi_i\cdot\eta_i\| < \varepsilon$ where

$$\psi_i = \varphi_i\cdot\eta_0\cdot\eta_1\ldots\eta_{i-1} - \eta_0\cdot\eta_1\ldots\eta_{i-1}.$$

Let $\eta_{(\sigma,\varepsilon)} = \eta_0\cdot\eta_1\ldots\eta_k$. Then

$$\|\varphi\cdot\eta_{(\sigma,\varepsilon)} - \eta_{(\sigma,\varepsilon)}\| < \varepsilon$$

for all $\varphi \in \sigma$. So any weak* cluster point of the net $\{\eta_{(\sigma,\varepsilon)}\}$ is a TLIM on A^*.

Corollary 4.8. *Let A be an F-algebra. Then A is left amenable if and only if* $|\psi(e)| = \inf\{\|\psi\cdot\varphi\|;\ \varphi \in P_1(A)\}$ *for each* $\psi \in A$.

Corollary above is due to Reiter for $A = L_1(G)$ of a locally compact group G [17, section 3.7], and to Wong [42, Theorem 3.1 (2)$\Leftrightarrow$(4)] for $A = M(S)$ of a locally compact semigroup S.

Since the Fourier algebra $A(G)$ of a locally compact group G is always left amenable, we have the following analogue of Reiter's result:

Corollary 4.9. *For any locally compact group G, and $\psi \in A(G)$, we have*

$$|\psi(u)| = \inf\{\|\psi\cdot\varphi\|;\ \varphi \in A(G) \cap P(G) \text{ and } \varphi(u) = 1\},$$

where u is the indentity of G.

Our next result is also due to Reiter [32] for $A = L_1(G)$ of a locally compact group G (see also [22, Proposition 2.6]). Note that Reiter's notion of bounded right approximate identity is slightly different from ours. However, a Banach algebra B has a bounded right approximate identity as defined in [32] if and only if B has a bounded right approximate identity $\{\varphi_\alpha\}$ in the usual sense, i.e. $\{\varphi_\alpha\}$ is a bounded net such that $\|\varphi\cdot\varphi_\alpha - \varphi_\alpha\| \to 0$ for each $\varphi \in B$ (see [2, p. 58]).

Theorem 4.10. *Let A be an F-algebra. Then $I_0(A)$ has a bounded right approximate identity if and only if A is left amenable and has a bounded right approximate identity.*

Proof. Assume that $\psi_\alpha \in I_0(A)$ is a bounded right approximate identity for $I_0(A)$. Let $\varphi_0 \in P_1(A)$ be fixed. Form the net

$$\theta_\alpha = \varphi_0\cdot\psi_\alpha - \varphi_0.$$

Then $\{\theta_\alpha\}$ is also bounded, and $\theta_\alpha(e) = -1$ for each α. Also, if $\varphi \in P_1(A)$, then

$$\|\varphi\cdot\theta_\alpha - \theta_\alpha\| = \|(\varphi\cdot\varphi_0 - \varphi_0)\cdot\psi_\alpha - (\varphi\cdot\varphi_0 - \varphi_0)\| \to 0$$

since $\varphi\cdot\varphi_0 - \varphi_0 \in I_0(A)$. Let η be a weak* cluster point of in A^{**}. Then n is non-zero since $\eta(e) = -1$, and $\varphi\odot n = n$ for each $\varphi \in P_1(A)$. An argument similar to that for the proof of Theorem 4.1 shows that A has a TLIM m. So A is left amenable.

Let r be a weak* cluster point of $\{\psi_\alpha\}$ in A^{**}. Then for each $\varphi \in P_1(A)$,

$$\varphi\odot(m + r - m\odot r) = \varphi\odot m + \varphi\odot r - \varphi\odot m\odot r = m + \varphi\odot r - m\odot r$$
$$= m + (\varphi - m)\odot r = m + (\varphi - m) = \varphi$$

since $\varphi - m$ is the weak* limit of a net in $I_0(A)$. So $m + r - m\odot r$ is a right identity in A^{**}. Consequently A has a bounded right approximate identity [2, p. 146].

Conversely if A is left amenable and A has a bounded right approximate identity $\{\varphi_\alpha\}$, let m be a TLIM on A^* and p be a weak* cluster point of the net $\{\varphi_\alpha\}$ in A^{**}. Then $p(e) = 1$. Let $q = p - m$. Clearly $I_0(A) \subseteq I_0(A^{**})$. Also if $n \in I_0(A^{**})$, then $n\odot m = 0$. Indeed, if $n = \sum_{i=1}^k \lambda_i n_i$, n_i are states, then $n(e) = \sum_{i=1}^k \lambda_i = 0$. Hence

$$n\odot m = \sum_{i=1}^k \lambda_i(n_i\odot m) = \sum_{i=1}^n \lambda_i m = 0$$

by the proof of Proposition 4.4. Consequently, if $n \in I_0(A^{**})$, we have

$$n \odot q = n \odot (p-m) = n \odot p - n \odot m = n \odot p = n.$$

Since $q \in I_0(A^{**})$, it follows that q is a right identity in $I_0(A)$. However $I_0(A^{**})$ can be identified with the second conjugate algebra of $I_0(A)$ with the Arens product, it follows that $I_0(A)$ has a bounded right approximate identity.

The following is an analogue of Reitier's result [32]:

COROLLARY. 4.11. *Let G be a locally compact group and let u be the identity of G. Then G is amenable if and only if the ideal $\{\varphi \in A(G);\ \varphi(u) = 0\}$ in $A(G)$ has a bounded approximate identity.*

Proof. This follows from Theorem 4.10 and the fact that G is amenable if and only if $A(G)$ has a bounded approximate identity (see Leptin [27]).

Given an F-algebra A, let $N(A)$ denote all $x \in A^*$ such that $\inf\{\|\varphi \cdot x\|:\ \varphi \in P_1(A)\} = 0$. Then as readily checked, $N(A)$ is closed under scalar multiplication. Furthermore, $N(A)$ includes all elements z of the form $\psi \cdot x - x$, $\psi \in P_1(A)$ and $x \in A^*$. In fact, if $n = 1, 2, \ldots$, let $\varphi_n = (1/n)\sum_1^n \psi^j$. Then $\varphi_n \in P_1(A)$ and $\|\varphi_n \cdot z\| = (1/n)\|\psi^{n+1} \cdot x - \psi \cdot x\| \leqslant (2/n)\|x\| \to 0$ as $n \to \infty$.

If $I_1, I_2 \subseteq P_1(A)$, let $d(I_1, I_2) = \inf\{\|\varphi_1 - \varphi_2\|;\ \varphi_1 \in I$ and $\varphi_2 \in I_2\}$. The following is an analogue of Theorem 1.7 of Emerson [12]. It also implies Theorem 2.1.7 of Riazi [31] when A is the mesure algebra of a locally compact semigroup.

THEOREM 4.12. *Let A be an F-algebra. The followings are equivalent:*

(a) *A is left amenable.*

(b) *$N(A)$ is closed under addition.*

(c) *$d(I_1, I_2) = 0$ for any two right ideals I_1, I_2 of the semigroup $P_1(A)$.*

Proof. (a)$\Rightarrow$(c). If A is left amenable, there exist a net $\psi_\alpha \in P_1(A)$ such that $\|\varphi\psi_\alpha - \psi_\alpha\| \to 0$ for all $\varphi \in P_1(A)$ (by Theorem 4.6). Hence if $\varphi_1 \in I_1$ and $\varphi_2 \in I_2$, then $\|\varphi_1\psi_\alpha - \varphi_2\psi_\alpha\| \to 0$.

(c)$\Rightarrow$(b). Let $x_1, x_2 \in N(A)$ and $\varepsilon > 0$. Choose $\varphi_1, \varphi_2 \in P_1(A)$ such that $\|\varphi_1 \cdot x_1\| \leqslant \varepsilon$ and $\|\varphi_2 \cdot x_2\| \leqslant \varepsilon$. Pick $\psi_1, \psi_2 \in P_1(A)$ such that $\|\varphi_1\psi_1 - \varphi_2\psi_2\| \leqslant \varepsilon$. Then

$$\begin{aligned}\|\psi_1\varphi_1 \cdot (x_1+x_2)\| &\leqslant \|\psi_1\varphi_1 \cdot x_1\| + \|\psi_1\varphi_1 \cdot x_2 - \psi_2\varphi_2 \cdot x_2\| + \|\psi_2\varphi_2 \cdot x_2\| \\ &\leqslant \varepsilon(2+\|x_2\|).\end{aligned}$$

Hence $x_1 + x_2 \in N(A)$ also.

(b)$\Rightarrow$(a). If (b) holds, then $N(A)$ is subspace of A^* such that $\varphi \cdot x - x \in N(A)$ for any $x \in A^*$ and $\varphi \in P_1(A)$. Let E be the self-adjoint elements in A^*. Then E is a real vector subspace of A^*. Let K denote all $x \in E$ such that $\inf\{\varphi(x);\ \varphi \in P_1(A)\} > 0$. Then K is open in E, $e \in K$ and $K \cap N(A) = \emptyset$. By a Hahn Banach separation theorem, there exists a continuous (real) linear functional θ on E such that $\theta(e) = 1$ and $\theta(x) = 0$ for all $x \in E \cap N(A)$. In particular, $\theta(\varphi \cdot x) = \theta(x)$ for all $\varphi \in P_1(A)$ and $x \in E$. Define $n(x) = \theta(u) + i\theta(v)$ when $x = v + iv$, $u, v \in E$. Then $n \in A^{**}$, $n(e) = 1$ and n is topologically left invariant. An argument similar to that for Theorem 4.1 shows that A^* has a TLIM.

A semigroup S is *left reversible* if any two right ideals in S has nonempty intersectin. Commutative semigroups (or more generally left amenable semigroups) and groups are left reversible.

COROLLARY 4.13. *Let A be an F-algebra. If $P_1(A)$ is left reversible, then A is left amenable.*

Finally we state a few facts concerning the set TLIM(A^*) for a left amenable F-algebra A.

PROPOSITION 4.14. *If A is a commutative F-algebra and A^* has a* TLIM *in $P_1(A)$, then A^* has a unique* TLIM.

Proof. If $n \in P_1(A)$ is a TLIM on A^* and $m \in$ TLIM(A^*), let $\varphi_\alpha \in P_1(A)$ be a net converging to m in the weak*-topology. Then for each $x \in A^*$, we have

$$\begin{aligned} m(x) = m(x \cdot n) &= \lim_\alpha \varphi_\alpha(x \cdot n) = \lim_\alpha n \cdot \varphi_\alpha(x) = \lim_\alpha \varphi_\alpha \cdot n(x) \\ &= \lim_\alpha n(x \cdot \varphi_\alpha) = n(x).\end{aligned}$$

Hence $m = n$.

PROPOSITION 4.15. *Let A be an F-algebra.*

(a) *If A^* has a* TLIM *in $P_1(A)$, then the identity e of A^* is an isolated point in the spectrum of A.*

(b) *If A^* has a unique* TLIM *and A is norm separable, then A^* has a* TLIM *in $P_1(A)$.*

Proof. (a) Assume that $\psi \in P_1(A)$ is a TLIM. Let $x \in \sigma(A)$ and $x \neq e$. Choose $\varphi \in P_1(A)$ such that $\varphi(x) \neq 1$. Then $\psi(x) = (\varphi \odot \psi)(x) = \varphi(x)\psi(x)$. Consequently $\psi(x) = 0$. Hence $\{y \in A^*;\ \psi(y) > \frac{1}{2}\} \cap \sigma(A) = \{e\}$ and e is isolated.

The proof of part (b) is similar to that of Granirer [16, Theorem 7], we omit the details.

COROLLARY 4.16. *If A is a norm separable commutative F-algebra, then A^* has a unique* TLIM *if and only if A^* has a* TLIM *in $P_1(A)$.*

Proof. This follows from Propositions 4.14 and 4.15.

In the case that $A = l_1(S)$ of a discrete semigroup S, or that A is either the Fourier algebra $A(G)$ or the group algebra $L_1(G)$ of a locally compact group G, then much stronger results then Proposition 4.15 are known to hold (see Granirer [14], [15], Klawe [25], Renauld [33]).

References

[1] R. F. Arens, *The adjoint of a bilinear operation*, Proc. Amer. Math. Soc. 2 (1951), pp. 839–848.

[2] F. F. Bonsall and J. Duncan, *Complete normed algebras*, Springer Verlag, 1973.

[3] J. Bunce, *Characterization of amenable and strongly amenable C*-algebras*, Pacific J. Math. 43 (1972), pp. 563–572.

[4] — *Representations of strongly amenable C*-algebras*, Proc. Amer. Math. Soc. 32 (1972), pp. 241–246.

[5] — *Finite operators and amenable C*-algebras*, Proc. Amer. Math. Soc. 56 (1976), pp. 145–151.

[6] A. Connes, *A factor not anti-isomorphic to itself*, Ann. of Math. 101 (1975), pp. 536–554.

[7] M. M. Day, *Amenable semigroup*, III. J. Math. 1 (1957), pp. 509–544.

[8] — *Lumpy subsets in left-amenable locally compact semigroups*, Pacific J. Math. 62 (1976), pp. 87–92.

[9] N. Dunford and J. T. Schwartz, *Linear operators* I, Interscience Publishers, 1957.

[10] C. F. Dunkl and D. E. Ramirez, *Existence and nonuniqueness of invariant means on* $L^{\infty}(G)$, Proc. Amer. Math. Soc. 32 (1972), pp. 525–530.

[11] — *The measure algebra of a locally compact hypergroup*, Trans. Amer. Math. Soc. 179 (1973), pp. 331–348.

[12] W. F. Emerson, *Characterizations of amenable groups*, Trans. Amer. Math. Soc. 241 (1978), pp. 183–194.

[13] P. Eymard, *L'algèbre de Fourier d'un groups localement compact*, Bull. Soc. Math. France 92 (1964), pp. 181–236.

[14] E. Granirer, *Criteria for compactness and discreteness of locally compact amenable groups*, Proc. Amer. Math. Soc. 40 (1973), pp. 615–624.

[15] — *Properties of the set of topological invariant means on P. Eymard's W*-algebra* VN(*G*), Indag. Math. 36 (1974), pp. 116–121.

[16] — *Weakly almost periodic and uniformly continuous functionals on the Fourier algebra of any locally compact group*, Trans. Amer. Math. Soc. 189 (1974), pp. 371–382.

[17] F. P. Greenleaf, *Invariant means on Topological Groups and their Applications*, Van Nostrand Reinhold, New York 1969.

[18] E. Hewitt and H. S. Zuckerman, *The* l_1*-algebra of a commutative semigroup*, Trans. Amer. Math. Soc. 83 (1956), pp. 70–97.

[19] — and K. A. Ross, *Abstract Harmonic Analysis* I, Springer Verlag, 1963.

[20] A. Hulanicki, *Means and Følner conditions on locally compact groups*, Studia Math. 27 (1966), pp. 87–104.

[21] R. J. Jewett, *Spaces with an abstract convolution of measures*, Advances Math. 18 (1975), pp. 1–101.

[22] B. E. Johnson, *Cohomology in Banach algebras*, Memoir Amer. Math. Soc. 127 (1972).

[23] — *Approximate diagonals and cohomology of certain annihilator Banach algebras*, Amer. J. Math. 94 (1972), pp. 685–698.

[24] R. V. Kadison, *Isometries of operator algebras*, Ann. of Math. 54 (1951), pp. 325–338.

[25] M. Klawe, *On the dimension of left invariant means and left thick subsets*, Trans. Amer. Math. Soc. 231 (1977), pp. 507–518.

[26] A. T. Lau, *Semigroup of operators on dual Banach spaces*, Proc. Amer. Math. Soc. 54 (1976), pp. 393–396.

[27] H. Leptin, *Sur l'algèbre de Fourier d'un localement compact*, C. R. Acad. Sci. Paris Sér. A-B 266 (1968), pp. A1180–A1182.

[28] S. A. McKilligan and A. J. White, *Representations of L-algebras*, Proc. London Math. Soc. 25 (3) (1972), pp. 655–674.

[29] T. Mitchell, *Constant functions and left invariant means on semigroups*, Trans. Amer. Math. Soc. 119 (1965), pp. 244–261.

[30] I. Namioka, *Følner's conditions for amenable semi-groups*, Math. Scand. 15 (1964), pp. 18–28.

[31] A. Riazi, *Characterizations of amenability of locally compact separately continuous semigroups*, Ph. D. Thesis, University of Calgary (1979).

[32] H. Reiter, *Sur certains ideaux dans* $L^1(G)$, C. R. Acad. Sci. Paris 267 (1968), pp. A882–885.

[33] P. E. Renaud, *Invariant means on a class of von Neuman algebras*, Trans. Amer. Math. Soc. 170 (1972), pp. 285–291.

[34] S. Sakai, *C*-algebras and W*-algebras*, Springer Verlag, 1971.

[35] M. Takesaki, *Duality and von Neuman algebras*, Lectures on Operator Algebras, Lecture Notes in Mathematics 247, Springer Verlag, pp. 66–779.

[36] J. L. Taylor, *The structure of convolution measure algebras*, Trans. Amer. Math. Soc. 119 (1965), pp. 150–166.

[37] — *Noncommutative convolution measure algebra*, Pacific J. Math. 31 (1969), pp. 809–826.

[38] — *Measure algebras*, Regional conference series in mathematics 16 (1972), Amer. Math. Soc.

[39] M. E. Walter, *W*-algebras and nonabelian harmonic analysis*, J. Func. Analysis 11 (1972), pp. 17–38.

[40] J. G. Wendel, *Left centralizers and isomorphisms of group algebras*, Pacific J. Math. 2 (1952), pp. 251—261.

[41] J. C. S. Wong, *Topologically stationary locally compact groups and amenability*, Trans. Amer. Math. Soc. 144 (1969), pp. 351–363.

[42] — *An ergotic property of locally compact amenable semigroups*, Pacific J. Math. 48 (1973), pp. 615–619.

[43] — *A characterization of topological left-thick subsets in locally compact left amenable semigroups*, Pacific J. Math. 62 (1976), pp. 295–303.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ALBERTA
Edmonton, Alberta, Canada
T6G 2G1

Accepté par la Rédaction le 2.2.1981