Reduced powers of $\aleph_2$-trees

by

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Abstract. Let T be an $\aleph_2$-tree, D a uniform filter on ω. Assuming CH, the reduced power tree T^ω/D is an $\aleph_2$-tree. We investigate the relationship between T and T^ω/D with regard to the properties of being Souslin, Aronszajn, and Kurepa.

The first work on this topic was done by Devlin in 1978, who proved all of the results given here, assuming V = L and using fine structure arguments for the two main theorems — Theorems 4 and 5 of this paper. Baumgartner and Laver subsequently found much simpler proofs of these results, and it is their proofs which we present here.

§ 1. **Preliminaries.** We work in ZFC set theory throughout. Our notation is standard. A *tree* is a poset $T = \langle T, \leqslant_T\rangle$ such that for each $x \in T$ the set

$$\hat{x} = \{y \in T \mid y <_T x\}$$

is well-ordered by $<_T$. The order-type of $\hat{x}$ (under $<_T$) is called the *height* of x in T, denoted by $\mathrm{ht}(x)$. The αth *level* of T is the set

$$T_\alpha = \{x \in T \mid \mathrm{ht}(x) = \alpha\}.$$

We define

$$T \restriction \alpha = \bigcup_{\beta<\alpha} T_\beta,$$

and denote by $T \restriction \alpha$ the restriction of T to the set $T \restriction \alpha$. (In practice, we often do not bother to distinguish between a tree and its domain, however.)

An $\aleph_2$-*tree* is a tree T such that:

(i) $T_{\omega_2} = \emptyset$ and $(\forall \alpha < \omega_2)(T_\alpha \neq \emptyset)$;

(ii) $(\forall \alpha < \omega_2)(|T_\alpha| \leqslant \aleph_1)$;

(iii) $(\forall \alpha < \beta < \omega_2)(\forall x \in T_\alpha)(\exists y, z \in T_\beta)(y \neq z \;\&\; x <_T y \;\&\; x <_T z)$.

The point to notice about the above definition is that it is possible that at limit levels α of T there are distinct $x, y \in T_\alpha$ such that $\hat{x} = \hat{y}$. It is quite common in the literature to exclude this possibility in the definition of an "$\aleph_2$-tree", but in the present circumstances this restricted notion is of no use to us.

A *branch* of a tree T is a totally ordered initial segment of T. If it has order-type α (under $<_T$) it is called an α-*branch*.

An *antichain* of a tree T is a pairwise incomparable subset of T.

An $\aleph_2$-tree with no ω_2-branch is said to be *Aronszajn*. An $\aleph_2$-tree with no antichain of cardinality $\aleph_2$ is said to be *Souslin*. It is easily seen that a Souslin

$\aleph_2$-tree is necessarily Aronszajn. An $\aleph_2$-tree with at least $\aleph_3$ ω_2-branches is said to be *Kurepa*.

Assuming CH, it is easy to construct an Aronszajn $\aleph_2$-tree. Assuming V = L, one may also construct Souslin $\aleph_2$-trees and Kurepa $\aleph_2$-trees though in neither case can the assumption of V = L be weakened as far as CH. For further details, see [1].

§ 2. **Tree reduced powers.** Let T be any $\aleph_2$-tree, and let D be a (non-principal) uniform filter on ω. For each $\alpha < \omega_2$, define an equivalence relation on T_α^ω by

$$s \sim t \quad \text{iff} \quad \{n \in \omega | \ s(n) = t(n)\} \in D.$$

Let T_α^* denote the set of all equivalence classes from T_α^ω under $\sim$, and let $[s]$ denote the equivalence class of $s \in T$. Let

$$T^* = \bigcup_{\alpha < \omega_2} T_\alpha^*.$$

Define a binary relation $<^*$ on T^* by

$$[s] <^* [t] \text{ iff } \quad \{\cdot \in \omega | \ s(n) <_T t(n)\} \in D.$$

It is easily seen that $\langle T^*, <^* \rangle$ is a tree, and that if we assume CH, it is in fact an $\aleph_2$-tree. $\boldsymbol{T^*}$ is the *reduced ω-power of T by D*, denoted by T^ω/D (or $\boldsymbol{T^*}$ if it is clear which D is involved).

LEMMA 1. *There is a canonical embedding e: $\boldsymbol{T} \to \boldsymbol{T^*}$ such that $e[T]$ is an initial segment of $\boldsymbol{T^*}$.*

Proof. Set

$$e(x) = [\langle x | \ n < \omega \rangle], \qquad x \in T.$$

It is a routine matter to check that e is as claimed. ∎

By Lemma 1, we see that up to a canonical isomorphism, $\boldsymbol{T^*}$ is an extension of $\boldsymbol{T}$. In section 3 we investigate the relationship between $\boldsymbol{T}$ and $\boldsymbol{T^*}$ with regards to the properties of being Aronszajn, Souslin, and Kurepa. By and large we shall show that the only relationships are trivial ones. In particular, assuming V = L we shall construct a Souslin $\aleph_2$-tree $\boldsymbol{T}$ such that T^ω/D is Kurepa. In the meantime, we conclude this section with a technical result of some interest.

Let T be any $\aleph_2$-tree. We say $\boldsymbol{T}$ is *σ-closed* iff, whenever $\alpha < \omega_2$ is a limit level of cofinality ω and b is an α-branch of $\boldsymbol{T} \upharpoonright \alpha$, there is a point of T_α with extends b.

LEMMA 2. *$\boldsymbol{T^*}$ is always σ-closed.*

Proof. Let $\alpha < \omega_2$ be a limit ordinal of cofinality ω, and let B be an α-branch of $\boldsymbol{T^*} \upharpoonright \alpha$. Let $\langle \alpha_n | \ n < \omega \rangle$ be a strictly increasing sequence of ordinals cofinal in α. Let $B \cap T_{\alpha_n}^* = \{[b_n]\}$, where $b_n = \langle b_{nm} | \ m < \omega \rangle \in T_{\alpha_n}^\omega$. Thus for $n_1 < n_2 < \omega$, $\{m < \omega | \ b_{n_1 m} <_T b_{n_2 m}\} \in D$. It suffices to find a sequency $b_\omega = \langle b_{\omega m} | \ m < \omega \rangle \in T_\alpha^\omega$ such that for all $n < \omega$, $\{m \in \omega | \ b_{nm} <_T b_{\omega m}\} \in D$.

By recursion we define sets $A_n \in D$, $n < \omega$, $A_0 \supseteq A_1 \supseteq A_2 \supseteq \ldots$, and elements $b_{\omega m} \in T_\alpha$, $m \in A_n - A_{n+1}$. Set $A_0 = \omega$. Let $A_1 = \{m \in A_0 | \ b_{0m} <_T b_{1m}\}$. For each $m \in A_0 - A_1$, pick $b_{\omega m} \in T_\alpha$ so that $b_{0m} <_T b_{\omega m}$. In general, let $A_{n+1} = \{m \in A_n | \ b_{nm} <_T b_{n+1,m}\}$ and for each $m \in A_n - A_{n+1}$ pick $b_{\omega m} \in T_\alpha$ so that $b_{nm} <_T b_{\omega n}$. Let $b_\omega = \langle b_{\omega m} | \ m < \omega \rangle$. For each $n < \omega$, we have $A_{n+1} \subseteq \{m \in \omega | \ b_{nm} <_T b_{\omega m}\}$, so $[b_n] <^* [b_\omega]$. Hence $[b_\omega]$ extends B on T_α^*. ∎

§ 3. **The main results.** We see what effect on T^ω/D it has for T to be Aronszajn, Souslin, or Kurepa. It turns out that the exact choice of D is irrelevant. All we need to know is that D contains all cofinite sets.

Our first result is a negative one.

THEOREM 1. *$\boldsymbol{T^*}$ is not Souslin.*

Proof. For each $\alpha < \omega_2$ such that $\omega\alpha = \alpha$ and each $x \in T_\alpha$ we can pick elements $x_n \in T_{\alpha+\omega}$, $n < \omega$, such that $x <_T x_n$ and $x_n \neq x_m$ for $n < m < \omega$. Let A be the set of all $[\langle x_n | \ n < \omega \rangle]$ for all such α, x. It is easily seen that A is an antichain of $\boldsymbol{T^*}$ of cardinality $\aleph_2$. (This uses Lemma 1: for each $x \in T_\alpha$ as above, $[\langle x_n | \ n < \omega \rangle]$ is an extension of $e(x)$ in $\boldsymbol{T^*}$ which is not in $e[T]$.) ∎

Broadly speaking, the only positive result possible is the following:

THEOREM 2. *Assume* CH. *If $\boldsymbol{T}$ is Kurepa, so too is $\boldsymbol{T^*}$.*

Proof. By Lemma 1. (We require CH in order for $\boldsymbol{T^*}$ to be an $\aleph_2$-tree.) ∎

Assuming V = L, we construct $\aleph_2$-trees which violate all reasonable implications.

THEOREM 3. *Assume* V = L. *There is a Souslin $\aleph_2$-tree, T, such that $\boldsymbol{T^*}$ is Aronszajn.*

Proof. Let

$$E = \{\alpha \in \omega_2 | \ \lim(\alpha) \,\&\, \mathrm{cf}(\alpha) = \omega_1\},$$

and let $\langle f_\alpha | \ \alpha \in E \rangle$ be (by $\diamondsuit_{\omega_2}(E)$) a sequence of functions f_α: $(\omega+1) \times \alpha \to \alpha$ such that whenever f: $(\omega+1) \times \omega_2 \to \omega_2$ the set $\{\alpha \in E | \ f \upharpoonright (\omega+1) \times \alpha = f_\alpha\}$ is stationary in ω_2.

We construct an $\aleph_2$-tree $\boldsymbol{T}$ by recursion on the levels, using the elements of ω_2.

To commence we set $T_0 = \{0\}$. If T_α is defined, $T_{\alpha+1}$ is obtained by appointing two new ordinals as successors to each element of T_α.

There remains the case when $\lim(\alpha)$ and $\boldsymbol{T} \upharpoonright \alpha$ is defined. There are two cases.

Case 1. $\mathrm{cf}(\alpha) = \omega$. Let T_α consist of one-point extensions (by new ordinals) of every α-branch of $\boldsymbol{T} \upharpoonright \alpha$. By CH, there are at most $\aleph_1$ of these, so this will not prevent $\boldsymbol{T}$ from being an $\aleph_2$-tree.

Case 2. $\mathrm{cf}(\alpha) = \omega_1$. Thus $\alpha \in E$. There are three subcases.

Case 2.1. $f_\alpha(\omega, 0) = 0$ and $A = f_\alpha''(\{0\} \times \alpha)$ is a maximal antichain of $\boldsymbol{T} \upharpoonright \alpha$. For each $x \in T \upharpoonright \alpha$, let b_x be an α-branch of $\boldsymbol{T} \upharpoonright \alpha$ containing x (e.g., take the $<_L$-

least such). Since α is the limit of a strictly increasing ω_1-sequence of ordinals, continuous at limit, our behaviour in Case 1 ensures that such a b_x can always be found.

Let T_α consist of one-point extensions (by new ordinals) of each b_x such that x lies above an element of A in $T \upharpoonright \alpha$. By the properties of A, every element of $T \upharpoonright \alpha$ will have an extension on T_α.

Case 2.2. α is not of the form $\beta+\omega_1, f_\alpha(\omega, 0) = 1$, and $\langle [c_\nu] |\ \nu < \alpha \rangle$ is an α-branch of $(T \upharpoonright \alpha)^\omega/D$, where $c_\nu = \langle f_\alpha(n, \nu) |\ n < \omega \rangle$. Let $\langle \gamma_\nu |\ \nu < \omega_1 \rangle$ be a strictly increasing, continuous sequence or ordinals, cofinal in α, such that $\mathrm{cf}(\gamma_{\nu+1}) = \omega_1$ for each $\nu < \omega$. For each $x \in T \upharpoonright \alpha$, pick an α-branch, b_x, of $T \upharpoonright \alpha$ which contains x and is such that for each $\nu < \omega_1$, if $x \in T \upharpoonright \alpha_{\nu+1}$ then $b_x \cap T_{\gamma_{\nu+1}} \cap \{f_\alpha(n, \gamma_{\nu+1}) |\ n < \omega\} = \emptyset$. It is easily seen that such a branch b_x can always be found. (In particular, if $a \in T \upharpoonright \gamma_{\nu+1}$ there are $\aleph_1$ extensions of a in $T_{\gamma_{\nu+1}}$.) Let T_α consist of one-point extensions (by new ordinals) of each b_x, $x \in T \upharpoonright \alpha$.

Case 2.3. Otherwise. In this case, for each $x \in T \upharpoonright \alpha$ pick any α-branch, b_x, of $T \upharpoonright \alpha$, and extend each b_x, $x \in T \upharpoonright \alpha$, onto T_α. Again, our behaviour in Case 1 enables us to do this.

The construction is complete. Clearly, $T = \bigcup_{\alpha<\omega_2} T_\alpha$ is an $\aleph_2$-tree. We show that T is Souslin. Suppose not, and let A be a maximal antichain of T of cardinality $\aleph_2$. Let h: $\omega_2 \to \omega_2$ enumerate A monotonically. Let f: $(\omega+1) \times \omega_2 \to \omega_2$ be such that $f(\omega, 0) = 0$ and $f(0, \xi) = h(\xi)$ for all $\xi < \omega_2$.

Now the set

$$C = \{\alpha \in \omega_2 |\ f''(\omega+1) \times \alpha \subseteq \alpha \text{ and } f''(\{0\} \times \alpha) \text{ is a maximal antichain of } T \upharpoonright \alpha\}$$

is clearly club in ω_2. So we can find an $\sigma \in E$ such that $f \upharpoonright (\omega+1) \times \alpha = f_\alpha$. Thus case 2.1 applied in the definition of T_α. But then every point of T_α lies above a point of $A \cap (T \upharpoonright \alpha)$, so we must have $A = A \cap (T \upharpoonright \alpha)$, contrary to the choice of A. Hence T is Souslin.

We finish by showing that T^* is Aronszajn. Suppose not, and let $\langle [c_\nu] |\ \nu < \omega_2 \rangle$ be an ω_2-branch of T^*, where $c_\nu = \langle f(n, \nu) |\ n < \omega \rangle$. Extend f to a function from $(\omega+1) \times \omega_2$ into ω_2 by setting $f(\omega, \nu) = 1$ for all $\nu < \omega_2$. We can find an ordinal $\alpha \in E$ now such that α is a limit of ordinals of cofinality ω_1 and $f \upharpoonright (\omega+1) \times \alpha = f_\alpha$. Now, for each $n < \omega_1$ $f(n, \alpha)$ extends some α-branch b_{x_n} of $T \upharpoonright \alpha$, where $x_n \in T \upharpoonright \alpha$, and where b_{x_n} was chosen as in Case 2.2 above (which clearly must have applied in the construction of T_α). With $\langle \gamma_\nu |\ \nu < \omega_1 \rangle$ as in Case 2.2 now, we can find a $\nu < \omega_1$ such that $x_n \in T \upharpoonright \gamma_\nu$ for all $n < \omega$. But then each branch b_{x_n} misses the set $\{f(n, \gamma_{\nu+1}) |\ n < \omega\}$, contrary to the fact that $\{n \in \omega |\ f(n, \gamma_{\nu+1}) <_T f(n, \alpha)\} \in D$. The proof is complete. ∎

In contrast to Theorem 3, we have:

THEOREM 4. *Assume* V = L. *There is a Souslin $\aleph_2$-tree, T, such that T^* is not Aronszajn.*

Proof (Baumgartner). Let $\langle C_\alpha |\ \alpha < \omega_2 \,\&\, \lim(\alpha) \rangle$ and $E \subseteq \omega_2$ be such that:

(i) $\alpha \in E \to \lim(\alpha) \,\&\, \mathrm{cf}(\alpha) = \omega$;
(ii) E is stationary in ω_2;
(iii) C_α is closed in α and has order-type at most ω_1;
(iv) $\mathrm{cf}(\alpha) = \omega_1 \to C_\alpha$ is unbounded in α and $\mathrm{otp}(C_\alpha) = \omega_1$;
(v) $\bar{\alpha} \in C_\alpha \to C_{\bar{\alpha}} = \bar{\alpha} \cap C_\alpha$;
(vi) $C_\alpha \cap E = \emptyset$.

The existence of such C_α, E follows from the combinatorial principle $\square$ (see [2]).

Let $\langle S_\alpha |\ \alpha \in E \rangle$ be a $\diamondsuit(E)$-sequence; i.e., $S_\alpha \subseteq \alpha$ and for any $X \subseteq \omega_1$ the set $\{\alpha \in E |\ X \cap \alpha = S_\alpha\}$ is stationary.

Construct an $\aleph_2$-tree, T, by recursion on the levels, using ordinals from ω_2 as elements, simultaneously constructing on ω_2-branch $\langle [\langle t_{\alpha n} |\ n < \omega \rangle] |\ \alpha < \omega_2 \rangle$ of T^* by defining elements $t_{\alpha n} \in T_\alpha$, $n < \omega$.

To commence we set $T_0 = \{0\}$ and let $t_{0n} = 0$ for all $n < \omega$. If T_α is defined, we obtain $T_{\alpha+1}$ by appointing two new ordinals to extend each member of T_α and choosing $t_{\alpha+1,n}$ to extend $t_{\alpha n}$ for each $n < \omega$. There remains the case when $\lim(\alpha)$ and $T \upharpoonright \alpha$ is defined. There are several subcases to consider.

Case 1. $\mathrm{cf}(\alpha) = \omega \,\&\, \alpha \notin E$. Appoint a successor to each α-branch of $T \upharpoonright \alpha$. By GCH, $|T_\alpha| \leqslant \aleph_1$ here. If now C_α is bounded in α, using the technique of Lemma 2 we can pick $t_{\alpha n} \in T_\alpha$, $n < \omega$, so that, if $\gamma = \max(C_\alpha)$, $t_{\gamma n} <_T t_{\alpha n}$ for all $n < \omega$, and for any $\beta < \alpha$, $t_{\beta n} <_T t_{\alpha n}$ for almost all $n < \omega$. If, on the other hand C_α is unbounded in α, and if it is the case that $(\forall n < \omega)(t_{\gamma n} <_T t_{\delta n})$ wherever $\gamma, \delta \in C_\alpha$ and $\gamma < \delta$, then we let $t_{\alpha n}$ be the extension on T_α of the α-branch of $T \upharpoonright \alpha$ determined by $\langle t_{\gamma n} |\ \gamma \in C_\alpha \rangle$, for each $n < \omega$. In any other case the construction breaks down.

Case 2. $\mathrm{cf}(\alpha) = \omega \,\&\, \alpha \in E$.

For each $x \in T \upharpoonright \alpha$, let b_x be an α-branch of $T \upharpoonright \alpha$ containing x, and to obtain T_α appoint an extension to each b_x such that x lies above an element of S_α, if S_α is a maximal antichain of $T \upharpoonright \alpha$, and to each b_x, $x \in T \upharpoonright \alpha$, otherwise. Use the technique of Lemma 2 to pick $t_{\alpha n} \in T_\alpha$, $n < \omega$, so that for any $\gamma < \alpha$, $t_{\gamma n} <_T t_{\alpha n}$ for almost all $n < \omega$.

Case 3. $\mathrm{cf}(\alpha) = \omega_1$. C_α is a club subset of α of order type ω_1, disjoint from E, so for each $x \in T \upharpoonright \alpha$ we can easily construct an α-branch, b_x, of $T \upharpoonright \alpha$ containing x, and extend each b_x, $x \in T \upharpoonright \alpha$, to obtain T_α. And providing that $(\forall n < \omega)(t_{\gamma n} <_T t_{\delta n})$ wherever $\gamma, \delta \in C_\alpha$ and $\gamma < \delta$, we may ensure that the α-branches of $T \upharpoonright \alpha$ determined by $\langle t_{\gamma n} |\ \gamma \in C_\alpha \rangle$, $n < \omega$, are amongst those extended, whence we can take $t_{\alpha n}$ to be the extension on T_α of the branch defined by $\langle t_{\gamma n} |\ \gamma \in C_\alpha \rangle$. In any other event the construction breaks down.

That completes the definition. Using the properties of the sequence $\langle C_\alpha |\ \alpha < \omega_2 \,\&\, \lim(\alpha)\rangle$, it is easy to see that the construction never breaks down, and that for any α,

$$\gamma \in C_\alpha \to (\forall n < \omega)(t_{\gamma n} <_T t_{\alpha n}).$$

The proof that $T = \bigcup_{\alpha<\omega_2} T_\alpha$ is a Souslin $\aleph_2$-tree is standard, and $\langle [\langle t_{\alpha n} |\ n < \omega\rangle] |\ \alpha < \omega_2\rangle$ is clearly an ω_2-branch of T^*. ∎

Remark. Using observations of Gregory and others (including ourselves), we see that all that we needed to assume above was GCH+□. (See [2].)

THEOREM 5. *Assume* V = L. *Then there is a Souslin $\aleph_2$-tree, T, such that T^* is Kurepa.*

Proof. (Laver) By V = L we can pick some Kurepa $\aleph_2$-tree, K. We now construct a Souslin $\aleph_2$-tree, T, much as above. The only difference is that we embed K into T^* as we proceed. That is, instead of simply defining *one* ω_2-branch $\langle [\langle t_{\alpha n} |\ n < \omega\rangle] |\ \alpha < \omega_2\rangle$ of T, we define an entire copy of K. The details are easily worked out by comparison with the proof of Theorem 4, so we shall not go beyond these few remarks. ∎

Remark. The assumption of V = L above can be weakened to GCH+ +□+ "there is a Kurepa $\aleph_2$-tree."

References

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Topological degree and Sperner's lemma

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Abstract. Starting with a combinatorial theorem of Ky Fan on pseudomanifolds we define a topological degree mod 2 for a certain class of continuous maps from an n-dimensional finite polyhedron Π^n into $R^{n+1}-\{0\}$. In case Π^n is an n-simplex the strong version of Sperner's lemma is used for finding conditions under which the degree of a map does not vanish. In this way we can generalize some well-known topological results including the fixed point theorems of Brouwer and Kakutani.

Introduction. Sperner's lemma has turned out to be useful in different mathematical fields. Usually this combinatorial result is applied via the celebrated covering theorem for simplexes due to B. Knaster, C. Kuratowski, S. Mazurkiewicz [9]. It is remarkable, however, that the proof of this covering theorem makes use only of the weak form of Sperner's lemma, i.e. only the existence of at least one completely labeled subsimplex is needed, whereas Sperner's lemma states that the number of completely labeled subsimplexes is odd. It is the goal of this paper to improve the mentioned covering theorem by utilizing the strong version of Sperner's lemma and thus to extend classical topological results such as the fixed point theorems of L. E. J. Brouwer and S. Kakutani.

Our approach is closely related to an idea of M. A. Krasnosel'skii [10] who introduced the Brouwer degree $\beta(f)$ of a continuous map f of an n-dimensional closed finite orientable polyhedron into the n-dimensional unit sphere on a combinatorial basis. A famous result of K. Borsuk giving a sufficient condition for $\beta(f) \neq 0$ is derived there from a combinatorial antipodal point theorem. In this paper, starting with a combinatorial theorem of Ky Fan [5, Theorem 2] on pseudomanifolds, we shall assign one of the numbers 0, 1 to each cnntinuous map $F\colon \Pi^n \to R_*^{n+1}$ of an n-dimensional finite polyhedron Π^n (not necessarily closed or orientable) into $R_*^{n+1} := R^{n+1}-\{0\}$ that satisfies a certain boundary condition; this number $\gamma(F)$ will be called *degree of* F here. For the sake of simplicity we shall do without orientation consideration, so γ will be a degree mod 2. Particularly important for the utility of our degree are sufficient conditions for $\gamma(F) = 1$. In the most interesting case $\Pi^n = \Sigma^n$, where Σ^n denotes the n-dimensional unit simplex, such conditions will be obtained from Sperner's lemma.