Mappings with 1-dimensional absolute neighborhood retract fibres

by

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Abstract. It is shown that if $f\colon X \to Y$ is a proper surjection between metric spaces where $\dim X \leqslant 2$ and where each $f^{-1}(y)$ is an absolute neighborhood retract having dimension $\leqslant 1$, then Y is countable dimensional. Two corollaries are: (i) if $f\colon X \to Y$ is a proper surjection between complete metric spaces where $\dim X \leqslant 2$ and where each $f^{-1}(y)$ is an absolute retract having dimension $\leqslant 1$, then $\dim Y \leqslant 2$; and (ii) if $f\colon S^3 \to Y$ is a surjection where each $f^{-1}(y)$ is an absolute neighborhood retract having dimension $\leqslant 1$, then Y is countable dimensional.

1. Introduction. The 1956 paper of E. Dyer [Dy_1] implicity contains the following result: if $f\colon X \to Y$ is an open map between compacta and there is an $\varepsilon > 0$ such that each $f^{-1}(y)$ is a 1-dimensional ANR (absolute neighborhood retract) containing no simple closed curve of diameter less than ε, then the cohomological dimension of Y is one less than the cohomological dimension of X. The fact that cohomological dimension and covering dimension are known to agree when the former is $\leqslant 1$ yields: if $f\colon X \to Y$ is an open cell-like map between compacta with each $f^{-1}(y)$ a 1-dimensional AR (absolute retract) or a point and $\dim X \leqslant 2$, then $\dim Y \leqslant 2$. In turn a reduction described in [KW_2] based on results from [Si] yields: if $f\colon X \to Y$ is an open cell-like map with each $f^{-1}(y)$ a 1-dimensional AR or a point and X is an ANR with $\dim X \leqslant 3$, then $\dim Y \leqslant 3$ or, equivalently, Y is an ANR.

Examples are described below which show that Dyer's result fails dramatically without the assumption that the map is open. In this paper, we supply a "generalization" which remedies this failure.

MAIN THEOREM. *If $f\colon X \to Y$ is a proper surjection between metric spaces where $\dim X \leqslant 2$ and where each $f^{-1}(y)$ is an* ANR *having dimension $\leqslant 1$, then Y is countable dimensional.*

COROLLARY A. *If $f\colon X \to Y$ is a proper surjection between metric spaces where X is complete and $\dim X \leqslant 2$ and where each $f^{-1}(y)$ is an* AR *having dimension $\leqslant 1$, then $\dim Y \leqslant 2$.*

COROLLARY B. *If $f\colon X \to Y$ is a proper surjection where X is a locally compact* ANR *and $\dim X \leqslant 3$ and where each $f^{-1}(y)$ is an* AR *having dimension $\leqslant 1$, then $\dim Y \leqslant 3$ or, equivalently, Y is an* ANR.

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COROLLARY C. *If $f\colon E^3 \to Y$ is a proper surjection where each $f^{-1}(y)$ is an* ANR *having dimension $\leqslant 1$, then Y is countable dimensional.*

Before presenting two widely known examples, we make several comments.

The first two corollaries add to the current knowledge of cell-like maps which do not raise dimension. Corollary B can be viewed as an extension of the result in [KW_1] that cell-like maps defined on 3-manifolds do not raise dimension and of the result in [KRW] that cell-like maps defined on 3-dimensional polyhedra with 1-dimensional point-inverses do not raise dimension. The techniques used in this paper are radically different from those used in the latter two papers.

At the heart of the paper in the following version of the result of Dyer mentioned above which does not assume openness. Let $f\colon X \to Y$ be a proper surjection between metric spaces where each $f^{-1}(y)$ is an ANR having dimension $\leqslant 1$ and where $\dim X \leqslant n$. If Y is not countable dimensional, then there is a subset $Z \subset Y$ and a subset $B \subset f^{-1}(Z)$ with $f(B) = Z$, $\dim Z = \infty$, the cohomological dimension of $Z \leqslant n-1$, and the restriction of f to B a cell-like map.

Even if X is locally compact, the set Z need not be locally compact. It is this fact and not simply an attempt to achieve generality which leads us into the category of proper maps defined on metric spaces.

A comment on the assumption of completeness in Corollary A is in order. A complete countable dimensional space has weak transfinite inductive dimension and dimension agrees with cohomological dimension for spaces of the latter type (see [Ku]).

FIRST EXAMPLE. Let $I = [0, 1]$ and let $\alpha\colon I \to I$ be a surjection whose non-degenerate point-inverses are the arcs which are the closures of the components of the complement of a Cantor set C. Set $X = (I \times \{0\}) \cup (C \times I)$ and define $f\colon X \to I$ by $f(s, t) = \alpha(s)$. Each point-inverse of f is an arc but $\dim X = \dim I = I$.

SECOND EXAMPLE. The basis for this construction is that for each separable metric space Y with $\dim Y \leqslant n$, there is a subset Z of a Cantor set an a proper surjection $\alpha\colon Z \to Y$ which is at most $(n+1)$-to-1 (see [Na]). Let $C * C \subset E^3$ be the join of a pair of Cantor sets contained in non-adjacent edges of a tetrahedron. For α and Z above, set $X = \bigcup \{\alpha^{-1}(y) * \alpha^{-1}(y)\colon y \in Y\}$ and define $f\colon X \to Y$ by requiring that $f(\alpha^{-1}(y) * \alpha^{-1}(y)) = y$. The map f is a proper surjection and each $f^{-1}(y)$ is a 1-complex. Examples where Y is not finite dimensional are obtained by taking a countable disjoint union.

2. Preliminaires. Spaces are assumed to be metric. A space X is *countable dimensional* provided $X = \bigcup_{i=1}^{\infty} Z_i$ and $\dim Z_i \leqslant 0$. A map is *proper* provided the inverse image of each compact set is compact. A map is *cell-like* provided it is proper and the inverse image of each point is a cell-like set or, equivalently, has trivial shape.

The cohomology theory used throughout the paper is Čech theory. Since the coefficient group is always the integers, it is suppressed. The notation c-$\dim X$ is used to denote the *integral cohomological dimension* of X and is defined as follows:

(i) c-$\dim X \leqslant n$ if for each closed subset $A \subset X$ the inclusion induced homomorphism $i^*\colon H^n(X) \to H^n(A)$ is surjective;

(ii) c-$\dim X = n$ if c-$\dim X \leqslant n$ but c-$\dim X \nleqslant n-1$.

The reader is referred to [Ku] for a treatment of cohomological dimension. We content ourselves with listing the following basic facts.

(a) $\dim X \geqslant c$-$\dim X$.

(b) If $\dim X < \infty$, then $\dim X = c$-$\dim X$.

(c) If c-$\dim X \leqslant n$, then c-$\dim X \leqslant n+1$.

(d) If $A \subset X$, then c-$\dim A \leqslant c$-$\dim X$.

(e) c-$\dim X \leqslant 1$ if and only if $\dim X \leqslant 1$.

The compact AR's which have dimension $\leqslant 1$ are precisely the dendrites (= compact, connected, locally connected metric spaces which contain no simple closed curves). A compact ANR which has dimension $\leqslant 1$ is characterized by being a compact, connected, locally connected metric space which, for some $\varepsilon > 0$, does not contain a simple closed curve having diameter $< \varepsilon$. Two facts needed are that each subcontinuum (= compact and connected subset) of a dendrite is a dendrite and that a subcontinuum of a 1-dimensional ANR is a dendrite provided it contains no simple closed curves. The reader is referred to [Wh] for the details.

We remind the reader that, since spaces are not assumed to be separable, it is important to interpret $\dim X$ to be the covering dimension of X. Since metric spaces have σ-discrete bases [Ke; p. 127], an application of the Sum Theorem [Na; p. 14] shows that if a space is not countable dimensional, then each open cover contains a set which is not countable dimensional.

3. A sequence of lemmas.

LEMMA 3.1. *Let $f\colon X \to Y$ be a proper surjection where X is countable dimensional. If Y is not countable dimensional, then there is a closed set $C \subset Y$ and an $\varepsilon > 0$ such that C is not countable dimensional and $\operatorname{diam} f^{-1}(y) \geqslant \varepsilon$ for each $y \in C$.*

Proof. Let $Y_0 = \{y \in Y\colon f^{-1}(y) \text{ is a point}\}$ and, for $n \geqslant 1$, let $Y_n = \{y \in Y\colon \operatorname{diam} f^{-1}(y) \geqslant 1/n\}$. Since the restriction of f yields a homeomorphism between $f^{-1}(Y_0)$ and Y_0, the latter set is countable dimensional. Since $Y = \bigcup_{n \geqslant 0} Y_n$ and Y_0 is countable dimensional, there is an integer $n \geqslant 1$ such that Y_n is not countable dimensional. It is easily checked that Y_n is closed and setting $C = Y_n$ and $\varepsilon = 1/n$ completes the proof.

LEMMA 3.2. *Let $f\colon X \to Y$ be a proper surjection with each $f^{-1}(y)$ an* ANR *having dimension $\leqslant 1$. If Y is not countable dimensional, then there is a set $A \subset X$*

such that the restriction $f|_A$: $A \to f(A)$ *is a proper surjection with each point-inverse an* AR *having dimension* $\leqslant 1$ *and* $f(A)$ *is not countable dimensional.*

Proof. For each pair of positive integers m and n, let $Y_{m,n}$ be the set of points $y \in Y$ satisfying:

(i) each subset of $f^{-1}(y)$ having diameter $< 1/m$ is contained in a subcontinuum of $f^{-1}(y)$ having diameter $< 1/3n$;

(ii) each subcontinuum of $f^{-1}(y)$ having diameter $< 1/n$ is a dendrite.

It is easy to verify that $Y = \bigcup Y_{m,n}$ and, therefore, there are integers, say m and n, such that $Y_{m,n}$ is not countable dimensional.

For each $y \in Y_{m,n}$ let $\mathscr{V}_y$ be a finite cover of $f^{-1}(y)$ by relatively open subsets of $f^{-1}(Y_{n,m})$ which have diameters $< 1/m$. Since f is proper, $f(\bigcup\{V \in \mathscr{V}_y\})$ is a neighborhood (relative to $Y_{m,n}$) of y and, therefore, there is a y such that $f(\bigcup\{V \in \mathscr{V}_y\})$ is not countable dimensional. Since $\mathscr{V}_y$ is finite, there is a $V \in \mathscr{V}_y$ such that $f(V)$ is not countable dimensional. Set $C = \bar{V} \cap f^{-1}(Y_{m,n})$.

For an element $y \in f(C)$, we have that $\operatorname{diam}(C \cap f^{-1}(y)) < 1/m$. Condition (i) yields that $C \cap f^{-1}(y)$ is contained in a dendrite having diameter $< 1/3n$ and, in turn, combines with Condition (ii) to insure that there is a unique smallest dendrite D_y such that $C \cap f^{-1}(y) \subset D_y \subset f^{-1}(y)$.

Let $A = \left(\overline{\bigcup_{y \in f(C)} D_y}\right) \cap \left(f^{-1}(f(C))\right)$; since the restriction of f to $f^{-1}(f(C))$ is proper and A is a relatively closed subset of the latter set, the restriction of f to A is proper. It remains to show that $(f|_A)^{-1}(y)$ is a dendrite for each $y \in f(A)$ $(= f(C))$ and, for this, it suffices to show that $(f|_A)^{-1}(y)$ is connected and has diameter $< 1/n$.

If $x \in (f|_A)^{-1}(y)$, then either $x \in D_y$ or there is a sequence $y(1), y(2), \ldots$ in $f(A)$ converging to y such that $D_{y(1)}, D_{y(2)}, \ldots$ converge in the Hausdorff metric to a continuum D with $x \in D \subset (f|_A)^{-1}(y)$. Since $C \cap f^{-1}(y(i)) \subset D_{y(i)}$ and C is a relatively closed subset of $Y_{m,n}$, $C \cap D \neq \emptyset$ and, therefore, $D \cap D_y \neq \emptyset$. Since D is connected, we conclude that $(f|_A)^{-1}(y)$ is connected. Since $\operatorname{diam} D \leqslant 1/3n$ and $\operatorname{diam} D_y < 1/3n$, we conclude that $\operatorname{diam}((f|_A)^{-1}(y)) < 1/n$.

Lemma 3.3. *Let* f: $X \to Y$ *be a proper surjection with each* $f^{-1}(y)$ *a locally connected continuum and suppose that* X *is countable dimensional. If* Y *is not countable dimensional, then there is a subset* $Z \subset Y$ *which is not countable dimensional and there are disjoint relatively closed subsets* $X(1)$ *and* $X(2)$ *of* $f^{-1}(Z)$ *such that, for* $j = 1, 2$, *the restriction* $f|_{X(j)}$: $X(j) \to Z$ *is a proper surjection and has connected point-inverses.*

Proof. Using Lemma 3.1 to replace Y by a closed subset if necessary, we assume that there is an $\varepsilon > 0$ such that $\operatorname{diam} f^{-1}(y) \geqslant \varepsilon$ for each $y \in Y$.

The argument used in the second paragraph of Lemma 3.2 insures the existence of an open subset $U_1 \subset X$ having diameter $< \varepsilon$ such that the set $L = f(U_1) = \{y \in Y: f^{-1}(y) \cap U_1 \neq \emptyset\}$ is not countable dimensional. Let U_2, V_1, and V_2 be additional open sets chosen so that $\bar{U}_j \subset V_j$ for $j = 1, 2$, $\bar{V}_1 \subset U_2$, and $\dim V_2 < \varepsilon$. For each $y \in L$, we have that:

(i) for $j = 1, 2$, $f^{-1}(y) \cap \operatorname{Fr} U_j \neq \emptyset$ and $f^{-1}(y) \cap \operatorname{Fr} V_j \neq \emptyset$;

(ii) for $j = 1, 2$, there is at least one and at most finitely many components of $f^{-1}(y) \cap \overline{V_j - U_j}$ which meet both $\operatorname{Fr} U_j$ and $\operatorname{Fr} V_j$.

The fact that $f^{-1}(y)$ is a continuum which has diameter $> \varepsilon$ insures that (i) holds and, in turn, guarantees that there is at least one component satisfying (ii) (see [Wi; p. 209]). The local connectivity of $f^{-1}(y)$ precludes there being infinitely many such components.

For $j = 1, 2$, let $K(j) = \bigcup\{C$: for some $y \in L$, C is a component of $f^{-1}(y) \cap \overline{V_j - U_j}$ which meets both $\operatorname{Fr} U_j$ and $\operatorname{Fr} V_j\}$. Observe that $K(j)$ is a relatively closed subset of $f^{-1}(L)$ and that $f(K(j)) = L$ for $j = 1, 2$. In particular, each restriction $f_j = f|_{K(j)}$ is a proper map and, therefore, can be factored as $f_j = l_j \circ m_j$ where m_j: $K(j) \to D(j)$ is proper and has connected point-inverses and l_j: $D(j) \to L$ is proper and has totally disconnected point-inverses. (Just as in the Monotone-Light Factorization Theorem for maps between compact spaces [Ei], the collection of components of point-inverses of f_j forms an upper semi-continuous decomposition of $K(j)$. The space $D(j)$ is the associated decomposition space which is metrizable [Mc], [St].)

For $j = 1, 2$ and for each positive integer k, set

$N_j(k) = \{y \in L$: cardinality of $l_j^{-1}(y) = k\}$ and set

$M_j(k) = \{y \in L$: each two points of $l_j^{-1}(y)$ have a distance apart $\geqslant 1/k\}$.

A consequence of condition (ii) above is that

$$L = \bigcup_{k,m,n} [N_1(m) \cap N_2(n) \cap M_1(k) \cap M_2(k)].$$

Since L is not countable dimensional, there is a choice of integers k, m, n such that the set $Q = N_1(m) \cap N_2(n) \cap M_1(k) \cap N_2(k)$ is not countable dimensional.

It is an easy matter to verify that the restriction of l_j to $l_j^{-1}(Q)$ is a covering map for each $j = 1, 2$; and, therefore, there is a relatively open subset $Z \subset Q$ which is not countable dimensional and which is evenly covered by both l_1 and l_2. For $j = 1, 2$, choose an embedding e_j: $Z \to l_j^{-1}(Z)$ with $l_j \circ e_j = Id_Z$ and set $X(j) = m_j^{-1}(e_j(Z))$. The sets $X(1)$ and $X(2)$ are easily seen to satisfy the conclusion of the lemma.

Lemma 3.4. *Let* f: $X \to Y$ *be a proper surjection with each* $f^{-1}(y)$ *an* AR *having dimension* $\leqslant 1$ *and suppose that* X *is countable dimensional. If* Y *is not countable dimensional, then there is a subset* $Z \subset Y$ *and relatively closed subsets* $X(1)$, $X(2)$, $X(3)$, $X(4)$ *of* $f^{-1}(Z)$ *satisfying*:

(a) Z *is not countable dimensional*;

(b) $X(1) \cap X(2) = \emptyset$, $X(3) \cap X(4) = \emptyset$, *and* $X(3) \cup X(4) \subset X(1)$;

(c) *each restriction* $f|_{X(j)}\colon X(j)\to Z$ *is a proper surjection with connected point-inverses.*

Proof. An application of Lemma 3.3 produces a subset $Z'\subset Y$ which satisfies (a) and sets $X'(1)$ and $X'(2)$ which satisfy the relevant features of (b) and (c). Since each $f^{-1}(y)$ is an AR having dimension $\leqslant 1$, the point-inverses of the restriction $f|_{X'(1)}$ are AR's. Applying Lemma 3.3 to this restriction produces a subset $Z\subset Z'$ which satisfies (a) and sets $X(3)$ and $X(4)$ which satisfy the relevant features of (b) and (c). Setting $X(1)=f^{-1}(Z)\cap X'(1)$ and $X(2)=f^{-1}(Z)\cap X'(2)$ completes the proof.

Main Proposition. *Let* $f\colon X\to Y$ *be a proper surjection with each* $f^{-1}(y)$ *an* AR *having dimension* $\leqslant 1$ *and suppose that* X *is countable dimensional. If* Y *is not countable dimensional, then there is a subset* $Z\subset Y$ *which is not countable dimensional and for which* $\dim X\geqslant c\text{-}\dim Z+1$.

Proof. The interesting case is that with $\dim X<\infty$. An easy consequence of the Vietoris–Begle Mapping Theorem [Sp] is that $c\text{-}\dim Y\leqslant\dim X$ and, therefore, each subset of Y has finite cohomological dimension. Let $Z\subset Y$ and $X(1),\ldots,X(4)$ be sets obtained using Lemma 3.4. Set $n=c\text{-}\dim Z$ and let $A\subset Z$ be a relatively closed subset such that the inclusion induced homomorphism $H^{n-1}(Z)\to H^{n-1}(A)$ is not onto and, consequently, $H^n(Z,A)\neq 0$.

For $j=1,\ldots,4$, let $f_j=f|_{X(j)}$ and $A(j)=f_j^{-1}(A)$. The f_j's have point-inverses which are AR's and an immediate consequence of the Vietoris–Begle Mapping Theorem (used several times below) is that these maps induce isomorphisms in cohomology. The diagram

$$\begin{array}{ccc}
H^{n-1}(A(1)) & \xrightarrow{(\delta_3,\delta_4)} & H^n(A(1)\cup X(3),A(1))\oplus H^n(A(1)\cup X(4),A(1))\\
\simeq\uparrow f^* & & \simeq\uparrow f_3^*\oplus f_4^*\\
H^{n-1}(A) & \xrightarrow{(\delta,\delta)} & H^n(Z,A)\oplus H^n(Z,A)
\end{array}$$

commutes where δ and the δ_j's are "coboundary" homomorphisms. Since $H^n(Z,A)\neq 0$, the diagonal map (δ,δ) is not onto and, therefore, the map (δ_3,δ_4) is not onto.

The diagram

$$\begin{array}{ccc}
H^n(X(3)\cup X(4)\cup A(1),A(1)) & \xrightarrow{\simeq} & H^n(X(3)\cup A(1),A(1))\oplus H^n(X(4)\cup A(1),A(1))\\
\bar\delta\nwarrow & & \nearrow(\delta_3,\delta_4)\\
 & H^{n-1}(A(1)) &
\end{array}$$

commutes upto sign where $\bar\delta$ is the "coboundary" homomorphism and the horizontal isomorphism comes from the Mayer–Vietoris sequence for the "couple of pairs" $\{(X(3)\cup A(1),A(1)),(X(4)\cup A(1),A(1))\}$. Since (δ_3,δ_4) is not onto, $\bar\delta$ is not onto and, therefore, $H^n(X(3)\cup X(4)\cup A(1))\neq 0$.

Since $\dim X\geqslant\dim X(1)\geqslant c\text{-}\dim X(1)$ and $\dim X\geqslant\dim f^{-1}(Z)\geqslant c\text{-}\dim f^{-1}(Z)$, it remains to show that either $c\text{-}\dim X(1)\geqslant n+1$ or $c\text{-}\dim f^{-1}(Z)\geqslant n+1$.

If the inclusion induced homomorphism $H^n(X(1))\to H^n(X(3)\cup X(4)\cup A(1))$ is not onto, then (by definition!) $c\text{-}\dim X(1)\geqslant n+1$. If this homomorphism is onto, then $H^n(X(1))\neq 0$ and the isomorphism f_1^* yields that $H^n(Z)\neq 0$. The diagram

$$\begin{array}{ccccc}
H^n(f^{-1}(Z)) & \xrightarrow{i^*} & H^n(X(1)\cup X(2)) & \xrightarrow[\simeq]{(i_1^*,i_2^*)} & H^n(X(1))\oplus H^n(X(2))\\
 & \underset{\simeq}{\nwarrow}(f|_{f^{-1}(Z)})^* & & \simeq\nearrow f_1^*\oplus f_2^* & \\
 & & H^n(Z)\xrightarrow{\Delta}H^n(Z)\oplus H^n(Z) & &
\end{array}$$

commutes where i_1^*, i_2^*, and i^* are inclusion induced homomorphisms, where $\Delta(e)=(e,e)$ is the diagonal map, and where the homomorphism (i_1^*,i_2^*) is an isomorphism since $X(1)\cap X(2)=\emptyset$. Since we are assuming that $H^n(Z)\neq 0$, the diagonal map Δ is not onto and, hence, i^* is not onto. As above, we conclude that $c\text{-}\dim f^{-1}(Z)\geqslant n+1$.

4. Proofs of main theorem and corollaries.

Proof of the Main Theorem. If the space Y is not countable dimensional, then Lemma 3.2 and the Main Proposition combine to produce a subset $Z\subset Y$ which is not countable dimensional and for which $\dim X\geqslant c\text{-}\dim Z+1$. Since $\dim X\leqslant 2$, the inequality implies that $c\text{-}\dim Z\leqslant 1$. Basic fact (e) from the list in Section 2 yields that $\dim Z\leqslant 1$ but this contradicts the fact that Z is not countable dimensional.

Proof of Corollary A. Since each $f^{-1}(y)$ is an AR, the Vietoris–Begle Mapping Theorem applies to show that $c\text{-}\dim Y\leqslant c\text{-}\dim X\leqslant 2$. Since $\dim X\leqslant 2$ and each $f^{-1}(y)$ is an AR having dimension $\leqslant 1$, Y is countable dimensional by the Main Theorem. Since the proper image of a complete space is complete [Va], the space Y is both complete and countable dimensional. Since dimension agrees with cohomological dimension for complete countable dimensional spaces, $\dim Y=c\text{-}\dim Y\leqslant 2$.

We will need the result of Sieklucki [Si] that an n-dimensional locally compact separable ANR does not contain uncountably many pairwise disjoint n-dimensional closed subsets. We also will need the observation that a *separable* metric space Y is countable dimensional provided each point of Y has arbitrarily small neighborhoods whose frontiers are countable dimensional. (Choose a countable base $\mathcal{U}$ for Y consisting of open sets whose frontiers are countable dimensional and write $Y = [\bigcup_{U\in\mathcal{U}} \operatorname{Fr} U] \cup [Y - \bigcup_{U\in\mathcal{U}} \operatorname{Fr} U]$ as a countable union of countable dimensional subsets and the 0-dimensional subset $Y - \bigcup_{U\in\mathcal{U}} \operatorname{Fr} U$.)

Proof of Corollary C. Sieklucki's result which was just mentioned implies that, for each $y \in Y$, there is arbitrarily small $\varepsilon > 0$ such that $\dim f^{-1}(\operatorname{Fr}(N_\varepsilon(y)) \leqslant 2$ and, hence, the Main Theorem yields that $\operatorname{Fr}(N_\varepsilon(y))$ is countable dimensional. The observation made above completes the proof.

Remark. If E^3 is replaced by a locally compact ANR having dimension $\leqslant 3$, then the preceding argument shows that Y is locally countable dimensional and, consequently, countable dimensional. The emphasis on E^3 is intended to establish a ready comparison with the second examples constructed in the introduction.

Proof of Corollary B. The preceding argument combines with Corollary A to show that, for each $y \in Y$, there is arbitrarily small $\varepsilon > 0$ such that $\dim \operatorname{Fr}(N_\varepsilon(y)) \leqslant 2$. Therefore, we conclude that $\dim Y \leqslant 3$ and appeal to [Ko] or [La] in order to deduce that Y is an ANR.

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