Approximating homotopy equivalences of surfaces by homeomorphisms

by

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Abstract. We prove the 2 and 3-dimensional version of the "Splitting Theorem" of Chapman and Ferry [2]. The consequence of this is the 2-dimensional analogue of the α-approximation theorem [2], and the equivalence of the 3-dimensional α-approximation theorem and of the Poincaré conjecture.

1. Introduction. The aim of this note is to extend some of the high-dimensional theorems of Chapman and Ferry to dimensions 2 and 3. More precisely, we prove the "Splitting theorem" from [2] in these dimensions. The 2-dimensional version of this theorem implies the 2-dimensional analogues of the "α-approximation theorem" and the "Bundle theorem" from [2], and theorem (1) from [3]. The 3-dimensional "Splitting Theorem" proves that the 3-dimensional "α-approximation theorem" is equivalent to the classical Poincaré conjecture.

The additional motivation for the proof of the 2-dimensional "α-approximation theorem" was [6], where it was used to study the fixed point sets of the close PL involutions of 3-manifolds.

We adopt from [2] the following notation: Let X, Y be two spaces and let α be an open cover of Y. We say that the *maps* $f, g\colon X \to Y$ *are* α-*homotopic* (written $f \overset{\alpha}{\simeq} g$) if there is a homotopy $F_t\colon f \simeq g$, $t \in [0, 1]$ such that the track of each point $\{F_t(x)\colon 0 \leqslant t \leqslant 1\}$ lies in some element of α. If $h\colon X \to Y$ is a map and Y is given a fixed metric then $f^{-1}(\varepsilon)$ denotes the cover $\{U \subset X\colon U$ is open and $\operatorname{diam} f(U) < \varepsilon\}$ of X. More generally $f^{-1}(\alpha)$ denotes $\{U \subset X\colon U$ is open in X and there exists a $V \in \alpha$ such that $f(U) \subset V\}$ whenever α is a cover of Y. If A is a subset of Y and α is a cover of Y, then we say that $f\colon X \to Y$ is an α-*equivalence over* A *with the* α-*inverse* g if g is a map of A into X, $fg|A$ is α-homotopic to the inclusion id_A, and $gf|f^{-1}(A)$ is $f^{-1}(\alpha)$-homotopic to the inclusion $\mathrm{id}_{f^{-1}(A)}$. If $A = Y$, then we say that f is an α-*equivalence*. If β is a cover of Y and $f\colon X \to Y$ is a proper map, then we say that f is a β-*map* if for every $y \in Y$ there is a $U \in \beta$ such that $f^{-1}(y) \subset U$. If X is a metric space then we say that f is an ε-*map* if for every

$y \in Y$, $f^{-1}(y)$ has a diameter $\leqslant \varepsilon$. By $f_{\#}$ and f_{*} we shall denote homomorphisms induced by the map f on homotopy and homology groups respectively.

We assume the following data:

(1.0) W is an n-manifold without boundary, $n = 2$ or 3, and W is orientable if $n = 2$. $S = S^{n-1}$ is an $(n-1)$-dimensional sphere. We put: $B_a^0 = S \times (-a, a)$ and $B_a = S \times [-a, a]$ for $a \in R$, $a > 0$. Let p: $S \times R \to R$ denote the usual projection. f: $W \to S \times R$ is a proper map, which is a $p^{-1}(\varepsilon)$-equivalence over B_2, with $p^{-1}(\varepsilon)$-inverse g: $B_2 \to W$.

Splitting Theorem (1.1). *Suppose that (1.0) is satisfied. Then if ε is sufficiently small, then there is an $(n-1)$-sphere $S_0 \subset (pf)^{-1}((-1, 1))$ such that $f|S_0$: $S_0 \to S^{n-1} \times R$ is a homotopy equivalence, S_0 is bicollared, and S_0 separates the component of W containing $(pf)^{-1}([-1, 1])$ into two components, one containing $(pf)^{-1}(-1)$ and the other containing $(pf)^{-1}(1)$.*

Addendum. It also follows that if C_0 is the closure of the component of $(pf)^{-1}((-1, \frac{4}{3}))\backslash S_0$ containing $(pf)^{-1}(1)$, and C_1 is the closure of the component of $(pf)^{-1}((-1, \frac{5}{3}))\backslash S_0$ containing $(pf)^{-1}(1)$, then C_0 deforms into S_0 rel S_0, with the deformation taking place in C_1 (i.e. there is a homotopy H_t: $C_0 \to C_1$ such that H_0 is an inclusion and $H_1(C_0) \subset C_1$, and $H_t|S_0 = \mathrm{id}_{S_0}$).

Note that ε depends neither on W nor on f.

In dimension 2 the Splitting Theorem and the torus argument imply, as in [2], the following theorems (see Section 3):

α-Approximation Theorem (1.2). *Let N^2 be a surface. For every open cover α of N there is an open cover β of N such that for any surface M and proper β-equivalence f: $M \to N$, which is already a homeomorphism from ∂M onto ∂N, f is α-close to a homeomorphism h: $M \to N$ (i.e. for every $m \in M$ there is a $U \in \alpha$ containing $f(m)$ and $h(m)$).*

Bundle Theorem (1.3). *Let p: $E \to B$ be a Hurewicz fibration such that E and B are locally compact metric spaces, B is locally path connected and locally finite dimensional, and the fibres $p^{-1}(b)$ are compact surfaces. Define $\partial E = \bigcup \{\partial p^{-1}(b)|\ b \in B\}$ and assume that $p|\partial E$: $\partial E \to B$ is a locally trivial bundle. Then p is also a locally trivial bundle.*

(1.3) gives another partial answer to the question raised by Raymond [8]. Then we can apply the proof used in [3] to get

Theorem (1.4). *If M is a surface and α is an open cover of M, then there is an open cover β of M such that, if N is a surface and g: $(M, \partial M) \to (N, \partial N)$ is a proper β-map, then g is homotopic through α-maps to a homeomorphism.*

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2. The proof of the "Splitting Theorem". To prove (1.1) we shall need lemmas. All the manifolds considered have dimension $\leqslant 3$, and so by [7] we can consider them as PL manifolds. In particular, we assume that $S^{n-1} \times R$ has a natural PL structure in which every submanifold $S^{n-1} \times [a, b]$ is PL.

The following lemma is easy, and so we omit the proof.

Lemma (2.1). *Let N be a surface, $x_0 \in N$, and let $\mathfrak{N}$ be a subgroup of $\pi_1(N, x_0)$ such that $\mathfrak{N} \neq \pi_1(N, x_0)$. Then there exists a PL embedding ξ: $S^1 \to N$ which determines an element $[\xi]$ of $\pi_1(N, x_0)\backslash\mathfrak{N}$.*

Lemma (2.2). *Given $a, b \in (-2, 2)$, $a < b$ there exists an ε such that whenever (1.0) holds, then there exists a PL $(n-1)$-sphere $S_0 \subset (pf)^{-1}((a, b))$ satisfying $k \cdot [S_0] \neq 0$ for every integer $k \neq 0$, where $[S_0]$ is the image in $H_{n-1}(f^{-1}(B_2))$ of the fixed generator of $H_{n-1}(S_0) \simeq Z$ by the homomorphism induced by the inclusion $S_0 \hookrightarrow f^{-1}(B_2)$. Moreover if $n = 3$, then S_0 disconnects the component of $f^{-1}(\bar{B}_2)$ containing S_0.*

Proof of 2.2. Let $c = \frac{1}{2}(a+b) \in (a, b)$ and $\varepsilon < \frac{1}{2}(b-a)$, and let f: $W \to S \times R$ satisfy our requirements for this choice of ε. Then $N = (pf)^{-1}((a, b))$ is a PL n-submanifold of W, and $N \supset (pf)^{-1}(\{c\}) \cup g(p^{-1}(\{c\}))$ (this follows from the fact that pfg is ε-close to p on $p^{-1}(\{c\})$). Let $g_0 = g|p^{-1}(\{c\})$. Then g_0 is a map into N of the $(n-1)$-sphere $p^{-1}(\{c\}) = S \times \{c\}$. Take $x_0 \in g(p^{-1}(\{c\}))$ and let i: $N \to W$ denote the inclusion and $i_{\#}$: $\pi_{n-1}(N, x_0) \to \pi_{n-1}(W, x_0)$ denote the induced homomorphism. Then $\mathfrak{N} = \mathrm{Ker}(i_{\#})$ is a π_1-invariant subgroup of $\pi_{n-1}(W, x_0)$. We claim that $[g_0] \in \pi_1(N, x_0)\backslash\mathfrak{N}$. In fact otherwise fg_0: $p^{-1}(\{c\}) \to S \times R$ would be homotopic to a constant, which is not the case since it is homotopic to the inclusion $p^{-1}(\{c\}) \to S \times R$. So by the sphere and projective plane theorem (see [5], p. 54) in the case of $n = 3$, and by Lemma (2.1) in the case of $n = 2$, there exists a covering map ξ: $S \to S_0 \subset N$ (which for $n = 2$ must be a homeomorphism), where S_0 is a PL 2-sphere S^2 or a projective plane P^2 if $n = 3$, and S_0 is a 1-sphere if $n = 2$, $x_0 \in S_0$, and ξ determines an element $[\xi]$ of $\pi_{n-1}(N, x_0)$. In particular S_0 is not contractible in N.

Now we show that, if $n = 3$, then S_0 disconnects the component of $f^{-1}(B_2)$ containing S_0. Suppose that it does not. Then there exists a simple closed curve α: $S^1 \to f^{-1}(B_2)$ such that $\alpha(S^1) \cap S_0 = x_0$, and $\alpha(S^1)$ is transversal to S_0.

Now we use the theory of the intersection index, as described in [1], pp. 97 and 114. To avoid assumptions concerning orientability, we consider homology with Z_2-coeficients. It is easy to see that $\alpha(S^1)$ and S_0 support the 1-cycle and 2-cycle respectively with Z_2-coefficients such that the corresponding homology classes $z_1 \in H_1(W, Z_2)$ and $z_2 \in H_2(W, Z_2)$ have an intersection index $z_1 \cdot z_2 = 1$. This implies that $z_1 \neq 0$, and so α is not homotopic to a constant. Since $\pi_1(B_2) = 0$ if $n = 3$, it follows on the other hand that $f\alpha$ and $gf\alpha$ are homotopic to constant maps which contradicts the fact that $gf\alpha \simeq \alpha$. So S_0

disconnects the component of $f^{-1}(B_2)$ which contains it. In particular S_0 is bicollared in W.

Now we prove that, in the case of $n=3$, S_0 is a sphere, and not a projective plane. Suppose on the contrary, that $S_0 \cong P^2$. Then there exists a closed curve $\alpha\colon S^1 \to S_0$, which reverses the orientation of $S_0 \cdot S_0$ is bicollared in W and so α reverses the orientation of W. This implies that α is not homotopic to a constant and we get a contradiction as before. So S_0 is a sphere.

Finally we prove that for any integer $k \neq 0$ we have $k \cdot [S_0] \neq 0$. Suppose that, for some $k \neq 0$, $k \cdot [S_0]$ is equal to 0. $[S_0] = i_*(e)$, where i_* is induced by the inclusion $i\colon S_0 \hookrightarrow f^{-1}(B_2)$ and e is a generator of $H_{n-1}(S_0) \cong Z$, and so $(f|S_0)_* \circ (r_k)_*\colon H_{n-1}(S_0) \to H_{n-1}(B_2)$ must be 0 for any map $r_k\colon S_0 \to S_0$ of degree k. But this is not the case: if it were, then $(f|S_0) \circ r_k$ and hence $f|S_0$ would be homotopic to constant maps, contradicting the fact that $gf|S_0$ is homotopic to the inclusion $S_0 \to W$ and S_0 is not contractible in W.

LEMMA (2.3). *Let T be a compact, orientable surface and let S_1 and S_2 be two disjoint 1-spheres, not contractible in T. Suppose that there exists a homotopy $h_t\colon S^1 \to T$, $t \in [1, 2]$ such that $h_i(S^1) = S_i$, and that the maps $h_i\colon S^1 \to S_i$ have a non-zero degree. Then there exists an embedding $h'\colon S^1 \times [1, 2] \to T$ such that $h'(S^1 \times \{i\}) = S_i$ for $i = 1, 2$.*

Proof of (2.3). S_2 is a PL sphere in T, and so we can find a PL embedding $u\colon S_2 \times [0, 1] \to T$ such that $S_2 = u(S_2 \times \{0\})$ and $u(S_2 \times [0, 1]) \cap S_1 = \emptyset$. We consider the decomposition space $\tilde{T} = T/G$ with non-degenerate points $a_t = u(S_2 \times \{t\})$, $t \in [0, 1]$, and with the projection map $q\colon T \to \tilde{T}$. Then $q|T \setminus u(S_2 \times [0, 1])$ is an embedding, and $A = \bigcup_{t \in [0,1]} a_t$ is an arc and $\check{T} = \tilde{T} \setminus \bigcup_{t \in (0,1)} a_t$ is a compact, orientable surface. Let T_1 be the component of $\check{T}$ containing $q(S_1)$. Then it is easy to construct a map $f\colon \check{T} \to T_1$ such that $f|T_1 = \mathrm{id}_{T_1}$. Let $s \in \pi_1(T_1)$ be determined by the inclusion $q(S_1) \hookrightarrow T$. Then fqh_t is a homotopy between $fqh_1\colon S^1 \to T$ and a constant map $fqh_2\colon S^1 \to f(a_t) \in T_1$. So $s^k = 1$ for a certain $k \neq 0$. But then we have $s = 1$. Suppose the opposite, i.e. $s^k = 1$. Then a subgroup of $\pi_1(T_1)$ generated by s is finite, cyclic, and so the covering $\tilde{T}_1$ of T_1 corresponding to it satisfies $H_1(\tilde{T}_1) = \pi_1(\tilde{T}_1) = Z_k$, which is impossible. This implies that $q(S_2) = \partial D$ for some disc $D \subset T_1$, whence $S_1 \cup S_2$ bounds either an annulus $q^{-1}(D)$ or an annulus $q^{-1}(D \cup A)$, depending on whether D contains a_0 or a_1.

LEMMA 2.4. *Let M be a connected 3-manifold, $\partial M = \emptyset$, and let S_1 and S_2 be two disjoint 2-spheres in M such that the elements $[S_1]$ and $[S_2]$ of $H_2(M)$ determined by S_1 and S_2 (as in (2.2)) satisfy the following condition: there are integers k_1, k_2 such that $k_1 \cdot [S_1] = k_2 \cdot [S_2] \neq 0$. Moreover, we assume that each S_i, $i = 1, 2$, disconnects M. Let L be a closure of a component of $M \setminus S_1 \setminus S_2$ such that $L \supset S_1 \cup S_2$. Then L is a compact manifold with the boundary $\partial L = S_1 \cup S_2$.*

Proof of (2.4). Suppose that L is non-compact. Let M_1 and M_2 be the closures in M of two components of $M \setminus L$ (note that $M \setminus S_2 \setminus S_2$ has precisely 3 components) such that $L \cap M_i = S_i$ for $i = 1, 2$. We consider the exact homology sequence of the pair $(M, M_1 \cup M_2)$:

$$H_3(M, M_1 \cup M_2) \to H_2(M_1 \cup M_2) \xrightarrow{j} H_2(M) \to H_2(M, M_1 \cup M_2).$$

If L is non-compact, then $H_3(M, M_1 \cup M_2) = H_3(L, S_1 \cup S_2) = 0$, and so j is a monomorphism. But $H_2(M_1 \cup M_2) = H_2(M_1) \oplus H_2(M_2)$ and, for each $i = 1, 2$, there is a $z_i \in H_2(M_i)$ such that $(j_i)_*(z_i) = [S_i]$, where $j_i\colon M_i \to M$ is an inclusion. So the fact that j is a monomorphism implies that $k_1 \cdot ((j_1)_*(z_1)) - k_2((j_2)_*(z_2)) = k_1 \cdot [S_1] - k_2 \cdot [S_2] \neq 0$. But we know that $k_1 \cdot [S_1] - k_2 \cdot [S_2] = 0$. So L is compact. This and the fact that $\partial M = \emptyset$ imply that $\partial L = S_1 \cup S_2$.

LEMMA (2.5). *Let $0 < \varepsilon < 1$, and $a_i, b_i \in (-2, 2)$, $i = 1, 2$, be such that $-2+2\varepsilon < a_1 < b_1 < a_2 < b_2 < 2-2\varepsilon$. Assume that (1.0) is satisfied, and let $S_i \subset (pf)^{-1}((a_i, b_i))$ be a PL $(n-1)$-sphere such that $k \cdot [S_i] \neq 0$ for $k \neq 0$, and if $n = 3$ then S_i disconnects the component of $f^{-1}(B_2)$ which contains it. $[S_i]$ is the image in $H_{n-1}(f^{-1}(B_{2-\varepsilon}))$ of the fixed generator of $H_{n-1}(S_i)$. Then there is a compact PL n-submanifold L of W such that L is an h-cobordism from S_1 to S_2.*

Proof of (2.5). First we prove that there exists a homotopy $h_t\colon S^{n-1} \to f^{-1}(B^0_{2-\varepsilon})$, $t \in [1, 2]$, such that $h_i(S) = S_i$ and that $h_i\colon S \to S_i$ has non-zero degree for $i = 1, 2$. Let $f_i = f|S_i\colon S_i \to B^0_{2-2\varepsilon}$ and let m_i be the degree of f_i (we define the degree of f_i as a number equal to the degree of $p_S \circ f_i$ where $p_S\colon S \times R \to S$ is a projection). Since $S_i \subset f^{-1}(B^0_{2-2\varepsilon})$ and $gf|S_i \overset{(pf)^{-1}(\varepsilon)}{\simeq} \mathrm{id}_{S_i}$, the map $gf_i\colon S_i \to f^{-1}(B^0_{2-\varepsilon})$ is homotopic to id_{S_i} in $f^{-1}(B^0_{2-\varepsilon})$. This and the fact that $[S_i] \neq 0$ imply that $m_i \neq 0$ for $i = 1, 2$. Let k_1, k_2 be the integers such that $m_1 \cdot k_1 = m_2 \cdot k_2 \neq 0$, and $r_i\colon S \to S_i$ be any map of degree k_i for $i = 1, 2$. fr_1 and fr_2 have the same degree, so they are homotopic in $S \times (a_1, b_2)$. Therefore gfr_1 and gfr_2 are homotopic in $g(S \times (a_1, b_2)) \subset f^{-1}(S \times (a-\varepsilon, b+\varepsilon))$. Since g is a $p^{-1}(\varepsilon)$-inverse for f over B_2, there are $p^{-1}(\varepsilon)$-small homotopies between gfr_i and r_i, $i = 1, 2$; their values lie in $f^{-1}(S \times (a_1 - \varepsilon, b_2 + \varepsilon))$. Thus there is a homotopy $h_t\colon S \to f^{-1}(B^0_{2-\varepsilon})$, $t \in [1, 2]$ such that $h_1 = r_1$ and $h_2 = r_2$. This is the homotopy we were looking for.

Then suppose $n = 2$. $\bigcup_{t \in [1,2]} h_t(S^1)$ is a compact space, and so it is contained in some compact surface $T \subset f^{-1}(B^0_{2-\varepsilon})$. Then by Lemma (2.3) we can find an annulus $L \subset T$, with $\partial L = S_1 \cup S_2 \cdot L$ is the required h-cobordism.

If $n = 3$, then the existence of h_t implies that the homology classes $[S_1]$, $[S_2] \in H_2(f^{-1}(B_{2-\varepsilon}))$ satisfy the condition $k_1 \cdot [S_1] = k_2 \cdot [S_2] \neq 0$, for some integers k_1, k_2. Of course $f^{-1}(B_{2-\varepsilon})$ is a 3-manifold with an empty boundary, and so we can use Lemma (2.4) to prove that S_1 and S_2 bound in

$f^{-1}(B_{2-\varepsilon})$ a compact manifold L. We only have to show that L is a h-cobordism. By [5], p. 26, we need to show that L is simply connected. Let $\alpha\colon S^1 \to L$ be any map. We may assume that α is PL and that $\mathrm{Im}(\alpha) \subset \mathrm{Int}(L)$. Then $f\alpha\colon S^1 \to B_{2-\varepsilon}$ is homotopic to a constant map, because $B_{2-\varepsilon}$ has a homotopy type of S^2. So $gf\alpha\colon S^1 \to W$ is homotopic to a constant map. Moreover, by (1.0), $\alpha \simeq gf\alpha$ and hence α is homotopic to aconstant map in W. Let $\bar{\alpha}\colon D^2 \to W$ be any map of the 2-disc D^2 into W which extends $\alpha(S^1 = \partial D^2)$, and which is transversal to $S_1 \cup S_2$. Then $\bar{\alpha}^{-1}(S_1 \cup S_2)$ is a finite collection of circles in D^2, and the component P of $\bar{\alpha}^{-1}(L)$ which contains ∂D^2 is a PL submanifold of D^2 bounded by a finite family l of circles. For any $c \in l$, $c \neq \partial D^2$, $\alpha^2|c$ can be extended to the map $P_c \to S_i$, $i = 1$ or 2, where P_c is a disc bounded in D^2 by c. The union of these extensions and of $\bar{\alpha}|P$ gives a map $D^2 \to L$ which extends α. This proves that L is simply connected.

Now we can prove Theorem (1.1):

Proof of (1.1). Let $\varepsilon \in (0, 1)$ and $a_i, b_i \in (-2, 2)$, $i = 1, 2$, or 3, satisfy the inequality

$$-2+2\varepsilon < a_1 < b_1 < -1-\varepsilon < -1+\varepsilon < a_2 < b_2 < 1-\varepsilon < 1+\varepsilon < \tfrac{4}{3}+\varepsilon < a_3 < b_3 < \tfrac{5}{3}-\varepsilon < 2-2\varepsilon.$$

Assume in addition that ε is so small that the hypothesis of (2.2) is satisfied for $(a, b) = (a_i, b_i)$, $i \in \{1, 2, 3\}$. Let $S_i \subset (pf)^{-1}((a_i, b_i))$ be PL spheres provided by (2.2) and our choice of ε, $i = 1, 2$, or 3, and let L be an h-cobordism from S_1 to S_3 provided by (2.5). In the sequel we shall use the following.

Claim. *Let Q be a connected n-submanifold of W with the boundary ∂Q. If $pf(\partial Q)$ misses a segment (a, b) contained in $(-2, 2)$ and intersects each component of $R \setminus (a, b)$, then $Q \supset f^{-1}(S \times [a+\varepsilon, b-\varepsilon])$.*

Proof of the claim. Fix $x_0 \in W$ with $pf(x_0) \in [a+\varepsilon, b-\varepsilon]$. By the connectedness of $pf(Q)$ there is a point $x_1 \in Q$ with $pf(x_0) = pf(x_1)$. Let $\alpha\colon [0, 1] \to pf(x_0) \times S$ be a path connecting $f(x_0)$ and $f(x_1)$. Then $\beta = g\alpha$ is a path between $gf(x_0)$ and $gf(x_1)$ lying in $f^{-1}(S \times (a, b))$ (we use here (1.0), which implies that on B_2 the maps p and pfg are ε-close). Using (1.0) again, we get paths β_i connecting x_i and $gf(x_i)$ for $i = 1, 2$ and such that $\mathrm{diam}\big(pf(\mathrm{Im}(\beta_i))\big) < \varepsilon$. Then $\mathrm{Im}(\beta_i) \subset f^{-1}(S \times (a, b))$ and $\beta = \beta_1 \cup g\alpha \cup \beta_2$ is a path in $f^{-1}(S \times (a, b))$ connecting x_0 and x_1. Then $\mathrm{Im}(\beta)$ misses ∂Q, and since $x_1 \in Q$, it follows that $x_0 \in Q$. This finishes the proof of the claim.

Since pfg is ε-close to p on B_2 it follows from the claim applied to $Q = L$ that $g(B_1) \subset f^{-1}(B_{1+\varepsilon}) \subset L$. In particular $S_2 \subset L$ and S_2 separates L into two components such that their closures L_1 and L_2 form h-cobordisms from S_1 to S_2 and from S_2 to S_3 respectively. Applying the claim to $Q = L_i$ for $a \in R$, we infer that $L_1 \supset (pf)^{-1}(-1)$ and $L_2 \supset (pf)^{-1}(1)$.

Now to prove that S_2 disconnects the component A of W cutting it into two components, one containing $(pf)^{-1}(-1)$ and the other containing $(pf)^{-1}(1)$, we need only to show that $A \setminus L$ is not connected. Suppose it is. Then there is a curve $\alpha\colon [0, 1] \to W$, such that $\alpha(0) \in S_3$, $\alpha(1) \in S_1$, and $\alpha([0, 1]) \cap L = \alpha(\{0, 1\})$. Hence there is a point $y \in \alpha((0, 1))$ such that $f(y) \in p^{-1}([-1, 1])$. Then $y \in f^{-1}(B_1) \subset L$, which gives a contradiction.

Finally we prove that $f|S_2\colon S_2 \to S \times R$ is a homotopy equivalence. First we notice that $f(S_2) \subset B_{1-\varepsilon}$. This, as we have shown, implies that $gf(S_2) \subset L \cap f^{-1}(B_{1+\varepsilon})$. From the fact that $gf|f^{-1}(B_2) \overset{(pf)^{-1}(\varepsilon)}{\simeq} \mathrm{id}$ it follows that there is a homotopy $h_t\colon S_2 \to W$, with $h_0 = \mathrm{id}_{S_2}$, $h_1 = gf|S_2$, such that the track $\{h_t(x)\colon t \in [0, 1]\}$ of each point $x \in S_2$ has image by pf of diameter $< \varepsilon$. This implies that for every $x \in S_2$, $\{h_t(x)\colon t \in [0, 1]\} \cap \partial L = \emptyset$, so $gf|S_2$ is a homotopy equivalence between S_2 and L, and so $f|S_2$ is a homotopy equivalence.

Now we can put $S_0 = S_2$. As we have shown it satisfies all the conditions of (1.1).

The addendum may be proved as follows: First we note that $(pf)^{-1}([-1, \tfrac{4}{3}]) \subset L$. This can be shown in the precisely the same way in which we have shown that $(pf)^{-1}(1) \subset L_2$, using the claim. The only difference is that we replace $\{1\}$ by $[-1, \tfrac{4}{3}]$ and L_2 by L, and we use the fact that $b_1 + \bar{\varepsilon} < -1 < \tfrac{4}{3} < a_3 - \bar{\varepsilon}$. This implies that the component C_0 of $(pf)^{-1}([-1, \tfrac{4}{3}]) \setminus S$ containing $(pf)^{-1}(1)$ is contained in L_2. But $S_0 = S_2$ is a deformation retract of L_2, and so C_0 can be deformed to S_0 in L_2. Finally we notice that $L_2 \subset C$. This easily follows from the fact that $b_3 < \tfrac{5}{3} - \varepsilon$. This finishes the proof of (1.1).

3. Remarks on the proof of the α-approximation theorem for dimension 2. The proof is only slightly different from the one given by Chapman and Ferry in [2].

First, using the "orientable" splitting theorem, we prove the "handle Lemma" as in [2] p. 589, with $n = 2$, and an orientable V^2. The proof is analogous to the one given in [2]. We need only to note that the surface W_0 ([2], p. 591) is immersed in V, and V is orientable in our case, so W_0 is orientable, and hence we can construct the orientable W_1, W_2, W_3 as in [2].

As in [2] we prove the "Main theorem" (p. 595 in [2]), with $n = 2$ and an orientable V^2. Then we prove the following, weaker version of the "α-approximation theorem":

Lemma (3.1). *Let N^2 be a surface, and let γ be any open cover of N. Then, for every open cover α of N, there is an open cover β of N such that for any β-equivalence $f\colon M \to N$, which is already a homeomorphism from ∂M onto ∂N, and is such that for any $c \in \gamma$, $f^{-1}(c)$ is an orientable surface, and f is α-close to a homeomorphism $h\colon M \to N$.*

The proof proceeds as in [2], pp. 597, 598. First, we prove the version of Lemmas (5.1) and (5.2) of [2] with $n = 2$ and an orientable M.

Then, the proof of Lemma (3.1) proceeds as the proof of the α-approximation theorem in [2], pp. 598, and the only change is that by our assumption, we require that a star finite cover $\{N_i\}$ found in the proof of the α-approximation theorem in [2] be such that all sets $f^{-1}(N_i)$ are orientable.

Lemma (3.1) easily implies Theorem (1.2) if we use the following Lemma (3.2):

LEMMA 3.2. *Let N be a surface. Then there is an open cover α of N such that if M is a surface and $f\colon M \to N$ is an α-equivalence, then for every $c \in \alpha$, $f^{-1}(c)$ is an orientable surface.*

Proof of (3.2). Suppose that γ is any cover of N by the open discs. Then we can easily find an open cover α of N such that to every $c \in \alpha$, there is $d \in \gamma$ such that $c \subset d$. Suppose that for a certain $c \in \alpha$ and some α-equivalence $f\colon M \to N$, $f^{-1}(c)$ is a non-orientable surface. Then there exists an element z of $H_1(f^{-1}(c), Z)$ such that $z \neq 0$, and $0 \neq i_*(z) \in H_1(M)$, where $i\colon f^{-1}(c) \to M$ is an inclusion. We can take for z an element of $H_1(f^{-1}(c))$ determined by the curve reversing orientation of $f^{-1}(c)$. Let g be the inverse of f. Then $g_*(f|f^{-1}(c))_*(z) = (g_* f_* i_*(z)) = i_*(z) \neq 0$, and on the other hand $(f|f^{-1}(c))_*(z) = 0$, because $c \subset d \in \gamma$, which gives a contradiction.

4. The equivalence of the α-approximation theorem and the Poincaré conjecture in dimension $n = 3$. It is very easy to construct for every $\varepsilon > 0$, the ε-equivalence from the homotopy sphere $\neq S^3$ (if one exists) onto S^3. This equivalence obviously cannot be approximated by homeomorphisms.

On the other hand if the Poincaré conjecture is satisfied, then we can use our "Splitting Theorem" in dimension 3 to prove the α-approximation theorem, as in [2]. The only difference is that in the construction of h in the "Handle Lemma" (step V) we use the theorem of Waldhausen [10] in the form described in [4] (Lemma 3, p. 65 in [4]). Note that the manifold W_3 in the step V of the construction in the "Handle Lemma" in [2] is homotopy equivalent to $B^k \times T^m$, $m+k = 3$, and we assume that the Poincaré conjecture holds, whence W_3 is irreducible.

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