Approximating homotopy equivalences of surfaces by homeomorphisms

by

W. Jakobsche (Warszawa)

Abstract. We prove the 2 and 3-dimensional version of the “Splitting Theorem” of Chapman and Ferry [2]. The consequence of this is the 2-dimensional analogue of the $\alpha$-approximation theorem [2], and the equivalence of the 3-dimensional $\alpha$-approximation theorem and of the Poincaré conjecture.

1. Introduction. The aim of this note is to extend some of the high-dimensional theorems of Chapman and Ferry to dimensions 2 and 3. More precisely, we prove the “Splitting theorem” from [2] in these dimensions. The 2-dimensional version of this theorem implies the 2-dimensional analogues of the “$\alpha$-approximation theorem” and the “Bundle theorem” from [2], and theorem (I) from [3]. The 3-dimensional “Splitting theorem” proves that the 3-dimensional “$\alpha$-approximation theorem” is equivalent to the classical Poincaré conjecture.

The additional motivation for the proof of the 2-dimensional “$\alpha$-approximation theorem” was [6], where it was used to study the fixed point sets of the close PL involutions of 3-manifolds.

We adopt from [2] the following notation: Let $X, Y$ be two spaces and let $\alpha$ be an open cover of $Y$. We say that the maps $f, g: X \to Y$ are $\alpha$-homotopic (written $f \equiv_\alpha g$) if there is a homotopy $F_t: f \equiv g$, $t \in [0, 1]$ such that the track of each point $F_t(x)$, $0 \leq t \leq 1$, lies in some element of $\alpha$. If $h: X \to Y$ is a map and $Y$ is given a fixed metric then $f^{-1}(a)$ denotes the cover $\{U \subset X: U$ is open and $f(U) \subset a\}$ of $X$. More generally $f^{-1}(a)$ denotes $\{U \subset X: U$ is open in $X$ and there exists a $V \in \alpha$ such that $f(U) \subset V\}$ whenever $\alpha$ is a cover of $Y$. If $A$ is a subset of $Y$ and $\alpha$ is a cover of $Y$, then we say that $f: X \to Y$ is an $\alpha$-equivalence over $A$ with the $\alpha$-inverse $g$ if $g$ is a map of $A$ into $X$, $f|_A$ $\alpha$-homotopic to the inclusion $id_A$, and $g|^{-1}(A) = f^{-1}(a)$ for each $a \in \alpha$. If $A = Y$, then we say that $f$ is an $\alpha$-equivalence. If $\beta$ is a cover of $Y$ and $f^*: X \to Y$ is a proper map, then we say that $f$ is a $\beta$-map if for every $y \in Y$ there is a $U \in \beta$ such that $f^{-1}(y) \subset U$. If $X$ is a metric space then we say that $f$ is an $\varepsilon$-map if for every...
2. The proof of the “Splitting Theorem”. To prove (1.1) we shall need lemmas. All the manifolds considered have dimension $\leq 3$, and so by [7] we can consider them as PL manifolds. In particular, we assume that $S^{n-1} \times R$ has a natural PL structure in which every submanifold $S^{n-1} \times \{a, b\}$ is PL.

The following lemma is easy, and so we omit the proof.

**Lemma (2.1).** Let $N$ be a surface, $x_0 \in N$, and let $\mathfrak{g}$ be a subgroup of $\pi_1(N, x_0)$ such that $\mathfrak{g} \neq \pi_1(N, x_0)$. Then there exists a PL embedding $\xi: S^1 \to N$ which determines an element $[\xi]$ of $\pi_1(N, x_0) \backslash \mathfrak{g}$.

**Lemma (2.2).** Given $a, b \in \{e(-2, 2), a \leq b \} there exists an $\epsilon$ such that whenever $1.0$ holds, then there exists a PL $(n-1)$-sphere $S_0 \subset (g^{-1}(a, b))$, satisfying $k[S_0] \neq 0$ for every integer $k \neq 0$, where $[S_0]$ is the image in $H_{n-1}(f^{-1}(B))$ of the fixed generator of $H_{n-1}(S_0)$. The homorphism induced by the inclusion $S_0 \subset f^{-1}(B)$, Moreover if $n = 3$, then $S_0$ disconnects the component of $f^{-1}(B)$ containing $S_0$.

**Proof.** 2.2. Let $c \in \mathbb{Z} \cup \{(a+b)c, c \leq (b-a), c \}$ and let $f: W \to S^1 \times R$ satisfy our requirements for this choice of $c$. Then $N = (g^{-1}(a, b))$ is a PL $n$-submanifold of $W$, and $N = (g^{-1}(c))(c) \cup (p^{-1}(c))$ (this follows from the fact that $g_0$ is close to $p$ on $p^{-1}(c)$). Let $g_0 = g^{-1}(c)$. Then $g_0$ is a map into $S_0 \subset f^{-1}(c) \subset S^1 \times R$. Take $x_0 \in g^{-1}(c)$ and let $s_0 = N$ denote the inclusion and $s_0 = \pi_{n-1}(N, x_0) \to \pi_{n-1}(W, x_0)$ denote the induced homomorphism. Then $\mathfrak{g} = \ker(g_0)$ is a $\pi_1$-invariant subgroup of $\pi_{n-1}(W, x_0)$. We claim that $[g_0] \in \pi_1(N, x_0) \backslash \mathfrak{g}$. In fact otherwise $f(g_0): p^{-1}(c) \to S^1 \times R$ would be homotopic to a constant, which is not the case since it is homotopic to the inclusion $p^{-1}(c) \to S^1 \times R$. So by the sphere and projective plane theorem (see [5], p. 54) in the case of $n = 3$, and by Lemma (2.1) in the case of $n = 2$, there exists a covering map $\xi: S^0 \to S_0$. Then there is a covering map $\xi: S^0 \to N$ which for $n = 2$ must be a homeomorphism, where $S_0$ is a PL 2-sphere $S^2$ or a projective plane $P^2$ if $n = 3$, and $S_0$ is a 1-sphere when $n = 2$, $x_0 \in S_0$, and $\xi$ determines an element $[\xi]$ of $\pi_{n-1}(N, x_0)$ in particular $S_0$ is not contractible in $N$.

Now we show that, if $n = 3$, then $S_0$ disconnects the component of $f^{-1}(B)$ containing $S_0$. Suppose that it does not. Then there exists a simple closed curve $a: S^1 \to f^{-1}(B)$ such that $a(S^1) \cap S_0 = \emptyset$, and $a(S^1)$ is transversal to $S_0$.

Now we use the theory of the intersection index, as described in [1], pp. 97 and 114. To avoid assumptions concerning orientability, we consider homology with $Z_2$-coefficients. It is easy to see that $a(S^1)$ and $S_0$ support the 1-cycle and 2-cycle respectively with $Z_2$-coefficient such that the corresponding homology classes $z_1 \in H_1(W, Z_2)$ and $z_2 \in H_2(W, Z_2)$ have an intersection index $z_1 \cdot z_2 = 1$. This implies that $z_1 \neq 0$, and so $\alpha$ is not homotopic to a constant. Since $\pi_1(B_2) = 0$ if $n = 3$, it follows on the other hand that $f_\alpha$ and $g_{\alpha}$ are homotopic to constant maps which contradicts the fact that $g_{\alpha}$ is $\alpha$. So $S_0$...
disconnects the component of \( f^{-1}(B_2) \) which contains it. In particular \( S_0 \) is bicollared in \( W \).

Now we prove that, in the case of \( n = 3 \), \( S_0 \) is a sphere, and not a projective plane. Suppose on the contrary, that \( S_0 \cong \mathbb{P}^1 \). Then there exists a closed curve \( \alpha : S^1 \to S_0 \), which reverses the orientation of \( S_0 \). \( S_0 \) is bicollared in \( W \) and so \( \alpha \) reverses the orientation of \( W \). This implies that \( \alpha \) is not homotopic to a constant and we get a contradiction as before. So \( S_0 \) is a sphere.

Finally we prove that for any integer \( k \neq 0 \) we have \( k \cdot [S_0] \neq 0 \).

Suppose that, for some \( k \neq 0 \), \( k \cdot [S_0] = 0 \). Then \( [S_0] = 0 \), which is impossible by the inclusion \( i : S^1 \to f^{-1}(B_2) \) and \( e \) is a generator of \( H^i_1(\mathbb{R}^i, \mathbb{R}^i) \). Hence \( f^{-1}(B_2) \) must be separate for any map \( r_1 : S^0 \to S_0 \) of degree \( k \). But this is not the case: if \( f \) were, then \( f(S_0) \) would be homotopic to constant maps, contradicting the fact that \( f(S_0) \) is homotopic to the inclusion \( S_0 \to W \) and \( S_0 \) is not contractible in \( W \).

**Lemma 2.23.** Let \( T \) be a compact, orientable surface and let \( S_1 \) and \( S_2 \) be two disjoint 1-spheres, not contractible in \( T \). Suppose that \( T \) has a homotopy \( h_t : T \to T \), \( t \in \{1, 2\} \) such that \( h_t(S_s) = S_t \), and that the maps \( h_t : S^1 \to T_1 \) have a non-zero degree. Then there exists an embedding \( \iota : S^1 \times [1, 2] \to T_1 \) such that \( h_t(s^1 \times i) = S_t \), \( i = 1 \) or \( 2 \).

**Proof of (2.23).** \( S^1 \) is a PL sphere in \( T \) and so we can find a PL embedding \( u : S^1 \times [0, 1] \to T_1 \) such that \( S_1 = u(S^1 \times \{0\}) \) and \( u(S^1 \times \{1\}) \) is a homotopy. We consider the decomposition space \( T = T'/G \) with the homotopy \( G : S^1 \to T \). Then \( q(T') \equiv S^1 \times \{0\} \in T_1 \) is an embedding and \( A = \{a_0, \ldots, a_n\} \) is a compact, orientable surface. Let \( T_1 \) be the component of \( T \) containing \( q(S') \). Then it is easy to construct a map \( f : T_1 \to T \) such that \( f(T_1) = u(S^1 \times \{0\}) \). Let \( e \in \pi_1(T) \) be determined by the inclusion \( q(S') \to T_1 \). Then \( f(q(S')) \) is homotopic to \( e \in T_1 \) and \( q^{-1}(S^1 \times \{0\}) \) is a constant map \( p_{q(\alpha)} : S^1 \to T_1 \). Then \( f(S(\alpha)) \) is a homeomorphism for any \( \alpha \in \pi_1(T) \). But then we have \( S^1 \times \{0\} \neq 1 \). Suppose the opposite, \( S^1 \times \{0\} = 1 \). Then a subgroup \( \pi_1(T) \) generated by \( \alpha \) is finite, cyclic, and so the covering \( T_1 \) corresponding to \( S^1 \times \{0\} \) is a finite group. Therefore the maps \( q(S') \to T_1 \) have the same degree, so \( S_0 \) is homotopic to \( S_0 \) in \( T' \).

**Lemma 2.24.** Let \( M \) be a connected 3-manifold, \( \partial M = \emptyset \), and let \( S_1 \) and \( S_2 \) be two disjoint 2-spheres in \( M \) such that the elements \( [S_1] \) and \( [S_2] \) of \( H_2(M) \) determined by \( S_1 \) and \( S_2 \) are as in (2.22) satisfy the following condition: there are integers \( k_1, k_2 \) such that \( k_1 \cdot [S_1] = k_2 \cdot [S_2] \). Moreover, we assume that each \( S_i, i = 1, 2 \), disconnects \( M \). Let \( L \) be a closure of a component of \( M \) with \( L \supseteq S_i \cup S_2 \). Then \( L \) is a compact manifold with the boundary \( \partial L = S_1 \cup S_2 \).

**Proof of (2.24).** Suppose that \( L \) is non-compact. Let \( M_1 \) and \( M_2 \) be the closures in \( M \) of two components of \( M \). Note that \( M_1 \cup M_2 \) has precisely 3 components such that \( L \cap M_i = S_i \) for \( i = 1, 2 \). We consider the exact homology sequence of the pair \( (M_1, M_1 \cup M_2) \):

\[
H_2(M_1, M_1 \cup M_2) \to H_2(M_1 \cup M_2) \xrightarrow{\partial} H_2(M_1, M_1 \cup M_2) \to H_2(M_1, M_1 \cup M_2).
\]

If \( L \) is non-compact, then \( H_2(M_1, M_1 \cup M_2) = H_2(L, S_1 \cup S_2) = 0 \), and so \( j \) is a monomorphism. But \( H_2(M_1, M_1 \cup M_2) = H_2(M_1, M_2) \) and, for each \( i = 1, 2 \), there is a \( j_i \in H_2(M_i) \) such that \( (j_i)_i \cdot [S_i] = [S_i] \), where \( i : M_1 \to M_1 \) is an inclusion. So the fact that \( j \) is a monomorphism implies that \( k \cdot (j_i)_i \cdot [S_i] = k_i \cdot [S_i] = k \cdot [S_i] = 0 \). But we know that \( k \cdot [S_i] = k_i \cdot [S_i] = 0 \). So \( L \) is compact. This and the fact that \( \partial L = S_1 \cup S_2 \) imply that \( \partial L = S_1 \cup S_2 \).

**Lemma 2.25.** Let \( 0 < s < 1 \), and \( a_1, b_1 \in (-2, 2) \), \( i = 1, 2 \), be such that \( -2 + 2s < a_1 < b_1 < a_2 < b_2 < 2 - 2s \). Assume that (1.0) is satisfied, and let \( \lambda \in \gamma \) be a PL embedding such that \( \lambda (\mathbb{R}^i) \neq 0 \) for \( \mathbb{R}^i \), and if \( n = 3 \) then \( S_0 \) disconnects the component of \( f^{-1}(B_2) \) which contains it. Then \( [S_0] \) is the image of \( H^1_2 \) (or \( f^{-1}(B_2) \)) of the fixed generator of \( H^1_2 \). Then there is a compact PL submanifold \( L \) of \( W \) such that \( L \) is an \( h \)-cobordism from \( S_1 \) to \( S_2 \).

**Proof of (2.5).** First prove that there exists a homotopy \( h_t : S^1 \to f^{-1}(B_2) \), \( t \in \{1, 2\} \), such that \( h_t(S) = S_t \) and that \( h_t : S_t \to S_t \) has non-zero degree for \( i = 1, 2 \). Let \( f_t = f \circ h_t : S_t \to f^{-1}(B_2) \), and \( m_{t} = \text{deg} f \) (we define the degree of \( f \) as a number equal to the degree of \( p_{t} f \) where \( p_{t} : S_t \to S_t \) is a projection). Then \( S_t \neq S_t \) projected into \( \gamma \). Therefore \( S_t \subset f^{-1}(B_2) \) and \( g_t \), \( t \in \{1, 2\} \), the map \( g_t : S_t \to f^{-1}(B_2) \) is homotopic to \( i_d \) in \( f^{-1}(B_2) \). This and the fact that \( [S_0] \neq 0 \) imply that \( \theta_t \neq 0 \) for \( i = 1, 2 \). Let \( k_1, k_2 \) be the integers such that \( m_{t} = k_1 \cdot k_2 = k \neq 0 \), and \( k_t : S_t \to S_t \) be any map of degree \( k \) for \( i = 1, 2 \). Therefore \( g_f \) and \( g_{j_t} \) are homotopic in \( p_{t} : S_t \to S_t \). Therefore \( g_t \) is a \( s \)-infinite inverse for \( f \) over \( S_t \), there are \( \partial f \)-small homotopies between \( g_f \) and \( g_{j_f} \), \( i = 1, 2 \); their values lie in \( f^{-1}(B_2) \). Therefore \( g_f \) is a \( s \)-infinite inverse for \( f \) over \( S_t \), there are \( \partial f \)-small homotopies between \( g_f \) and \( g_{j_f} \), \( i = 1, 2 \). Therefore \( g_f \) is a 3-manifold with an empty boundary, and so we can use Lemma 2.4 to prove that \( S_1 \) and \( S_2 \) bound in
$f^{-1}(B_{2,2})$ a compact manifold $L$. We only have to show that $L$ is a $h$-cobordism. By [3], p. 26, we need to show that $L$ is simply connected. Let $x: S^3 \to L$ be any map. We may assume that $x$ is PL and that $\text{Im}(x) \subset \text{Int}(L)$. Then $f: S^3 \to B_{2,2}$ is homotopic to a constant map, because $B_{2,2}$ has a homotopy type of $S^3$. So $\text{ga}$; $S^3 \to W$ is homotopic to a constant map. Moreover, by (1.0), $x \simeq \text{ga}$ $x$ and hence $x$ is homotopic to a constant map in $W$. Let $z: D^2 \to W$ be any map of the $2$-disc $D^2$ into $W$ which extends $\text{ga}(S^3 \to D^2)$, and which is transversal to $S_1 \cup S_2$. Then $\text{ga}$; $S^3 \to W$ is a finite collection of circles in $D^3$, and the component $P$ of $\text{ga}$; $S^3 \to W$ which contains $\partial D^2$ is a PL submanifold of $D^3$ bounded by a finite family $I$ of circles. For any $x \in \partial D^2$, $\alpha \in \partial D^2$, $\alpha$ can be extended to the map $P_i \to S_i$, $i = 1$ or $2$, where $P_i$ is a disc bounded in $D^2$ by $\alpha$. The union of these extensions and of $\partial P$ gives a map $D^2 \to L$ which extends $x$. This proves that $L$ is simply connected.

Now we can prove Theorem (1.1) of the addendum may be proved as follows: First we note that $f^{-1}(1, 0, 0)$ is a deformation retract of $L_2$. Thus, $f^{-1}(1, 1, 0)$ is a deformation retract of $L_2$. This implies that $S_3 = S_3$ is a deformation retract of $L_2$. Now we can put $S_0 = S_2$. As we have shown it satisfies all the conditions of (1.1).

Now to prove that $S_3$ disconnects the component $A$ of $W$ cutting it into two components, one containing $(f^{-1})(1)$ and the other containing $(f^{-1})(1)$, we need only to show that $A$ is not connected. Suppose it is. Then there is a curve $x: [0, 1] \to W$, such that $x(0) = S_3$, $x(1) = S_1$, and $x(0, 1) \cup L = x([0, 1])$. Hence there is a point $y \in x([0, 1])$ such that $f(y) \in f^{-1}(1, 1, 1)$. Then $y \in f^{-1}(1, 1, 1) \subset L$, which gives a contradiction.

Finally we prove that $f(S_2) \subset S_2 \times R$ is a homotopy equivalence. First we notice that $f(S_2) \subset B_{2,2}$. This, as we have shown, implies that $g(f(S_2)) \subset L_2$. From the fact that $g(f^{-1}(B_{2,2})) \subset L_2$, it follows that there is a homotopy $h_t: S_2 \to W$, with $h_0 = \text{id}_{S_2}$, $h_1 = g(f(S_2))$, such that the track $h_t: t \in [0, 1]$ of each point $x \in S_2$ has image by $g$ of diameter $\varepsilon < \varepsilon$. This implies that for every $x \in S_2$, $h_t(x): t \in [0, 1]) \cup \partial L = \emptyset$, so $g(f(S_2))$ is a homotopy equivalence between $S_2$ and $L_2$, and so $f(S_2)$ is a homotopy equivalence.

Now we can put $S_0 = S_2$. As we have shown it satisfies all the conditions of (1.1).

The addendum may be proved as follows: First we note that $g^{-1}$ in the precisely the same way in which we have shown that $g(f^{-1}(1)) \subset L_1$ using the claim. The only difference is that we replace $[1, 0] \supset [1, 0, 0]$ by $L_2$ by $L_1$ and use the fact that $L_1 + \varepsilon < \varepsilon \subset \varepsilon < \varepsilon$. This implies that the component $C_0$ of $(f^{-1}(1, 0, 0), \partial S_3)$ containing $(f^{-1}(0, 1))$ is contained in $L_1$. But $S_3 = S_3$ is a deformation retract of $L_2$, and so $C_0$ can be deformed to $S_3$ in $L_2$.

Now we notice that $L_2 \subset C$. This easily follows from the fact that $b_3 \subset \varepsilon < \varepsilon$. This finishes the proof of (1.1).

3. Remarks on the proof of the $\alpha$-approximation theorem for dimension 2.

The proof is only slightly different from the one given by Chapman and Ferry [2]. First, using the "orientable" splitting theorem, we prove the "handle Lemma" as in [2] p. 589, with $n = 2$, and an orientable $V^2$. The proof is analogous to the one given in [2]. We need only to note that the surface $W_0$ (23), p. 591) is immersed in $V$, and $V$ is orientable in our case, so $W_0$ is orientable, and hence we can construct the orientable $W_1$, $W_2$, $W_3$ as in [2]. As in [2] we prove the "Main theorem" (p. 595 in [2]), with $n = 2$ and an orientable $V^2$. Then we prove the following, weaker version of the "$\alpha$-approximation theorem":

**Lemma.** Let $N^2$ be a surface, and let $y$ be any open cover of $N$. Then, for every open cover $x$ of $N$, there is an open cover of $N$ such that for any $\alpha$-equivalence $f: M \to N$, which is already a homomorphism from $\partial M$ onto $\partial N$, and is such that for any $y \in f^{-1}(c)$, $\alpha$ is orientable surface, and $f$ is $\alpha$-close to a homomorphism $h: M \to N$. 


The proof proceeds as in [2], pp. 597, 598. First, we prove the version of Lemmas (5.1) and (5.2) of [2] with $n = 2$ and an orientable $M$.

Then, the proof of Lemma (3.1) proceeds as the proof of the $\alpha$-approximation theorem in [2], pp. 598, and the only change is that by our assumption, we require that a star finite cover $\{N_i\}$ found in the proof of the $\alpha$-approximation theorem in [2] be such that all sets $f^{-1}(N_i)$ are orientable.

Lemma (3.1) easily implies Theorem (1.2) if we use the following Lemma (3.2):

**Lemma 3.2.** Let $N$ be a surface. Then there is an open cover $\alpha$ of $N$ such that if $M$ is a surface and $f: M \to N$ is an $\alpha$-equivalence, then for every $c \in \alpha$, $f^{-1}(c)$ is an orientable surface.

**Proof of (3.2).** Suppose that $\gamma$ is any cover of $N$ by the open discs. Then we can easily find an open cover $\alpha$ of $N$ such that to every $c \in \alpha$, there is $d \in \gamma$ such that $c \subset d$. Suppose that for a certain $c \in \alpha$ and some $\alpha$-equivalence $f: M \to N$, $f^{-1}(c)$ is a non-orientable surface. Then there exists an element $z$ of $H_i(f^{-1}(c), \mathbb{Z})$ such that $z \neq 0$, and $0 \neq i_\alpha(e) = H_i(M)$, where $i_\alpha(f^{-1}(c)) = M$ is an inclusion. We can take for $z$ an element of $H_i(f^{-1}(c))$ determined by the curve reversing orientation of $f^{-1}(c)$. Let $g$ be the inverse of $f$. Then $g_\gamma(f^{-1}_\alpha e) = (g_\alpha f^{-1}_\alpha e) = i_\gamma(e) = 0$, and on the other hand $f^{-1}(g_\gamma e) = 0$, because $c \subset d \in \gamma$, which gives a contradiction.

4. The equivalence of the $\alpha$-approximation theorem and the Poincaré conjecture in dimension $n = 3$. It is very easy to construct for every $\varepsilon > 0$, the $\varepsilon$-equivalence from the homotopy sphere $\approx S^3$ (if one exists) onto $S^3$. This equivalence obviously cannot be approximated by homeomorphisms.

On the other hand if the Poincaré conjecture is satisfied, then we can use our "Splitting Theorem" in dimension 3 to prove the $\alpha$-approximation theorem, as in [2]. The only difference is that in the construction of $h$ in the "Handle Lemma" (step V) we use the theorem of Waldhausen [10] in the form described in [4] (Lemma 5, p. 65 in [4]). Note that the manifold $W^k$ in the step V of the construction in the "Handle Lemma" in [2] is homotopy equivalent to $B^m \times T^k$, $m + k = 3$, and we assume that the Poincaré conjecture holds, whence $W^k$ is irreducible.

**References**