§ 4. **Complements of 0-dimensional sets in I^{∞}.** As remarked in the introduction, the question of equivalence of imbeddings of a 0-dimensional space into I^{∞} reduces to the question of homeomorphism of their complements.

4.1. THEOREM. *Let $X, Y \subset I^{\infty}$ be 0-dimensional, and $f\colon I^{\infty}\setminus X \to I^{\infty}\setminus Y$ a homeomorphism. Then f extends to a homeomorphism $\bar{f}\colon I^{\infty} \to I^{\infty}$.*

Proof. For each $x \in X$ and $i \geqslant 1$, let $N_i(x)$ be the open $1/i$-neighborhood of x in I^{∞}. We claim that $\bar{f}(x) = \bigcap_{i=1}^{\infty} \overline{f(N_i(x)\setminus X)}$ defines an extension of f. Since each $N_i(x)$ is open and connected, and X is 0-dimensional, $N_i(x)\setminus X$ is nonempty and connected. Thus $\bar{f}(x)$ is the intersection of a decreasing sequence of continua, and is therefore a continuum which must lie in Y. Since Y is 0-dimensional, $\bar{f}(x)$ is a point. Clearly, this defines a continuous extension $\bar{f}\colon I^{\infty} \to I^{\infty}$ of f. Similarly, one defines a continuous extension $\bar{g}\colon I^{\infty} \to I^{\infty}$ of $g = f^{-1}$. Since X and Y are nowhere dense in I^{∞}, each of the compositions $\bar{f}\circ\bar{g}$ and $\bar{g}\circ\bar{f}$ is the identity map, and $\bar{f}$ is a homeomorphism.

Remark. It is clear from the proof that this theorem can be generalized, by substituting for I^{∞} any locally connected continuum Z, and requiring only that X and Y be totally disconnected and locally non-separating in Z.

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Chains in Ehrenfeucht-Mostowski models

by

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Abstract. We study the relationship between the chain of indiscernibles in an Ehrenfeucht-Mostowski model and the subchains of the model. As an application we construct large families of almost disjoint models for some theories.

Introduction. This work arises from the following considerations: by means of Ehrenfeucht–Mostowski models, the class of chains can be represented in the class of models of any theory. In some cases, especially unstable theories this method allows the construction of a large number of models (see S. Shelah [10]). However the sort of relationship between these models seems to be interesting, also. And this representation of classes of chains in classes of models allows one to think, in particular, that the complexity of comparing models is as high as the complexity of comparing chains. This paper is an attempt to work out this idea.

There is one basic question: if for a theory T two Ehrenfeucht–Mostowski models $M(C)$, $M(C')$ are comparable in some sense (by extension or elementary extension), what is the relationship between the chains C and C' which generated them? the same question arises when C' is a subchain of $M(C)$. Here we give a partial answer to this question, assuming that $M(C)$ is partially ordered by a formula φ, we prove: *if C' is a chain of regular power $\varkappa$ in this partial order then there is some subchain C'' of C', with power $\varkappa$, which is isomorphic to a subchain of C or its converse C^** (Theorem II–1). This is not the best possible result, however, as we get that C' is a countable union of chains, each of them being isomorphic to a subchain of some finite lexicographical product of copies of C or C^*.

Nevertheless this result is enough to transfer some properties of chains to models. Let (P) be the following property of two chains C and C': "C and C' are of same power $\varkappa$ and there is no chain of power $\varkappa$ order or antiorder isomorphic to subchains in both C and C'". If we take two chains C and C' with the property (P), then the two Ehrenfeucht–Mostowski models generated by them also have the corresponding property for models. So if the orderings on C and C' are definable in the Ehrenfeucht–Mostowski models to which they give rise, these models are uncomparable. The situation occurs when the theory T has some model containing an infinite chain; in this case large families of chains satisfying pairwise the property (P) give rise to large families of uncomparable models with the corresponding property. We can get such families of chains by using the super-rigid chains introduced by R. Bonnet. This method allows us to build many models. It gives a partial approach to the well known result of S. Shelah on the number of models of unstable theories.

This text is divided into two sections. The first section is purely relation theory. It is the only part of the properties of Ehrenfeucht–Mostowski models that we need for our approach. The application to model theory is given in the second section.

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I. Relational ingredients. In the following we are concerned with partial and linearly ordered sets. Our definitions and terminology are, with a few exceptions, the same as those in the book by Kuratowski and Mostowski [8]. We shall use symbols such as $\leqslant, \preccurlyeq, \trianglelefteq, \ldots$ to denote order relations. If necessary we shall use other symbols, sometimes with subscripts. For instance if $\leqslant$ denotes some ordering of a set E we shall denote its converse by $\leqslant^*$, and by $<$, the strict order ($x < y$ means $x \leqslant y$ and $x \neq y$); we shall write $\boldsymbol{E}$ and $\boldsymbol{E}^*$ for the corresponding ordered sets $(E, \leqslant), (E, \leqslant^*)$. If the order is *linear* (or *total*) we shall say that the corresponding (linearly) ordered set is a *chain*, and that its subsets with the induced order are *subchains*. Preferably we shall denote chains by the letters $\boldsymbol{C}, \boldsymbol{D}, \boldsymbol{P}, \ldots$. For a chain $\boldsymbol{C} = (C, \leqslant)$ we shall denote by $\vec{a}$ any finite strictly increasing sequence of elements of C, and by $[C]^m$ the set of such $\vec{a}$ with length m. By a *local automorphism of* $\boldsymbol{C}$ we shall mean any strictly increasing map g with domain and range subsets of C and we shall denote by $\bar{g}(\vec{a})$, the sequence $(g(a_1), \ldots, g(a_m))$ for $\vec{a} = (a_1, \ldots, a_m)$ with $a_1 < a_2 \ldots < a_m$ in the domain of g.

I–1. Generated and invariant ordered sets. Let us consider an ordered set $\boldsymbol{E} = (E, \leqslant)$, a chain $\boldsymbol{C}$ and a set Φ such that each element φ of Φ is a map from $[C]^{m(\varphi)}$ into E where $m(\varphi)$ is an integer, which we call the *arity* of φ. We say that $\boldsymbol{E}$ is $\boldsymbol{C}$*-generated, modulo* Φ, when each element of E is the range of some φ of Φ. We say that $\boldsymbol{E}$ (or the order on E) is $\boldsymbol{C}$*-invariant, modulo* Φ when for any local automorphism g of $\boldsymbol{C}$, for any φ, ψ in Φ, and for any strictly increasing sequences $\vec{a}, \vec{b}$ from the domain of g, with respective lengths $m(\varphi)$ and $m(\psi)$, we have: $\varphi(\vec{a}) \leqslant \psi(\vec{b})$ iff $\varphi(\bar{g}(\vec{a})) \leqslant \psi(\bar{g}(\vec{b}))$.

EXAMPLE. The simplest case is when $\boldsymbol{E} = [\boldsymbol{C}]^m$ for some m and Φ contains only the identity function on $[C]^m$, in this case we say briefly that $[\boldsymbol{C}]^m$ is $\boldsymbol{C}$*-invariant*. Then these conditions mean that the comparison of $\vec{a}$ and $\vec{b}$ does not depend upon the choice of $\vec{a}$ and $\vec{b}$ but only on the relative position of their components. These are precisely the conditions that are equivalent to the definability of the order on $[C]^m$ from the order on C, by way of the free formula $F(\vec{X}, \vec{Y})$. As typical examples we have the cartesian order and the lexicographical order. We shall show that all examples of a $\boldsymbol{C}$-invariant *linear* ordering on $[\boldsymbol{C}]^m$ are lexicographically ordered.

Let P be a subset of C and Φ' a subset of Φ. We shall denote by $\bar{\boldsymbol{P}}^{\Phi'}$ the subset of E generated by $\boldsymbol{P}$ modulo Φ', with the induced order.

I–2. LEMMA. *The following are equivalent:*

(i) $\boldsymbol{E}$ *is* $\boldsymbol{C}$*-invariant modulo* Φ.

(ii) *For any subsets* P *and* P' *of* C, *if* g *is some order isomorphism from* $\boldsymbol{P}$ *onto* $\boldsymbol{P}'$, *then there is a unique order isomorphism* $\tilde{g}$ *from* $\bar{\boldsymbol{P}}^{\Phi}$ *onto* $\bar{\boldsymbol{P}}'^{\Phi}$ *such that* $\tilde{g}(\varphi(\vec{a})) = \varphi(\bar{g}(\vec{a}))$ *for any* $\varphi \in \Phi$ *and* $\vec{a} \in [\boldsymbol{P}]^{m(\varphi)}$.

(iii) *For any finite subsets* Φ' *of* Φ *and* P *of* C, $\bar{\boldsymbol{P}}^{\Phi'}$ *is* $\boldsymbol{P}$*-invariant modulo* Φ' (*in fact it is enough to consider subsets* P *of* C *with no more than* $4\operatorname{Max}\{m(\varphi)\colon \varphi \in \Phi'\}$ *elements*).

Sketch of proof. (i) $\Rightarrow$ (ii). For $x \in \bar{\boldsymbol{P}}^{\Phi}$ let us define $\tilde{g}(x) = \varphi(\bar{g}(\vec{a}))$ where $\vec{a}$ and φ are such that $x = \varphi(\vec{a})$ and note that if $\varphi(\vec{a}) = \psi(\vec{b})$ then:

$$\varphi(\bar{g}(\vec{a})) = \psi(\bar{g}(\vec{b})).$$

The converse and (i) $\Rightarrow$ (iii) are obvious.

(iii) $\Rightarrow$ (i). For $g, \varphi, \psi, \vec{a}, \vec{b}$ as defined previously, let P be a subset of C containing the components of $\vec{a}, \vec{b}, \bar{g}(\vec{a}), \bar{g}(\vec{b})$. As $\bar{\boldsymbol{P}}^{\{\varphi,\psi\}}$ is $\boldsymbol{P}$-invariant the result follows.

I–3. THEOREM. *Let the partial ordering* $\boldsymbol{E}$ *be* $\boldsymbol{C}$*-invariant, modulo* Φ. *If* $\boldsymbol{C}$ *is an infinite chain then the ordering on* $\boldsymbol{E}$ *can be strengthened* (¹) *to some linear ordering which is* $\boldsymbol{C}$*-invariant, modulo* Φ.

Proof. First step. Let us consider some linear strengthening $\preccurlyeq$ of the ordering $\leqslant$ on E. We prove the following lemma:

For each finite subset Φ' *of* Φ, *there is some infinite subchain* $\boldsymbol{C}'$ *of* $\boldsymbol{C}$ *such that* $\bar{\boldsymbol{C}}'^{\Phi'}$, *with the ordering induced by* $\preccurlyeq$, *is* C'*-invariant modulo* Φ'.

–1– Let Φ' be a *finite* subset of Φ and m' an integer. We say that two m'-element subsets P and Q of C are equivalent iff the order isomorphism $\tilde{g}$ from $\bar{\boldsymbol{P}}^{\Phi'}$ onto $\bar{\boldsymbol{Q}}^{\Phi'}$, induced by the unique isomorphism g from $\boldsymbol{P}$ onto $\boldsymbol{Q}$, is also an order isomorphism for the linear orderings induced by $\preccurlyeq$ on $\bar{P}^{\Phi'}$ and $\bar{Q}^{\Phi'}$. Obviously we have an equivalence relation with a finite number of classes (as Φ' is finite, such $\bar{\boldsymbol{P}}^{\Phi'}$ are finite, and if p is their common number of elements, then there is at most $p!$ classes). So if we apply Ramsey's theorem we get an infinite subset C' of C with all its m'-element subsets in the same class.

–2– Let us show that for such a C', all its m''-element subsets are equivalent for $m'' \leqslant m'$. The result will follow by induction, let m be an integer and let $\boldsymbol{D}$ be a subchain of $\boldsymbol{C}$ with at least $(m+1)$-elements such that the m-element subsets of D are equivalent as above. We can observe that the $(m-1)$-elements subsets of D are also equivalent: if D has $m+1$ elements, it is easy. If D has more than $m+1$

(¹) We say that an order $\leqslant'$ is a *strengthening* of the order $\leqslant$ if $x \leqslant y$ implies $x \leqslant' y$ for all possible x, y (we do not use the term *extension* to avoid confusion with its meaning in model theory). Let us recall that every partial ordering has some linear strengthening (Szpilrajn [11]).

elements then two m-element subsets P and Q of D are equivalent to P' and Q' respectively, which are two $(m+1)$-elements subsets of some $(m+1)$-elements subset of D. So by trivial induction the result follows.

–3– Let us take $m' = 2\,\mathrm{Max}\,\{m(\varphi)/\varphi \in \Phi'\}$ and consider some infinite chain C' given by the Ramsey's theorem. As its m''-element subsets are also equivalent for $m'' \leqslant m'$ it follows that this chain has the required property. To see this, take a local automorphism g of C' and elements φ and ψ in Φ' with $\vec{a} = (a_1, \ldots, a_{m(\varphi)})$ and $\vec{b} = (b_1, \ldots, b_{m(\psi)})$ in the domain of g. The two sets $\{a_1, \ldots, a_{m(\varphi)}\} \cup \{b_1, \ldots, b_{m(\psi)}\}$ and:

$$\{g(a_1), \ldots, g(a_{m(\varphi)})\} \cup \{g(b_1), \ldots, g(b_{m(\psi)})\}$$

have the same number of elements which is at most $m'' = m(\varphi)+m(\psi)$. Because $m'' \leqslant m'$, they are equivalent subsets of C'. Because g is the unique isomorphism between them we have $\varphi(\vec{a}) \preccurlyeq \psi(\vec{b})$ iff $\varphi(\bar{g}(\vec{a})) \preccurlyeq \psi(\bar{g}(\vec{b}))$.

Second step. By the above lemma it follows that *for any finite subsets P' of C and Φ' of Φ, the order on $\bar{P}'^{\Phi'}$ can be strengthened to some P'-invariant* (mod Φ') *linear ordering* (take some infinite subchain C' of C, as in the lemma. For any P' take some isomorphism g from P' into C'. As the ordering on $\bar{P}'^{\Phi'}$ take the isomorphic image under $\bar{g}^{-1}$ of the linear ordering on $\overline{g(P')}^{\Phi'}$ induced by $\preccurlyeq$. Because all finite subsets of C' are equivalent, this linear ordering does not depend of the choice of g.

By the compactness theorem it follows that *the ordering on $\bar{C}^{\Phi}$ can be strengthened by some C-invariant* (mod Φ) *linear ordering.*

By Szpilrajn's result, such strengthening can be extended to the whole set E in a strengthening of $\leqslant$. Hence we get the result.

Let us give a direct proof of the compactness argument: for any finite subsets P' of C and Φ' of Φ let us consider the set $\mathscr{U}_{P',\Phi'}$ of the binary relations on $E' = \bar{C}^{\Phi}$, each of which induces a P'-invariant (mod Φ') linear strengthening of the induced order on $\bar{P}^{\Phi'}$. These $\mathscr{U}_{P',\Phi'}$ sets are closed in the set of the binary relations on E', for the pointwise convergence topology, and they have the finite intersection property (note that $\mathscr{U}_{P',\Phi'} \cap \mathscr{U}_{P'',\Phi''} \supseteq \mathscr{U}_{P' \cup P'', \Phi' \cup \Phi''}$). So their intersection is non empty. Any element of this intersection is a relation having the required property. Note that if Φ has a finite size then this argument is useless: *to have the required strengthening on $\bar{C}'^{\Phi}$ it suffices to have it on some $\bar{C}'^{\Phi}$ for some C with at least* $4m(\Phi)$ *elements* (here $m(\Phi) = \mathrm{Max}\,\{m(\varphi)\ \ \varphi \in \Phi\}$). (Build a linear order on every $\bar{P}^{\Phi}$ with P of size $2m(\Phi)$, as in the beginning of the second step. Observe that for two such sets the orders are the same on their intersection). Although C' is of very small size we do not know how prove that C' has such a strengthening without Ramsey's theorem.

Remark. To construct on a set a linear strengthening of a partial order it is enough to order two incomparables elements, in an arbitrary way, and so on; Afterwards one has to use a set theoretical axiom. But this method does not work here. Let us look at the following example: suppose E is generated by two unary functions φ and ψ, ordered in such a way that $\varphi(c) < \psi(c')$ for $c \neq c'$. If we decide that $\psi(c) \leqslant \varphi(c)$ then, by C-invariance, we get the following cycle: $\psi(c) \leqslant \varphi(c) < \psi(c') \leqslant \varphi(c') < \psi(c)$.

Our method of proof is in same spirit as some of Fraissé's proofs concerning *enchainability*. Note that this notion (similar to the indiscernability one) and the use of Ramsey's theorem have occured in his thesis (1953, [4]).

Generalization. The conclusion of the above theorem still holds if we replace the partial ordering by a binary relation with no cycles: we get a C-invariant linear strengthening of this binary relation (take an arbitrary linear strengthening of this relation and do the two steps of the proof).

I–4. Theorem. *Let E be a chain which is C-generated and invariant modulo one function φ. If C is infinite then E is order isomorphic to asubchain of a finite lexicographical product of copies of C or C^*.*

Let us begin to prove the following lemmas.

Lemma. *Let E be a partial order which is C-generated and invariant modulo one function φ. If C has at least $3m(\varphi)$ elements then for every chain C_1, containing C there is some function φ_1 extending φ and some C_1-generated and invariant* (mod φ_1) *partial order E_1 which extends E.*

Proof. Let C_1 be a chain containing C. For $n = m(\varphi)$ let us say that two elements $\vec{a}$ and $\vec{b}$ of $[C_1]^n$ are equivalent iff there is some local isomorphism g of C_1 such that $\bar{g}(\vec{a})$, $\bar{g}(\vec{b})$ are in $[C]^n$ and:

$$\varphi(\bar{g}(\vec{a})) = \varphi(\bar{g}(\vec{b})).$$

(Note that if this equality holds for one g, it holds for every one.) By the bijective map θ between E and the set E' of equivalence classes of elements $\vec{a} \in [C]^n$ we can put on E' a copy of the ordering on E. And we can extend this ordering to the whole set of classes E'_1 in the following way: the classes of two elements $\vec{a}$, $\vec{b}$ of $[C_1]^n$ are ordered as the classes of $\bar{g}(\vec{a})$ and $\bar{g}(\vec{b})$ in E' (where g is any local automorphism which maps $\vec{a}$ and $\vec{b}$ into $[C]^n$). To get the result it is enough to extend the map θ to some ordered set E_1 in such a way so that this map will be an isomorphism between E_1 and E'_1.

Notation. For any $\vec{a} \in [C]^n$, any k-element subset K of $[1, \ldots, n]$, we denote by $\vec{a}|_K$ the element of $[C]^k$ obtained from $\vec{a}$ by deleting the components whose indices are not in K. For example if $\vec{a} = (a_1, a_2, a_3, a_4)$ and $K = \{2, 4\}$ then $\vec{a}|_K = (a_2, a_4)$.

Lemma 2. *Let E be some C-generated and invariant, modulo one function φ, partial ordering. If C is a dense*(1) *chain then there is some k-elements subset K of*

(1) We mean that a chain C is dense in the usual sense: for any $x, y \in C$, with $x < y$ there are u, v, w such that $u < x < v < y < w$.

$[1, \ldots, m(\varphi)]$ *and some bijective map* ψ *from* $[C]^k$ *onto* E *such that:*

1) $\varphi(\vec{a}) = \psi(\vec{a}|_K)$ *for every* $\vec{a} \in [C]^{m(\varphi)}$,

2) E *is* C*-generated and invariant* (mod ψ).

Proof. In the following we denote by $s_i(x)\vec{a}$ the element of $[C]^n$ which has the same components as $\vec{a}$ for the indices j, for $j \neq i$, and values x for the index i. (Of course, this notation supposes that x is between a_{i-1} and a_{i+1}). Suppose that we have the equality

$$\varphi(s_i(x)\vec{a}) = \varphi(s_i(y)\vec{a})$$

for some $\vec{a}$ and $x \neq y$. Then we have this equality for all possible choices of x, y (between a_{i-1}, a_{i+1}), and also for every $\vec{a}$. We say that the index i is *inactive* if the previous situation occurs. When φ is defined on C, by saying that the index 1 is inactive, we mean that φ is constant. In this case, to get the result, take for K the empty set. We assert that if φ is not injective then there is some inactive index: let $\vec{a}$, $\vec{b}$ be such that $\vec{a} \neq \vec{b}$ and $\varphi(\vec{a}) = \varphi(\vec{b})$. Suppose that $\vec{a} < \vec{b}$ in the lexicographic order and let i be the smallest index such that $a_i \neq b_i$ (so we have $a_i < b_i$). Let x be between a_{i-1} and a_i. The elements $s_i(x)\vec{a}$ and $\vec{b}$ are in the same position as $\vec{a}$ and $\vec{b}$ so we have $\varphi(s_i(x)\vec{a}) = \varphi(\vec{b})$ and then $\varphi(s_i(x)\vec{a}) = \varphi(\vec{a})$. This means that i is inactive. Now we work by induction: if φ is injective the lemma is proved; if φ is not injective, take some inactive index i, take $K = \{1, 2, \ldots, i-1, i+1, \ldots, n\}$ and define the function φ' from $[C]^{n-1}$ into E, by taking $\varphi'(\vec{a}|_K) = \varphi(\vec{a})$. (Because C is dense, this equation suffices to define φ'). Do this again on φ' and repeat the procedure as many times as it is possible. The resulting function ψ as the required properties (to see the C-invariance of E mod ψ, use the fact that, in a dense chain, every local automorphism with finite domain can be extended to any finite subset).

LEMMA 3 [1]. *If C is a dense chain, every linear C-invariant ordering on $[C]^n$ is a lexicographical ordering for some total ordering on $[1, \ldots, n]$, the components being ordered as C or C^*.*

Proof. Let us denote by $\leqslant$ the linear ordering of C and let $\preccurlyeq$ be any C-invariant linear ordering on $[C]^n$.

First part. 1) For every $i \in [1, \ldots, n]$, let us define a new ordering on C in the following way: for two elements $x, y \in C$, we say $x \underset{i}{\preccurlyeq} y$ iff there is some $\vec{a} \in [C]^n$ such that $s_i(x)\vec{a} \preccurlyeq s_i(y)\vec{a}$. If there is a such an $\vec{a}$ then this relation holds for every $\vec{a}$ where the substitutions of x and y are possible. So the relation $\underset{i}{\preccurlyeq}$ is antisymmetric. For every x, y in C, we can always choose $\vec{a}$ where the substitutions of x and y are possible (C is dense), so we have $x \underset{i}{\preccurlyeq} y$ or $y \underset{i}{\preccurlyeq} x$. If for some x, y in C we have $x \leqslant y$ and $x \underset{i}{\preccurlyeq} y$ then for every x', y' in C such that $x' \leqslant y'$, we get $x' \underset{i}{\preccurlyeq} y'$ (by a local automorphism, transform x in x', y in y' and extend it to some $\vec{a}$), so $\underset{i}{\preccurlyeq}$ in exactly the given ordering $\leqslant$ on C or the converse. And if $n = 1$ Lemma 3 is proved.

2) We denote by $a_{ij}(x, y)\vec{a}$ the element of $[C]^n$ with the same components as $\vec{a}$, except for the ith and jth components where the values are respectively x and y (this notation is only defined when we get $s_{ij}(x, y)\vec{a}$ as an element of $[C]^n$ and will be used only in this case). Of course the notation $s_{ijk}(x, y, z)\vec{a}$ has a corresponding meaning.

We define a total order $\trianglelefteq$ on the set of indices $[1, \ldots, n]$ in the following way: we say $i \trianglelefteq j$ iff $i = j$, or there is some x, x', y, y' and $\vec{a}$ such that $x \underset{i}{\prec} x'$, $y \underset{j}{\prec} y'$, and $s_{ij}(x, y')\vec{a} \preccurlyeq s_{ij}(x', y)\vec{a}$ (of course the substitution of x, y' and x', y makes sense iff they are ordered like i, j in their natural order). Because $\preccurlyeq$ is a total ordering we have $i \trianglelefteq j$ or $j \trianglelefteq i$. If in the above definition we have $s_{ij}(x, y')\vec{a} \preccurlyeq s_{ij}(x', y)\vec{a}$, then for $\vec{a}$ fixed this relation in still true for every possible choice of x, x', y, y', and for x, x', y, y' fixed it is true for every possible choice of $\vec{a}$. So the relation $\trianglelefteq$ is antisymmetric. Let us show that it is transitive: suppose $i, j, k \in [1, \ldots, n]$ such that $i \trianglelefteq j$ and $j \trianglelefteq k$. If $i = j$ or $j = k$ then $i \trianglelefteq k$. Suppose i, j, k are pairwise distinct elements of $[1, \ldots, n]$ (so $n \geqslant 3$). Let x, x', z, z' be in C and $\vec{a} \in [C]^n$ such that $x \underset{i}{\prec} x'$, $z \underset{k}{\prec} z'$ and $s_{ik}(x, z')\vec{a}$, $s_{ik}(x', z)\vec{a}$ are defined. Choose y in C such that $y \underset{j}{\prec} a_j$ and $s_{ijk}(x', y, z')\vec{a}$ is defined. From $i \triangleleft j$ it follows that

$$s_{ik}(x, z')\vec{a} \preccurlyeq s_{ijk}(x', y, z)\vec{a}.$$

From $j \triangleleft k$ it follows that $s_{ijk}(x', y, z')\vec{a} \preccurlyeq s_{ik}(x', z)\vec{a}$. So $s_{ik}(x, y')\vec{a} \preccurlyeq s_{ik}(x', y)\vec{a}$, namely $i \triangleleft k$.

Second part. Now, we show that the ordering $\preccurlyeq$ is exactly the lexicographical ordering $\underset{\text{lex}}{\leqslant}$ defined by $\trianglelefteq$ and the $\underset{i}{\preccurlyeq}$, $i = 1, \ldots, n$, (we recall that $\vec{a} \underset{\text{lex}}{\leqslant} \vec{b}$ iff $\vec{a} = \vec{b}$ or $a_{i_0} \underset{i_0}{\leqslant} b_{i_0}$ for the smallest index i_0, relative to $\trianglelefteq$, such that $a_{i_0} \neq b_{i_0}$).

As we have two linear orderings, it is enough to show that $\vec{a} \underset{\text{lex}}{\leqslant} \vec{b}$ implies $\vec{a} \preccurlyeq \vec{b}$. For two elements $\vec{a}$, $\vec{b}$ of $[C]^n$ let $d(\vec{a}, \vec{b})$ be the number of indices i such that $a_i \neq b_i$. We prove the previous implication by induction: *we suppose it true for all $\vec{a}$, $\vec{b}$ such that $d(\vec{a}, \vec{b}) < d \leqslant n$ and we prove it for all $\vec{a}$, $\vec{b}$ with $d(\vec{a}, \vec{b}) = d$.*

Let $\vec{a}$, $\vec{b}$ be such that $\vec{a} \underset{\text{lex}}{\leqslant} \vec{b}$, $d(\vec{a}, \vec{b}) = d$ and let i_0 be the smallest i, under $\trianglelefteq$, for which $a_i \neq b_i$.

$d = 1$. By the definition of the lexicographical ordering and of $\underset{i_0}{\leqslant}$, we get the result.

$d = 2$. Let j be the other index where $\vec{a}$ and $\vec{b}$ differ, thus $i_0 \triangleleft j$.

First case. $b_j \underset{j}{\prec} a_j$. By the definition of $\triangleleft$, we get $\vec{a} \preccurlyeq \vec{b}$.

Second case. $a_j \underset{j}{\prec} b_j$. Let $k = \mathrm{Min}(i_0, j)$, $k' = \mathrm{Max}(i_0, j)$ for the natural ordering on $[1, \ldots, n]$.

[1] W. Hodges has shown this lemma in his thesis, however he did not publish it. In fact this lemma is a consequence of a result due to K. Arrow and known as Arrow's Paradox.

If $a_k < b_k$ on $\boldsymbol{C}$, we have $\vec{a} \preccurlyeq s_k(b_k)\vec{a} = s_{k'}(a_{k'})\vec{b} \preccurlyeq \vec{b}$.
If $b_k < a_k$ on $\boldsymbol{C}$, we have $\vec{a} \preccurlyeq s_{k'}(b_{k'})\vec{a} = s_k(a_k)\vec{b} \preccurlyeq \vec{b}$.

$d > 2$. Let j_0 be the greatest index (for the natural ordering on $[1, \ldots, n]$) for which $\vec{a}$ and $\vec{b}$ differ. By the choice of i_0 we have $i_0 \trianglelefteq j_0$.

First case. Suppose $j_0 \neq i_0$.

First subcase. $a_{j_0} < b_{j_0}$, on $\boldsymbol{C}$.

If $a_{j_0} \underset{j_0}{<} b_{j_0}$, then $\vec{a} \underset{\text{lex}}{<} s_{j_0}(b_{j_0})\vec{a} \underset{\text{lex}}{<} \vec{b}$. So by induction we get the result.

If $b_{j_0} \underset{j_0}{<} a_{j_0}$. Let us choose some x satisfying $a_{i_0} \underset{i_0}{<} x \underset{i_0}{<} b_{i_0}$ and $a_{i_0-1} < x < b_{i_0+1}$. Because $\underset{i_0}{<}$ is the given ordering on $\boldsymbol{C}$ or its converse this is equivalent to choosing x both between a_{i_0} and b_{i_0}, and between a_{i_0-1} and a_{i_0+1} for the ordering on $\boldsymbol{C}$. This is possible because $\boldsymbol{C}$ is dense. So we get:

$$\vec{a} \underset{\text{lex}}{<} s_{i_0,j_0}(x, b_{j_0})\vec{a} \underset{\text{lex}}{<} \vec{b}.$$

Second subcase. $b_{j_0} < a_{j_0}$ on $\boldsymbol{C}$. Replace $\preccurlyeq$ by its converse $\preccurlyeq^*$. So we get $\underset{i}{<}^*$ instead of $\underset{i}{<}$, the ordering $\lhd$ remains the same, and $\underset{\text{lex}}{<}$ becomes $\underset{\text{lex}}{<}^*$. The result follows as in the first subcase.

Second case. $j_0 = i_0$. Let j_1 be the smallest index (for the natural ordering on $[1, \ldots, n]$) for which $\vec{a}$ and $\vec{b}$ differ; as $\vec{a}$ and $\vec{b}$ have at least two different components, we have $j_1 \neq j_0$.

Let us reverse the order on $\boldsymbol{C}$, and reverse the indexing (for example (a, b, c) becomes (c, b, a)); repeat for j_1 what was done for j_0. ■

Remark. The result is true even when $\boldsymbol{C}$ is not dense, and furthermore *when $\boldsymbol{C}$ is finite with at least $(2n+1)$-elements*. To get the result when $\boldsymbol{C}$ has at least $3n$ elements it is enough to know it when the chain is dense: by Lemma 1, we can extend $\boldsymbol{C}$ to a dense chain $\boldsymbol{C}_1$. We get an ordering, on $[\boldsymbol{C}_1]^n$, which extends the ordering on $[\boldsymbol{C}]^n$. By Lemma 3, $[\boldsymbol{C}_1]^n$ is lexicographically ordered, so is $[\boldsymbol{C}]^n$.

Proof of Theorem I–4. If $\boldsymbol{C}$ is a dense chain then, by Lemma 2, we get a bijective map from some $[\boldsymbol{C}]^m$ onto $\boldsymbol{E}$. Then Lemma 3 gives the answer.

If $\boldsymbol{C}$ is not dense, extend it to a dense chain $\boldsymbol{C}_1$; by Lemma 1 extend φ to some map φ_1 from $[\boldsymbol{C}_1]^n$ onto some E_1. By Lemma 2 take the one-one mapping ψ_1 from some $[\boldsymbol{C}_1]^m$ onto E_1. By Lemma 3, $\boldsymbol{E}_1$ is lexicographically ordered like $[\boldsymbol{C}_1]^m$. As $\psi_1(\vec{a}|_K) = \varphi_1(\vec{a})$ it follows that $\boldsymbol{E} \subseteq \psi_1([\boldsymbol{C}]^m)$. (NB: the equality does not hold, in general.) So $\boldsymbol{E}$ is ordered as a subset of $[\boldsymbol{C}]^m$.

I–5. Theorem. *Let $\boldsymbol{E}$ be an ordered set which is $\boldsymbol{C}$-generated and invariant modulo a countable set Φ. Then every chain $\boldsymbol{D}$ of $\boldsymbol{E}$, with regular cardinal $\varkappa$, contains a subchain $\boldsymbol{D}'$, with power $\varkappa$, which is order isomorphic to a subchain of $\boldsymbol{C}$ or of $\boldsymbol{C}^*$.*

Proof. If $\varkappa = \omega$, the theorem is obviously true but without interest. Because of Theorem I–3, we have only to work with the case where $\boldsymbol{E}$ is linearly ordered (if not, take a $\boldsymbol{C}$-invariant strengthening of $\boldsymbol{E}$; obviously, a chain of the first is still a chain of the second). Let $\boldsymbol{D}$ be a chain of $\boldsymbol{E}$. Because $\varkappa$ is regular (here $\operatorname{cof}(\varkappa) \neq \omega$ is enough) there is some $\varphi \in \Phi$, the range E_φ of which contains a subset D_1 of D with power $\varkappa$. By Theorem I–4, $\boldsymbol{E}_\varphi$ is order isomorphic to a lexicographically ordered product of copies of $\boldsymbol{C}$ or $\boldsymbol{C}^*$. As $\varkappa$ is regular, every subset of power $\varkappa$ of such product contains a subset, of power $\varkappa$, which is isomorphic to a subchain of $\boldsymbol{C}$, or $\boldsymbol{C}^*$. D_1 has this property. This proves the theorem.

Remark. This result has no interest for power $\varkappa$ such that every chain of power $\varkappa$ contains a subchain isomorphic to the ordinal $\varkappa$ or its inverse $\varkappa^*$; for a strongly inaccessible $\varkappa$, this property is equivalent to weak compactness.

It is false for singular $\varkappa$. Let us give a counterexample: let $\boldsymbol{C}$ be the ordinal ω_ω and $\boldsymbol{E} = [\boldsymbol{C}]^2$ ordered as follows: $(a, b) \preccurlyeq (a', b')$ iff $a \leqslant a'$ or $a = a'$ and $b' < b$. This set contains a chain $\boldsymbol{D}$ of type

$$\omega^* + \omega_1^* + \ldots + \omega_n^* \ldots$$

(which, of course, has no common subchain with $\boldsymbol{C}$ or $\boldsymbol{C}^*$, of power $\aleph_\omega$. Take $D = \{(i, j)\colon\ i = \omega_n,\ j = \omega_n + \alpha,\ \alpha < \omega_n,\ n \in N\}$.

Generalization. Obviously *the conclusion of the above theorem holds if we have only a preorder* (namely a reflexive and transitive relation $\leqslant$). This preorder has the same chains that the associate order (namely the order $\preccurlyeq$ for which $a \preccurlyeq b$ iff $a = b$ or $a < b$ and $b \leqslant a$).

As previously stated, this conclusion holds if we have a binary relation with no cycles (NB: it does not matter that the transitivisation of this relation may not be $\boldsymbol{C}$-invariant order. The theorem can still be applied to some $\boldsymbol{C}$-invariant linear strengthening of this relation).

Otherwise we should like to point out that this relationnal framework is the natural one in which to extend the notion of enchainability which was recently appeared in group theory [1].

II. Applications to model theory.

Let us recall that $\boldsymbol{C}$ is a chain of *indiscernibles* in a model $\mathscr{A}$ iff $\boldsymbol{C}$ is a subset of A, and for every formula $\theta(x_1, \ldots, x_n)$ of the language of $\mathscr{A}$ and for every $\vec{a}, \vec{b}$ in $[\boldsymbol{C}]^n$, $\mathscr{A} \models \theta[\vec{a}]$ iff $\mathscr{A} \models \theta[\vec{b}]\,x$. In the following we consider a countable language, for the sake of simplicity. Let us recall the well-known result:

Ehrenfeucht–Mostowski Theorem. *Let T be a theory which has an infinite model. For every chain $\boldsymbol{C}$ there is some model $\mathscr{A}$ of T in which $\boldsymbol{C}$ is a chain of indiscernibles.*

For a theory T, let us consider T^* a Skolemization of T. Let $\mathscr{A}^*$ be a Skolemization of $\mathscr{A}$. If C is a subset of A, let us note by $\mathscr{A}^*(\boldsymbol{C})$ the Skolem hull

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[1] G. Higman, *Homogeneous relations*, Oxford Quaterly J. 28 (2) (1977), pp. 31–39.

of C in $\mathscr{A}^*$ and let $\mathscr{A}(C)$ be its reduct to the language of $\mathscr{A}$. We call such a model $\mathscr{A}(C)$, where C is a chain of indiscernibles, *an Ehrenfeucht–Mostowski model of T.*

An application of the previous relational result arises from the fact that any $\mathscr{A}^*(C)$ is C-generated and C-invariant modulo the set of functions which are interpretations of the Skolem functions. Let us reformulate Theorem I–4 for theories with ordered models (namely theories containing the axioms for partial ordering or theories T for which there is a formula which defines a partial ordering on each model of T_1).

II–1. THEOREM. *Let T be a theory (in a countable language), with infinite partially ordered models; then in every Ehrenfeucht–Mostowski model of T, generated by a chain C, every chain whose power is a regular cardinal $\varkappa$ contains a subchain of power $\varkappa$ order isomorphic to a subchain of C or C^*.*

By this result, we can build models with properties similar to the given chains. The following will illustrate this fact. Let us consider a chain C of power $\varkappa$ which does not contain the ordinal $\varkappa$ and its inverse $\varkappa^*$. Such chains do exist for non weakly compact cardinals, especially for successor cardinals (for cardinal $\varkappa = \mu^+$, take any subchain of power μ^+ in 2^μ lexicographically ordered, cf. [8]). If $\varkappa$ is regular, then, by the above theorem, an Ehrenfeucht–Mostowski model generated by C has the same property. So we get:

II–2. THEOREM. *Let T be a theory (in a countable language) with infinite partially ordered models. For every regular and no weakly compact cardinal $\varkappa$, there is some model of T of power $\varkappa$ without a subchain ordered as $\varkappa$ or $\varkappa^*$.*

In fact this result can be found in a more general setting in W. Hodges paper [7]. Furthermore he gives a counterexample of this result for strong limit cardinals.

Now let us consider *two chains* C *and* C' *of power* $\varkappa$ *which satisfy the following property* (P): "*there is no chain of power $\varkappa$ order or antiorder isomorphic to subchains in both* C *and* C'". Then, by Theorem II–1, we are going to show that the Ehrenfeucht–Mostowski models generated by C and C' have the property (P)(1). To get a large family of chains such that every pair of them has the property (P), let us introduce the following chains studied by R. Bonnet [2]: following his terminology, we say that a chain C is *super-rigid* if for every strictly increasing or decreasing partial map in C, the set of non-fixed points has a strictly smaller power than C. He proved that such chains of power $2^{\aleph_0}$ exist, and indicated how to generalize this result to any successor cardinal, assuming GCH.

Let us recall that, assuming GCH, for every successor cardinal $\varkappa$, there is a family of power $2^\varkappa$ of pairwise almost disjoint subsets of $\varkappa$ (namely the power of the intersection of two of them is less than $\varkappa$).

(1) For an example, if $\varkappa = \aleph_1$, let us consider R and ω_1.

By considering a super-rigid chain of successor cardinal $\varkappa$, some family as above induces a family of power $2^\varkappa$ of chains of power $\varkappa$, such that any pair of them has the property (P). So we have the following theorem and its immediate corollary:

II–3. THEOREM (Assuming GCH). *Let T be a theory with infinite partially ordered models. If one of them contains an infinite chain, then, for every successor cardinal $\varkappa$ there is a family of $2^\varkappa$ models of T, each of which is of cardinality $\varkappa$ and contains a chain of power $\varkappa$, such that for any chain* D *of power $\varkappa$, there are no two models in the family, into both of which* D *can be embedded.*

Proof. Take a family of power $2^\varkappa$ of chains of power $\varkappa$ such that any pair of them has the property (P); for each chain C of this family, let us consider the Ehrenfeucht–Mostowski models of T, generated by C. Let us suppose that a chain D of power $\varkappa$ is embedded in two such models generated by C_1 and C_2 respectively. Let f_1 and f_2 be the isomorphisms which map D onto D_1 in $\mathscr{A}(C_1)$, and onto D_2 in $\mathscr{A}(C_2)$ respectively. By Theorem II–1, D_1 which is a chain of power $\varkappa$ in $\mathscr{A}(C_1)$, has a subchain D_1' of power $\varkappa$ which is embedded by g_1 in C_1 (or C_1^*). Let D_2' be the isomorphic image under $f_2 \circ f_1^{-1}$ of D_1' in $\mathscr{A}(C_2)$. It is a subchain of D_2. By the previous argument it contains a subchain D_2'' of power $\varkappa$ which is isomorphic by g_2 to a subchain of C_2 (or C_2^*).

Let us consider the map $g_1 \circ f_1 \circ f_2^{-1} \circ g_2^{-1}$: it is a monotonic partial map from C_2 into C_1, which contradicts the fact that C_2 and C_1 have the property (P).

Let us say that two models $\mathscr{M}$ and $\mathscr{M}'$, of a theory T and of the same cardinality, *are almost disjoint* iff there is no model $\mathscr{M}''$ of T of the same power which can be embedded into $\mathscr{M}$ and $\mathscr{M}'$ (e.g. two chains of power $\varkappa$ have property (P) iff their associated betweeness relations are almost disjoint).

COROLLARY (GCH). *Let T be a theory with partially ordered infinite models. If every model of T contains a chain of the same power, then, for every successor cardinal $\varkappa$, there is a family of $2^\varkappa$ models of T of power $\varkappa$, which are pairwise almost disjoint (even more strongly: their reducts to the ordering are pairwise almost disjoint).*

APPLICATIONS. This result works for Peano Arithmetic. It also does for ZFC, because in every model of ZFC the chain of the ordinals has the same power as the model.

Remark. To get a family of μ models of cardinal $\varkappa$ as in the above theorem and corollary, it is enough to have a family of μ chains of power $\varkappa$ which are almost disjoint. Use the following lemma:

LEMMA. *Let $\mathscr{L}$ be a family of pairwise almost disjoint chains, every one of the same power $\varkappa$. Then there is some family of chains $\mathscr{L}'$, of the same power as $\mathscr{L}$, such that each chain in $\mathscr{L}'$ is a subchain of power $\varkappa$ of some chain of $\mathscr{L}$ and such that any pair of them has the property* (P).

Proof. Let $\mathscr{C}$ be the set of pairs of elements of $\mathscr{L}$ not satisfying the property

(P). If $|\mathscr{C}| < |\mathscr{L}|$, the family $\mathscr{L}'$ obtained by deleting the chains occuring in such pairs, has the required property. If $|\mathscr{C}| = |\mathscr{L}|$, for each pair $\{C, D\}$, let us take some chain E of power λ which is order embedded into both C (or C^*) and D (or D^*). We get a new family of chains and it is easy to verify that they have pairwise the property (P) (use the fact that the elements of $\mathscr{L}$ are pairwise almost disjoint).

Now we would show how it is easy to compare models using the super-rigid chains and our Theorem II–1, though the below theorem is true in a more general setting (see the Shelah results about instable theories [10]).

II–4. Theorem (GCH). *Let T be a theory with infinite partially ordered models. If one of them contains an infinite chain, then, for every successor cardinal $\varkappa$, one can build a function which maps every subset A of $\varkappa$ to a model $\mathscr{M}_A$ of T in such a way that the following are equivalent*:

(i) *A is a subset of B*;
(ii) *$\mathscr{M}_A$ is an elementary submodel of $\mathscr{M}_B$*;
(iii) *$\mathscr{M}_A$ is isomorphic to some submodel of $\mathscr{M}_B$*.

Proof. Let k be an element not in $\varkappa$ and $C = (\varkappa \cup \{k\}) \times \varkappa$. Let us define F as a mapping from $\mathscr{P}(\varkappa)$ into $\mathscr{P}(C)$ by $F(A) = (A \cup \{\varkappa\}) \times \varkappa$. Obviously we have: $A \subset B$ iff $F(A) \subset F(B)$ and if $A \subset B$ then $|F(A) - F(B)| = \varkappa$. Let us order C in such a way that the corresponding chain C is super-rigid. Let T^* be a Skolemization of our theory T, and let $\mathscr{M}^*(C)$ be the model of T^* in which C is a chain of indiscernibles, and the ordering on C is the same as the ordering induced by the model. This is equivalent to say that the order on C is definable by a formula of the language of T. That is possible because there is some model of T with an infinite chain. For every subset A of $\varkappa$, let us consider the submodel of $M^*(C)$ generated by $F(A)$. It is a model of T^* (note that $F(A)$ is an infinite subchain of C), and its reduct to T, $\mathscr{M}_A$, is a model of T. Obviously if A is a subset of B, then $\mathscr{M}_A$ is an elementary submodel of $\mathscr{M}_B$.

So we have proved: (i) $\Rightarrow$ (ii). The implication (ii) $\Rightarrow$ (iii) is trivially true. Now let us prove (iii) $\Rightarrow$ (i). Let us suppose that $\mathscr{M}_A$ is isomorphic to a submodel of $\mathscr{M}_B$. If A is not a subset of B we have $|F(A) - F(B)| = K$. Then let us consider some embedding g of $\mathscr{M}_A$ into $\mathscr{M}_B$. It maps the chain $F(A) - F(B)$ onto a chain included in $\mathscr{M}_B$. This chain being of power $\varkappa$, Theorem II–1 says that it contains some subchain D of power $\varkappa$, order isomorphic to a subchain of $F(B)$ or to its converse. Let h be some monotonic function from D into $F(B)$. The function $h \circ g$ mapping $F(A) - F(B)$ into $F(B)$ is a monotonic partial mapping of C which moves $\varkappa$ elements of C. This contradicts the super-rigidity of C.

Some remarks and problems. Counting the number of models is not the only way to describe the complexity of a class of models. We can preorder the class of models of cardinality $\varkappa$ by defining, for example $\mathscr{M} \leqslant \mathscr{M}'$ iff $\mathscr{M}$ is isomorphic to a submodel of $\mathscr{M}'$. And a question is: *how "large" is the associate order*? In considering the question, let us recall the notion of *dimension* introduced by B. Dushnik, E. W. Miller [1]: the *dimension* of a partially ordered set E is the

smallest cardinal λ such that E is isomorphic to subset of a direct product of λ chains. It is known that the dimension of $2^\varkappa$ is $\varkappa$, so results such as Theorem II–4 asserts that *the dimension of the associate ordering to the class of models of cardinality $\varkappa$ is at least $\varkappa$. We conjecture that this dimension is at least $\varkappa^+$*. (It is true for the theory of chains, because R. Laver proved in [6] that for every infinite $\varkappa$ the class of scattered chains of cardinality $\varkappa$ has dimension $\varkappa^+$). So the question is: *for an unstable theory T, is $\varkappa^+$ or $2^\varkappa$ the dimension of the class of models of T of power $\varkappa$ ($\varkappa > \omega$)*?

Further generalizations. To establish relations between comparisons of chains and comparisons of models, we have to get stronger results about the chains in Ehrenfeucht–Mostowski models.

Already by Lemma I–3, it follows that a chain in an Ehrenfeucht–Mostowski model generated by C is a part of a countable union of finite lexicographical products of C or C^*. For example, if C is a countable union of scattered chains then every chain in an Ehrenfeucht–Mostowski model generated by C is of the same kind. Important results about these chains are obtained by Laver and Galvin. Although some of these are usefull, we do not use these methods because we think that it is possible to get more: for instance *that a totally ordered Ehrenfeucht–Mostowski model generated by C is a subchain of a countable lexicographical product of copies of C or C^** (see below).

We would like to obtain the same results for chains in the case where the model is not ordered, so that we could apply the results to unstable theories. This problem can be summed up as follows: is a chain on a set with a C-invariant binary relation a part of a lexicographical product of C or C^*? Note, by Theorem 1, it is enough to assume that the binary relation is a tournament (namely an antisymmetric binary relation in which two arbitrary elements are connected).

Added in proof. Since the paper has been submitted, some of the problems mentionned above have been solved. Especialy we have a complete description of ordered E.M. models. By this description we get the following: A linearly ordered E.M. model generated by a chain C, which contains C^* and the chain Q of the rationnal numbers, is a part of the set $C^{<\omega}$ of finite sequences of elements of C, lexicographically ordered. With this, we get that the class of models with cardinality $\varkappa$, $\varkappa > \omega$ for a theory as above, contains a subclass order isomorphic to the class of chains of the same cardinality. Thus its dimension is at least $\varkappa^+$.

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A new construction of a Kurepa tree with no Aronszajn subtree

by

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Abstract. In 1969, we asked whether V = L implies the existence of a Kurepa tree having no Aronszajn subtrees. The affirmative answer to this question was supplied by Ronald Jensen in 1971, whose proof appeared in [2]. Jensen's proof was somewhat involved, and required some delicate argumentation. We present here a much simpler proof which has the same degree of complexity as the construction of any Kurepa tree in L.

Preliminaries. For terminology and notation covering trees we refer to either [1] or [2]. An in these references, for $\lambda \leqslant \omega_1$, by a *$\lambda$-tree* we mean a normal tree of height λ having countable levels. An *Aronszajn tree* is an ω_1-tree with no uncountable branch, a *Kurepa tree* is an ω_1-tree with at least $\aleph_2$ uncountable branches. Aronszajn trees can be constructed in ZFC. Kurepa trees can be constructed assuming V = L (Solovay) or $\diamondsuit^+$ – which is true if V = L (Jensen).

For background on constructibility we refer to [1]. We shall not require any fine structure theory.

The question as to whether V = L implies the existence of a Kurepa tree with no Aronszajn subtrees was raised by me in 1969, and answered affirmatively by Jensen in 1971. Jensen's (rather involved) proof appeared in [2], together with an application of such a tree to solve a problem in partition calculus. At the time, it seemed as though, my application to combinatorics not withstanding, such trees were merely a curiosity. (Indeed, my original question was little more than a "coffee room" variety.) That this was not the case was demonstrated by Juhász and Weiss ([3]), who proved that the existence or such a tree is equivalent to the existence of an ω_1-metrizable, ω_1-compact space of cardinality at least ω_2, resolving an old question of Sikorski.

The new construction of such a tree (from V = L) does not involve any new methods, rather a refinement of the known tricks of the trade. That a rather simple modification to the standard construction of a Kurepa tree in L would give the required result occurred to me after a discussion with Bill Fleissner on some work of Ken Kunen and himself on the normal Moore space problem.

[1] The result in this paper was obtained during the summer of 1980 whilst I was visiting the University of Toronto (Erindale College). My stay in Toronto was supported in part by a joint Nuffield Foundation/NSERC award.