Analysis on a class of Banach algebras with applications to harmonic analysis on locally compact groups and semigroups

by

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Abstract. In this paper we introduce and study a class of Banach algebras called $F$-algebras which includes the group algebra and the Fourier algebra of a locally compact group. It also includes the measure algebra of a locally compact semigroup.

1. Introduction. Let $G$ be a locally compact group with a fixed left Haar measure $\lambda$. Let $L^p(G)$ $(1 \leq p < \infty)$ denote the Banach space of measurable functions $f$ on $G$ such that $\int |f|^p \, d\lambda$ is integrable. Then $L^1(G)$ is a Banach algebra with norm $\|f\|_1 = \int |f| \, d\lambda$ and product defined by

$$ (f \ast g)(x) = \int f(y) g(y^{-1} x) \, d\lambda(y); \quad f, g \in L^1(G). $$

The dual of $L^1(G)$ is the commutative $W^*$-algebra $L^1(G)$ consisting of all essentially bounded measurable functions on $G$ as defined in [19, p. 141] with pointwise multiplication.

Associating with the locally compact group $G$ is another Banach algebra, $A(G)$, which can be defined as follows: $A(G)$ consists of all continuous functions $f$ on $G$ of the form $k \ast h$, where $k, h \in L^1(G)$, $k(x) = k(x)$, and $h(x) = h(x^{-1})$. Then $A(G)$ is contained in $C_0(G)$, the bounded continuous complex-valued functions on $G$ vanishing at infinity (see [19], p. 295). Let $\text{VN}(G)$ denote the von-Neumann algebra generated by the left regular representation of $G$, i.e. the closure of the operators $\pi(f) \ast h \in L^2(G)$, where $f \in L^2(G)$, the algebra of bounded linear operators from $L^2(G)$ onto $L^2(G)$ in the weak operator topology. Then...
each φ = k * h in A(G) can be regarded as a linear functional on VN(G) defined by
\[ \varphi(T) = \langle Th, k \rangle \quad \text{for each } T \in VN(G). \]
P. Eymard [13, p. 210 and p. 218] proved that each ultraweakly continuous linear functional on VN(G) is of this form. Furthermore A(G) with pointwise multiplication and norm
\[ ||φ|| = ||φ(x)|| : x \in VN(G) \text{ and } ||x|| \leq 1 \]
is a commutative Banach algebra called the Fourier algebra of G. When G is commutative, then A(G) = L1(G), where G is the dual group of G (see [13, p. 209]).

The two Banach algebras L1(G) and A(G) are important topological algebraic objects in the study of harmonic analysis on locally compact groups. They have been shown to have deep relation with the structure of the underlying group G (see Wendel [40] and Walter [39]). They also share a crucial common property: each of them is the predual of a W*-algebra and the identity of the W*-algebra is in the spectrum of the Banach algebra.

In this paper, we shall introduce and study a class of Banach algebras, called F-algebras, that includes the algebra L1(G) and A(G) of a locally compact group G. Roughly speaking, an F-algebra is a Banach algebra A which is the predual of a W*-algebra M (not necessarily unique) and the identity of M is in the spectrum of A. The class of F-algebras includes the Fourier-Stieljes algebra B(G) of a locally compact group G and the measure algebra M(S) of a locally compact semigroup S. It also includes the class of convolution measure algebras studied by Taylor [36], [37], [38]; the class of L-algebras (for which the identity of the dual algebra is in the spectrum of the L-algebra) considered by McKilligan and White [28]; predual algebra of a Hopt von Neumann algebra [35] and the measure algebra of a locally compact hypergroup [11] or semi-convoses [21]. Our approach provides a unified treatment in the study of various aspects of the specific Banach algebras mentioned above.

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2. Preliminaries and some notations. Let E be a linear space, and φ be a linear functional on E, then the value of φ at an element x in E will be written as φ(x) or (φ, x). If F is a subspace of the algebraic dual of E, then σ(E, F) will denote the weakest locally convex topology on E such that each of the functionals in F is continuous.

If K is a subset of a normed linear space E, the closure of K will be denoted by K when the closure is taken with respect to the norm topology, or by K when the closure is taken with respect to topology τ on E different from the norm topology. The continuous dual of E will be denoted by E*.

If M is a W*-algebra, then M* will denote its unique predual. The topology σ(M*, M) on M will be referred to as the ultraweak, or simply the σ-topology.

Let A be a Banach algebra. By the reversed algebra of A, denoted by A', we shall mean the Banach algebra with the same underlying Banach space as A and the multiplication reversed. The spectrum of A will be denoted by σ(A).

If A is a normed algebra, then for each φ ∈ A, and x ∈ A*, define the elements φ·x and x·φ in A* by
\[ \langle x·φ, γ \rangle = \langle x, φ·γ \rangle \quad \text{and} \quad \langle φ·x, γ \rangle = \langle x, γ·φ \rangle \]
for each γ ∈ A. We say that a subspace X of A* is topologically left (resp. right) invariant if X·φ ⊆ X (resp. φ·X ⊆ X) for each φ ∈ A; X is topologically invariant if it is both left and right topologically invariant.

If X is a topologically left invariant subspace of A* and m ∈ X*, we define an operator m_E from X into A* by
\[ \langle m_E(x), φ \rangle = \langle x·φ, m \rangle \quad \text{for each } φ ∈ A. \]

We say that X is topologically left introverted if m_E(X) ⊆ X for each m in X*. Similarly, we can define topologically right introverted subspaces of A*. A subspace X of A* is topologically introverted if it is both left and right topologically introverted.

In [1], Arens defined a product on the second conjugate space A** by
\[ \langle m ⊙ n, x \rangle = \langle m, n_L(x) \rangle \quad \text{for each } m, n ∈ A**, x ∈ A*. \]

Then A** with resp. to this product becomes a Banach algebra. If X is a topologically left invariant and left introverted subspace of A*, then the Arens product on X makes sense and renders X* into a Banach algebra.

3. F-algebras. By an F-algebra we shall mean a pair (A, M) such that A is a complex Banach algebra and M is a W*-algebra such that A = M*, the predual of M, and the identity of M is a multiplicative linear functional on A. If there is no confusion, we shall simply say that A is an F-algebra and we shall identify A* with M. The identity of A* will be denoted by e. Also P(A) will denote the cone of all positive functionals in A and P(1)(A) will denote the set of all φ in P(A) such that φ(e) = 1.

Note that the W*-algebra M of an F-algebra (A, M) need not be unique. In fact, let L be a W*-algebra such that the reversed algebra M' is not W*-isomorphic to M (see [6]). Define on A = M* = (M')_a the multipli-
As well known, the dual of $A_1 \oplus A_2$ can be identified with $(A_1)^* \times (A_2)^*$ with norm $\|x_1, x_2\| = \max \{\|x_1\|, \|x_2\|\}$, $x_i \in (A_i)^*$. In fact, if $(x_1, x_2) \in (A_1)^* \times (A_2)^*$ and $(\varphi_1, \varphi_2) \in A_1 \oplus A_2$, then

$$\langle x_1, x_2, (\varphi_1, \varphi_2) \rangle = \varphi_1(x_1) + \varphi_2(x_2).$$

**Proposition 3.4.** Let $A_1$ be a normed algebra and $A_2$ be any F-algebra. Then

(a) $A_1$ has a (right, left, two-sided) identity if and only if $A_1 \oplus A_2$ has a (right, left, two-sided) identity.

(b) $A_1$ has a (right, left, two-sided) bounded approximate identity if and only if $A_1 \oplus A_2$ has a (right, left, two-sided) bounded approximate identity.

**Proof.** We shall prove (b). The proof of (a) is similar. Let $(\varphi_{x_0})$ be a bounded right approximate identity for $A_2$. Then $\lim_{x \to x_0} \varphi_{x_0}(x) = 1$. Hence if $(\varphi_1, \varphi_2) \in A_1 \oplus A_2$, then

$$\|\varphi_1(x), \varphi_2(x) - (\varphi_1, \varphi_2)\| = \|\varphi_{x_0}(x) - 1\| + \|\varphi_{x_0}(x) - \varphi(x)\|$$

which converges to zero. Consequently, $(0, \varphi_{x_0})$ is a bounded right approximate identity for $A_1 \oplus A_2$.

Conversely, if $(\varphi_{x_0}, \varphi_{x_2})$ is a bounded right approximate identity for $A_2$, and $\varphi_2 \in P_1(A_2)$, then

$$\|\varphi_{x_0}(x), \varphi_{x_2}(x) - (0, \varphi_{x_0})\| = \|\varphi_{x_0}(x) - 1\| = \|\varphi_{x_0}(x) - \varphi(x)\|$$

which converges to zero. Hence $\|\varphi_{x_0}(x) - \varphi_{x_2}(x)\|$ also converges to zero. Consequently $(\varphi_{x_0})$ is a bounded right approximate identity for $A_2$. The other cases can be proved similarly.

**Proposition 3.5.** If $A_1$ is a normed algebra and $A_2$ is any F-algebra, then the Banach algebra $(A_1)^* \oplus (A_2)^*$ is isometric and algebra isomorphic to the second conjugate algebra $(A_1 \oplus A_2)^{**}$.

**Proof.** Note that $(A_1)^{**}$ is an F-algebra by Proposition 3.2. Define a linear map $J: (A_1)^{**} \otimes (A_2)^{**} \to (A_1 \oplus A_2)^{**}$ by

$$J(m_1, m_2, (x_1, x_2)) = m_1(x_1) + m_2(x_2)$$

for each $m_i \in (A_i)^{**}$, $x_i \in A_i$, $i = 1, 2$. Then $J$ is a linear isometry from $(A_1)^{**} \otimes (A_2)^{**}$ onto $(A_1 \oplus A_2)^{**}$. Furthermore, a routine calculation shows that $J$ is even an algebra homomorphism.

If $X_1, \ldots, X_n$ are topological spaces, let $X_i$ denote the $n$-tuple $(0, \ldots, 0, x_i, 1, \ldots, 1)$ with $x_i \in X_i$ appearing in the $i$th coordinate. Equip $X_i$ with the topology $\tau_i$ induced from $X_i$. By the direct sum of spaces $X_1, \ldots, X_n$, denoted by $X_1 \oplus \cdots \oplus X_n$, we shall mean the set $X = \bigcup X_i$ with
the topology $\tau$ on $X$ consisting of all subsets $0$ of $X$ such that $0\cap \hat{x}_i \in \tau$ for each $i = 1, \ldots, n$.

Let $A_1$ be a normed algebra, and $A_1, \ldots, A_n$ be $F$-algebras. We define inductively the direct sum $A_1 \oplus \cdots \oplus A_n$ by:

$A_1 \oplus \cdots \oplus A_n = (A_1 \oplus \cdots \oplus A_{n-1}) \oplus A_n.$

It is easy to see in this case that if $\phi = (\phi_1, \ldots, \phi_n)$ and $\psi = (\psi_1, \ldots, \psi_n)$ are elements in $A_1 \oplus \cdots \oplus A_n$, and $\phi * \psi = (\gamma_1, \ldots, \gamma_n)$, then

$\gamma_k = \phi_k * \psi_k + \left( \sum_{i=1}^{n} \phi_{k} \psi_{i} \right) \theta_k + \left( \sum_{i=1}^{n} \psi_{k} \phi_{i} \right) \theta_k$

where $\theta_k$ is the identity of $(A_k)^*$. Also $(A_1 \oplus \cdots \oplus A_n)^*$ consists of all elements in $(A_1)^* \times \cdots \times (A_n)^*$ of the form $(0, \ldots, 0, \theta_k, 0, \ldots, 0)$ where $\theta_k \in o(A_k)$.

\begin{proposition}
Let $A_i = (A_1 \oplus \cdots \oplus A_n)$ and $M = M_1 \times \cdots \times M_n$. Then $(A_i, M)$ is also an $F$-algebra. Furthermore $\sigma(A)$ is homeomorphic to $\sigma(A_1) \oplus \cdots \oplus \sigma(A_n)$.
\end{proposition}

\begin{proof}
For $n = 2$, define a map $h$ from

$\sigma(A_1 \oplus A_2) = (\sigma(A_1) \cup \{0, x_2\})$ \cup \{0, x_2 \}$

into $(A_1 \oplus A_2)^*$ by

$h(x_1, 1) = (x_1, e_2)$ and $h(0, x_2) = (0, x_2)$

for each $x_1 \in o(A_1)$ and $x_2 \in o(A_2)$. Then $h$ is a homeomorphism from $\sigma(A_1 \oplus A_2)$ onto $\sigma(A_1 \oplus A_2)$. The general case follows by induction.

\end{proof}

4. Left amenable $F$-algebras. Let $A$ be a Banach algebra. By a left \textit{Banach $A$-module} we shall mean a Banach space $X$ equipped with a bounded bilinear map from $A \times X \to X$, denoted by $(\phi, x) \mapsto \phi \cdot x$, $\phi \in A$, $x \in X$, such that $\phi_1 \cdot (\phi_2 \cdot x) = (\phi_1 \cdot \phi_2) \cdot x$ for all $\phi_1, \phi_2 \in A$, $x \in X$. A right Banach $A$-module is defined similarly. A two-sided Banach $A$-module is a left and right $A$-module such that

$(\phi_1 \cdot x) \cdot \phi_2 = \phi_1 \cdot (\phi_2 \cdot x)$ for all $\phi_1, \phi_2 \in A, x \in X$.

If $X$ is a two-sided Banach $A$-module, then $X^*$ becomes a two-sided Banach $A$-module with

$\langle \phi \cdot f, x \rangle = \langle f, x \cdot \phi \rangle$ and $\langle f \cdot \phi, x \rangle = \langle f, \phi \cdot x \rangle$

for all $f \in X^*$, $\phi \in A$.

Let $X$ be a two-sided Banach $A$-module. A \textit{derivation} from $A$ into $X^*$ is a linear map $D: A \to X^*$ such that

$D(\phi \cdot \psi) = D(\phi) \cdot \psi + \phi \cdot D(\psi)$

for all $\phi, \psi \in A$. Clearly if $f \in X^*$, then the map $D_f: A \to X^*$ defined by

$D_f(\phi) = \phi \cdot f - f \cdot \phi$

is a bounded derivation. Any derivation of this form is called an \textit{inner derivation}.

A Banach algebra $A$ is \textit{amenable} if for any two-sided Banach $A$-module $X$, any bounded derivation from $A$ into $X^*$ is an inner derivation. By Johnson proved in [22, Theorem 2.5] that a locally compact group $G$ is amenable if and only if the algebra $L_1(G)$ is amenable. The class of amenable Banach algebras has been studied extensively by Johnson in [22], [23] and Bunce in [3], [4] and [5].

However, Johnson’s theorem [22, Theorem 2.5] is no longer valid for semigroups. In fact, the semigroup $S$ of positive integers with addition is amenable, but the Banach algebra $l_1(S)$ is $\sigma$-finite defined as in [7] or [18] is not amenable [2, p. 244].

Let $A$ be an $F$-algebra. A \textit{topological left invariant mean} (abbreviated as TLIM) on $A^*$ is an element $m$ in $P_1(A^{**})$ such that

$m(\phi x) = m(x)$ for each $\phi \in P_1(A), x \in A^*$.

The set of TLIM on $A^*$ will be denoted by $TLIM(A^*)$. The notion of TLIM has been considered by many authors for various special cases of $A$ (see for example [5], [20], [33], [42]).

We say that an $F$-algebra $A$ is \textit{left amenable} if for any two-sided Banach $A$-module $X$ such that $\phi \cdot x = \phi(e) x$ for all $\phi \in A, x \in X$, every bounded derivation from $A$ into $X^*$ is inner.

\begin{theorem}
Let $A$ be an $F$-algebra. Then $A^*$ has TLIM if and only if $A$ is left amenable.
\end{theorem}

\begin{proof}
Let $m$ be a TLIM on $A^*$ and let $X$ be a two-sided Banach $A$-module with $\phi \cdot x = \phi(e) x$ for all $\phi \in A, x \in X$. Let $D$ be a bounded derivation from $A$ into $X^*$. Let $L$ be the restriction of $D^*$ to $X$, and let $f = D^* (m)$. We shall show that $D = D_{-f}$. Indeed, if $x \in X$, $\phi \in P_1(A)$ and $\psi \in A$, then

$\langle L(\phi x), \psi \rangle = \langle x \cdot \phi, D(\psi) \rangle = \langle x, D(\phi \cdot \psi) - D(\phi) \cdot \psi \rangle$

$\quad = \langle L(x), \phi \cdot \psi - \psi \phi \rangle.$

\end{proof}
Proposition 4. [p. 238] shows that there exists a non-zero \( n \in A^{**} \) such that \( n(x \cdot \varphi) = \varphi(e)(e)(x) \) for each \( \varphi \in A, x \in A^{*}. \) Hence \( \varphi \otimes n = n \) and \( \varphi \otimes n^{*} = n^{*} \) for all \( \varphi \in P_{1}(A). \) So we may assume that \( n \) is self-adjoint. Write \( n = n^{*} - n^{*} \), the orthogonal decomposition of \( n. \) If \( \varphi \in P_{1}(A), \) then \( \varphi \otimes n = \varphi \otimes n^{*} - \varphi \otimes n^{*}. \) Since \( \varphi \otimes n^{*}, \varphi \otimes n^{*} \) are positive and

\[
\|\varphi \otimes n^{*} + \varphi \otimes n^{*}\| = \|\varphi \otimes n^{*}\| = \|\varphi \otimes n^{*}\| = \|\varphi \otimes n^{*}\| = \|\varphi \otimes n^{*}\| = \|\varphi \otimes n^{*}\|
\]

it follows that \( \varphi \otimes n^{*} = n^{*} \) and \( \varphi \otimes n^{*} = n^{*} \) [34, Theorem 1.14.3]. Consequently, if \( n^{*} \neq 0 \) (say) and \( m = n^{*} - n^{*} \), then \( m \) is a TLIM on \( A^{*}. \)

The following is an analogue of Johnson's theorem [22, Theorem 2.5.2]:

**Corollary 4.2.** A semigroup \( S \) is left amenable if and only if the Banach algebra \( l_{1}(S) \) is left amenable.

**Corollary 4.3.** A locally compact group \( G \) is amenable if and only if the measure algebra \( M(G) \) is left amenable.

**Proof.** This follows from Theorem 4.1 and proof of Lemma 5.1 in [43].

**Examples.**

(1) Any commutative \( F \)-algebras are left (and right) amenable. In particular, the algebras \( A(G) \) and \( B(G) \) of any locally compact group and the measure algebra \( M(S) \) of any commutative locally compact semigroup \( S \) are left amenable.

In fact, if \( A \) is commutative, we consider the commutative semigroup \( \mathcal{S} = (s_{0}, \varphi \in P_{1}(A)) \) of affine continuous maps on the compact convex set \( (P_{1}(A^{*})), \text{weak}^{*}) \) into itself defined by \( \varphi \otimes m \) for each \( \varphi \in P_{1}(A), m \in P_{1}(A^{*}). \) Since \( A \) is commutative, \( \mathcal{S} \) is also commutative. Hence by the Markov-Kakutani fixed point theorem [9, p. 456], \( P_{1}(A^{*}) \) contains a common fixed point \( m \) for \( \mathcal{S}. \) Clearly \( m \in TLIM(A^{*}). \)

(2) Let \( M \) be a \( W^{*} \)-algebra and \( A = M_{s}. \) Then \( A \) is amenable if and only if \( A \) is amenable:

(i) left amenable if multiplication on \( A \) is defined by \( \varphi \cdot \psi = \psi(e)(e) \);  
(ii) right amenable if multiplication on \( A \) is defined by \( \varphi \cdot \psi = \psi(e)(e) \);  
(iii) both left and right amenable (but not amenable) if multiplication on \( A \) is defined by \( \varphi \cdot \psi = \psi(e)(e) \).

**Proposition 4.4.** Let \( A \) be an \( F \)-algebra. Then \( A \) is left amenable if and only if \( A^{**} \) is amenable.

**Proof.** Let \( m \in TLIM(A^{**}). \) If \( \varphi \in P_{1}(A^{**}), \) choose a net \( \varphi_{n} \in P_{1}(A) \) converging to \( \varphi \) in the weak* topology. Then for each \( x \in A^{*}, \) we have

\[
m(x) = \lim m(x \cdot \varphi_{n}) = \lim \varphi_{n}(m_{n}(x)) = m_{n}(x)
\]

hence \( m \in TLIM(A^{**}). \) Conversely, if \( \gamma \in TLIM(A^{**}), \) then \( \gamma \) restricted to \( A^{*} \) can easily be seen to be a TLIM on \( A^{*}. \)

**Proposition 4.5.** Let \( A_{1} \) and \( A_{2} \) be \( F \)-algebras. Then \( A_{1} \oplus A_{2} \) is left amenable if and only if \( A_{1} \) is left amenable in particular \( C \oplus A \) is left amenable for any \( F \)-algebra \( A. \)

**Proof.** If \( m_{1} \in TLIM(A_{1}) \) and \( \varphi_{1}, \varphi_{2} \in P_{1}(A_{1} \oplus A_{2}), \) then by Proposition 3.5, we have

\[
(\varphi_{1}, \varphi_{2}) \otimes (m_{1}, 0) = (\varphi_{1}(e)(m_{1} + \varphi_{2}(e) m_{2}, 0) = (\varphi_{1}(e)(m_{1} + \varphi_{2}(e) m_{2}, 0) = (m_{1}, 0).
\]

Hence \( (m_{1}, 0) \in TLIM((A_{1} \oplus A_{2})). \) Conversely, if \( (m_{1}, m_{2}) \) is a TLIM on \( (A_{1} \oplus A_{2})^{*}, \) let \( \varphi_{1} \in P_{1}(A_{1}), \) then \( (\varphi_{1}, 0) \in P_{1}(A_{1} \oplus A_{2}), \) and hence

\[
(m_{1}, m_{2}) = (\varphi_{1}, 0) \otimes (m_{1}, m_{2}) = (\varphi_{1}(m_{1}, m_{2})) = (m_{1}, m_{2})
\]

by Proposition 3.5. Consequently \( m_{2} = 0 \) and \( m_{1} \in \varphi \otimes m_{1}. \) So \( m_{1} \in TLIM(A_{1}). \)

**Theorem 4.6.** Let \( A \) be an \( F \)-algebra. Then the following are equivalent:

(a) \( A \) is left amenable.

(b) There exists a net \( \varphi_{n} \in P_{1}(A) \) such that \( \|\varphi_{n} \cdot \varphi_{n} - \varphi_{n}\| \to 0 \) for each \( \varphi \in P_{1}(A). \)

(c) For each \( x \in A^{*}, \) the set \( \overline{K(x)} \) contains \( \lambda \varepsilon \) for some complex number \( \lambda, \) where \( K(x) = \{x \mid x \in P_{1}(A) \}. \)

In this case \( \lim K(x) \) is and only if there exists a TLIM on \( A^{*} \) such that \( m(x) = \lambda \).

**Proof.** The equivalence of (a) and (b) can be proved by an argument similar to that of Namioka's elegant proof of Day's theorem [30, Theorem 2.2].

If (a) holds and \( m \) is a TLIM on \( A^{*}, \) let \( \{\varphi_{n}\} \) be a net in \( P_{1}(A) \) converging to \( m \) in the weak* topology. Then for each \( x \in A^{*}, \) the net \( \{\varphi_{n}(x)\} \) converges to \( m(x) \) in the \( \sigma \)-topology. Hence (c) holds.

That implies (a) follows easily from [26, Theorem 2.1] by considering the semigroup \( S = \{T_{\varphi} \mid \varphi \in P_{1}(A) \} \) of \( \sigma \)-continuous operators on \( A^{*} \) defined by \( T_{\varphi}(x) = \varphi(x), \varphi \in P_{1}(A), x \in A^{*}. \)

For the algebra \( L_{1}(S) \) of a discrete semigroup \( S, \) the equivalence of (a) and (b) was established by Day [7], and the equivalence of (a) and (c) and the last statement is due to Mitchell [29]. Our theorem implies [41, Theorem 5.4 with \( X = L_{w}(G) \)] and [42, Theorem 3.1 (1)> (2) = (3)] of Wong. Condition (c) has also been considered by Dunkl-Ramirez for the Fourier algebra of a locally compact group [10, Theorem 2].

Given an \( F \)-algebra \( A, \) let \( I_{0}(A) = \{\varphi \in A \mid \varphi(e) = 0\}. \) Then \( I_{0}(A) \) is a closed two-sided ideal in \( A. \)

**Theorem 4.7.** Let \( A \) be an \( F \)-algebra. Then the following are equivalent:

(a) \( A \) is left amenable.

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(1) In [26] "\( = \)" in condition (1) should be replaced by "\( \leq \).\)
(b) There exists a net \( \varphi_n \in P_1(A) \) such that \( \lim \| \psi \cdot \varphi_n \| = \| \psi \| \) for each \( \psi \in A \).

(c) For each \( \psi \in I_0(A) \) and \( \varepsilon > 0 \), there exists \( \varphi \in P_1(A) \) such that \( \| \psi \cdot \varphi \| < \varepsilon \).

**Proof.** (a) \( \Rightarrow \) (b). If \( A \) is left amenable, then there exists a net \( \{ \varphi_n \} \in P_1(A) \) such that \( \| \psi \cdot \varphi_n \| \rightarrow 0 \) for each \( \psi \in P_1(A) \) (Theorem 6.9). Let \( \varphi \in A \) and write \( \psi = \sum \lambda_i \varphi_i \), where \( \varphi_i \in P_1(A) \). Then \( \| \psi \| = \| \psi \cdot \varphi \| < \varepsilon \). Given \( \varepsilon > 0 \), choose \( a_0 \) such that if \( x > a_0 \), then \( \| \psi \cdot \varphi - \varphi \|_A < \varepsilon/n \lambda_i \) (of course we may assume that \( \lambda_i \neq 0 \)). Then

\[
\| \psi \cdot \varphi \| \leq \| \lambda_i \varphi_i \| \leq \sum_i \| \psi \cdot \varphi \|_A + \frac{1}{\varepsilon} < \varepsilon + \| \psi \| \leq \varepsilon \]

for all \( x \geq a_0 \). On the other hand

\[
\| \psi \| - \| \psi \cdot \varphi \| \leq \| \psi \|_A \| \varphi \|_A < \varepsilon \]

for all \( x \geq a_0 \).

(b) \( \Rightarrow \) (b) is clear.

(c) \( \Rightarrow \) (a). As in the proof of [17, Theorem 3.7.3] let \( \eta_\sigma \in P_1(A) \) be fixed. Given \( \varepsilon > 0 \), and \( \varepsilon = \| \hat{\psi} \| \) a finite subset of \( P_1(A) \), let \( \psi_\sigma = \psi_1 \cdot \eta_\sigma - \eta_0 \). Then \( \psi_\sigma \in I_0(A) \). Hence we may find \( \eta_1 \in P_1(A) \) such that \( \| \psi_\sigma - \eta_1 \| < \varepsilon \).

Now \( \psi_2 = \psi_2 \cdot \eta_0 - \eta_0 \cdot \eta_1 \) is in \( I_0(A) \). So we may find \( \eta_2 \in P_1(A) \) such that \( \| \psi_2 - \eta_2 \| < \varepsilon \). Inductively we may find \( \eta_r \in P_1(A) \) such that \( \| \psi_r - \eta_r \| < \varepsilon \) where

\[
\psi_r = \psi_1 \cdot \eta_0 \cdot \eta_1 \cdot \eta_2 \cdots \eta_r - \eta_0 \cdot \eta_1 \cdots \eta_r.
\]

Let \( n_{\text{for} \sigma} = \eta_0 \cdot \eta_1 \cdots \eta_r \). Then

\[
\| \psi - \eta_{\text{for} \sigma} \| < \varepsilon
\]

for all \( \sigma \in \sigma \). So any weak* cluster point of the net \( \{ \eta_{\text{for} \sigma} \} \) is a TLIM on \( A^\ast \).

**Corollary 4.8.** Let \( A \) be an F-algebra. Then \( A \) is left amenable if and only if \( \| \psi \| = \inf \{ \| \psi \cdot \varphi \| : \varphi \in P_1(A) \} \) for each \( \psi \in A \).

Corollary above is due to Reiter for \( A = L_1(G) \) of a locally compact group \( G \) [17, section 3.7], and to Wong [42, Theorem 3.1 (2) \( \Rightarrow \) (4)] for \( A = M(S) \) of a locally compact semigroup \( S \).

Since the Fourier algebra \( A(G) \) of a locally compact group \( G \) is always left amenable, we have the following analogue of Reiter's result:

**Corollary 4.9.** For any locally compact group \( G \), and \( \psi \in A(G) \), we have

\[
\| \psi \| = \inf \{ \| \psi \cdot \varphi \| : \varphi \in A(G) \cap P(G) \text{ and } \varphi(u) = 1 \},
\]

where \( u \) is the identity of \( G \).

Our next result is also due to Reiter [32] for \( A = L_1(G) \) of a locally compact group \( G \) (see also [22, Proposition 2.6]). Note that Reiter's notion of bounded right approximate identity is slightly different from ours. However, a Banach algebra \( B \) has a bounded right approximate identity as defined in [32] if and only if \( B \) has a bounded right approximate identity \( \{ \varphi_n \} \) in the usual sense, i.e. \( \varphi_n \) is a bounded net such that \( \| \psi \cdot \varphi_n \| \rightarrow 0 \) for each \( \varphi \in B \) (see [2, p. 53]).

**Theorem 4.10.** Let \( A \) be an F-algebra. Then \( I_0(A) \) has a bounded right approximate identity if and only if \( A \) is left amenable and has a bounded right approximate identity.

**Proof.** Assume that \( \psi \in I_0(A) \) is a bounded right approximate identity for \( I_0(A) \). Let \( \varphi \in P_1(A) \) be fixed. Form the net

\[
\theta_n = \varphi_0 \cdot \psi_n - \varphi_0.
\]

Then \( \{ \theta_n \} \) is also bounded, and \( \theta_n(u) = -1 \) for each \( u \). Also, if \( \varphi \in P_1(A) \), then

\[
\| \psi_0 \cdot \varphi - \theta_n \| = \| \psi_0 \cdot \varphi - \psi \cdot \varphi_0 \| = 0
\]

since \( \psi_0 \cdot \varphi_0 \in I_0(A) \). Let \( \eta \) be a weak* cluster point of \( \{ \psi_n \} \). Then \( \eta \) is non-zero since \( \psi_0 \eta = -1 \), and \( \varphi \eta = 0 \) for each \( \varphi \in P_1(A) \). An argument similar to that for the proof of Theorem 4.1 shows that \( A \) is a TLIM in \( M(S) \). Let \( A \) be left amenable.

Let \( r \) be a weak* cluster point of \( \{ \psi_n \} \) in \( A^** \). Then for each \( \varphi \in P_1(A) \),

\[
\varphi \circ (m + r - m \circ r) = \varphi \circ (m + r - (m \circ r) = m + r \circ m - m \circ r = m + (r - m) = (m - r)
\]

since \( \varphi - m \) is the weak* limit of a net in \( I_0(A) \). So \( m + r - m \circ r \) is a right identity in \( A^** \). Consequently \( A \) has a bounded right approximate identity [2, p. 146].

Conversely if \( A \) is left amenable and \( A \) has a bounded right approximate identity \( \{ \varphi_n \} \), let \( m \) be a TLIM on \( A^* \) and \( \varphi \) be a weak* cluster point of the net \( \{ \varphi_n \} \) in \( A^** \). Then \( \varphi(u) = 1 \). Let \( q = \varphi - m \). Clearly \( I_0(A) = I_0(A^** \).

Also if \( n \in I_0(A^** \), then \( n \circ m = 0 \). Indeed, if \( n = \sum \lambda_i n_i \), then \( n(u) = \sum \lambda_i n_i(u) = \sum \lambda_i m = 0 \).
by the proof of Proposition 4.4. Consequently, if \( n \in I_\emptyset(A^{**}) \), we have

\[
n\emptyset = n\emptyset = n\emptyset = n\emptyset = n\emptyset = n\emptyset = n.
\]

Since \( n \in I_\emptyset(A^{**}) \), it follows that \( n \) is a right ideal in \( I_\emptyset(A) \). However \( I_\emptyset(A^{**}) \) can be identified with the second conjugate algebra of \( I_\emptyset(A) \) with the Arens product, it follows that \( I_\emptyset(A) \) has a bounded right approximate identity.

The following is an analogue of Reiter's result [32]:

**Corollary 4.11.** Let \( G \) be a locally compact group and let \( u \) be the identity of \( G \). Then \( G \) is amenable if and only if \( \{ \phi \in \mathcal{A}(G) : \phi(u) = 0 \} \) in \( \mathcal{A}(G) \) has a bounded approximate identity.

**Proof.** This follows from Theorem 4.10 and the fact that \( G \) is amenable if and only if \( A(G) \) has a bounded approximate identity (see Loeptin [27]).

Given an \( F \)-algebra \( A \), let \( N(A) \) denote all \( x \in A^* \) such that \( \inf \|\phi \cdot x\| : \phi \in \mathcal{P}(A) \) = 0. Then as readily checked, \( N(A) \) is closed under scalar multiplication. Furthermore, \( N(A) \) includes all elements of the form

\[
\psi \cdot x, \psi \in \mathcal{P}(A), \quad x \in A^*.
\]

In fact, if \( n = 1, 2, \ldots \), let \( \phi_n = (1/n) \sum_{i=1}^{n} \psi_i \).

Then \( \phi_n \in \mathcal{P}(A) \) and \( \|\phi_n \cdot x\| = (1/n) \|\psi \cdot x\| \leq \|\psi \cdot x\| \rightarrow 0 \) as \( n \rightarrow \infty \).

If \( I_1, I_2 \subseteq \mathcal{P}(A) \), let \( d(I_1, I_2) = \inf \|\phi \cdot x\| : \phi \in \mathcal{P}(A) \cap I_1 \) and \( \phi \in I_2 \). The following is an analogue of Theorem 1.7 of Emerson [12]. It also implies Theorem 2.1.7 of Razi [31] when \( A \) is the mesure algebra of a locally compact semigroup.

**Theorem 4.12.** Let \( A \) be an \( F \)-algebra. The followings are equivalent:

(a) \( A \) is left amenable.

(b) \( N(A) \) is closed under addition.

(c) \( d(I_1, I_2) = 0 \) for any two right ideals \( I_1, I_2 \) of the semigroup \( \mathcal{P}(A) \).

Proof. (a) \( \Rightarrow \) (b). Assume that \( \psi \in \mathcal{P}(A) \) such that \( \inf \|\phi \cdot x\| \rightarrow 0 \) for all \( \phi \in \mathcal{P}(A) \) (by Theorem 4.6). Hence if \( \phi \in \mathcal{F} \), then \( \|\phi \cdot \psi \| \leq \|\phi \cdot x\| \rightarrow 0 \).

(b) \( \Rightarrow \) (c). Let \( x_1, x_2 \in N(A) \) and \( \varepsilon > 0 \). Choose \( \phi_1, \phi_2 \in \mathcal{P}(A) \) such that \( \|\phi_1 \cdot x_1\| \leq \varepsilon \) and \( \|\phi_2 \cdot x_2\| \leq \varepsilon \). Pick \( \psi_1, \psi_2 \in \mathcal{P}(A) \) such that \( \|\phi_1 \cdot \psi_1 \| = \|\phi_2 \cdot \psi_2 \| \leq \varepsilon \). Then

\[
\|\phi_1 \cdot (x_1 + x_2)\| \leq \|\phi_1 \cdot x_1\| + \|\phi_1 \cdot x_2\| + \|\phi_2 \cdot x_1\| + \|\phi_2 \cdot x_2\| \leq \varepsilon (2 + \|x_1\|).
\]

Hence \( x_1 + x_2 \in N(A) \).

(b) \( \Rightarrow \) (a). If (b) holds, then \( N(A) \) is a right \( A \)-subspace. Let \( \phi \cdot x \in N(A) \) for any \( x \in A^* \) and \( \phi \in \mathcal{P}(A) \). Let \( E \) be the self-adjoint elements in \( A^* \) and \( E \) is a real vector subspace of \( A^* \). Let \( K \) denote all \( x \in E \) such that \( \inf \|\psi \cdot x\| : \phi \in \mathcal{P}(A) \geq 0 \). Then \( K \) is open in \( E \), \( e \in K \) and \( K \cdot N(A) = 0 \). By

a Hahn Banach separation theorem, there exists a continuous (real) linear functional \( \theta \) on \( E \) such that \( \theta(e) = 1 \) and \( \theta(x) = 0 \) for all \( x \in E \). In particular, \( \theta \cdot x \cdot E = 0 \) for all \( \phi \in \mathcal{P}(A) \) and \( x = x \). Define \( n(x) = \theta(u) + \theta(x) \) when \( x = x + u \), \( u \in E \). Then \( n \in A^* \), \( n(e) = 1 \) and \( n \) is topologically left invariant. An argument similar to that for Theorem 4.1 shows that \( A^* \) has a TLIM.

A semigroup \( S \) is left reversible if any two right ideals in \( S \) have nonempty intersection. Commutative semigroups (or more generally left amenable semigroups) and groups are left reversible.

**Corollary 4.13.** Let \( A \) be an \( F \)-algebra. If \( \mathcal{P}(A) \) is left reversible, then \( A \) is left amenable.

Finally we state a few facts concerning the set TLIM\((A^*)\) for a left amenable \( F \)-algebra \( A \).

**Proposition 4.14.** If \( A \) is a commutative \( F \)-algebra and \( A^* \) has a TLIM in \( \mathcal{P}(A) \), then \( A^* \) has a unique TLIM.

Proof. If \( n \in \mathcal{P}(A) \) is a TLIM on \( A^* \) and \( n(\phi) = 0 \) for all \( \phi \in \mathcal{P}(A) \), then \( n(\phi) = 0 \) for all \( \phi \in \mathcal{P}(A) \). Hence \( n \) is a continuous linear functional on \( A^* \) and \( n(e) = 1 \). By the Hahn Banach separation theorem, there exists a continuous (real) linear functional \( \theta \) on \( A^* \) such that \( \theta(e) = 1 \) and \( \theta(x) = 0 \) for all \( x \in A^* \). In particular, \( \theta \cdot x \cdot A^* = 0 \) for all \( \phi \in \mathcal{P}(A) \) and \( x = x \). Define \( n(x) = \theta(u) + \theta(x) \) when \( x = x + u \), \( u \in E \). Then \( n \in A^* \), \( n(e) = 1 \) and \( n \) is topologically left invariant. An argument similar to that for Theorem 4.1 shows that \( A^* \) has a TLIM.

Hence \( m = n \).

**Proposition 4.15.** Let \( A \) be an \( F \)-algebra.

(a) If \( A^* \) has a TLIM in \( \mathcal{P}(A) \), then the identity \( e \) of \( A^* \) is an isolated point in the spectrum of \( A \).

(b) If \( A^* \) has a unique TLIM and \( A \) is norm separable, then \( A^* \) has a TLIM in \( \mathcal{P}(A) \).

Proof. (a) Assume that \( \psi \in \mathcal{P}(A) \) is a TLIM. Let \( x \in \sigma(A) \) and \( x \neq e \). Choose \( \phi \in \mathcal{P}(A) \) such that \( \phi(x) = \psi(x) \). Then \( \psi(x) = \phi(x) \psi(x) = \phi(x) \psi(x) \). Consequently \( \psi(x) = 0 \). Hence \( \|\phi \cdot \psi \| \geq \|\phi \cdot \psi \| = 0 \) and \( e \) is isolated.

The proof of part (b) is similar to that of Granirer [16, Theorem 7], we omit the details.

**Corollary 4.16.** If \( A \) is a norm separable commutative \( F \)-algebra, then \( A^* \) has a unique TLIM if and only if \( A^* \) has a TLIM in \( \mathcal{P}(A) \).

Proof. This follows from Propositions 4.14 and 4.15.

In the case that \( A = \mathcal{I}(S) \) of a discrete semigroup \( S \), or that \( A \) is either the Fourier algebra \( A(G) \) or the group algebra \( \mathcal{I}(G) \) of a locally compact group \( G \), much stronger results then Proposition 4.15 are known to hold (see Granirer [14], [15], Klawe [25], Renault [33]).
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