Maximal σ-independent families

by

Kenneth Kunen (Madison, Wis.)

Abstract. If there is a non-trivial maximal σ-independent family, then the Continuum Hypothesis holds but there is a weakly inaccessible cardinal between ω_1 and 2^{ω_1}. The existence of such a family is equiconsistent with the existence of a measurable cardinal.

§ 0. Introduction. Let $\varkappa$ and θ be infinite cardinals, with θ regular. A family $\mathcal{S} \subset \mathcal{P}(\varkappa)$ is called *θ-independent* iff whenever $\mathcal{S}_0, \mathcal{S}_1 \subset \mathcal{S}$, with $\mathcal{S}_0 \cap \mathcal{S}_1 = \emptyset$ and $|\mathcal{S}_0 \cup \mathcal{S}_1| < \theta$, we have

$$\bigcap \{A\colon A \in \mathcal{S}_0\} \cap \bigcap \{\varkappa \setminus A\colon A \in \mathcal{S}_1\} \neq \emptyset .$$

σ-independent means ω_1-independent.

The case $\theta = \omega$ is well-known, and the discussion of the elementary results for arbitrary θ is essentially the same, except for an occasional restriction on cardinal exponentiation. For example, the famous theorem of Hausdorff [H] says that for any $\varkappa$, there is an ω-independent $\mathcal{S} \subset \mathcal{P}(\varkappa)$ of size $2^{\varkappa}$, and the same proof yields a θ-independent $\mathcal{S}$ provided that $\varkappa^{<\theta} = \varkappa$.

The cases $\theta = \omega$ and $\theta > \omega$ diverge, however, when we consider the existence of *maximal* θ-independent families; here, we call $\mathcal{S} \subset \mathcal{P}(\varkappa)$ maximal iff it is θ-independent but no proper superset is. First, we must discard a trivial case. Suppose $\mathcal{S}$ is θ-independent, $|\mathcal{S}| < \theta$ and $|\bigcap \mathcal{S}| = 1$; then $\mathcal{S}$ is clearly maximal, and such $\mathcal{S}$ can easily be constructed. Thus, we ask: Is there a maximal θ-independent $\mathcal{S} \subset \mathcal{P}(\varkappa)$ with $|\mathcal{S}| \geqslant \theta$? For $\theta = \omega$, the answer is yes for any $\varkappa$ by Zorn's Lemma, whereas for $\theta > \omega$, Zorn's Lemma does not apply, and in fact the existence of maximal families immediately entails large cardinals. We shall show:

THEOREM 1. *Suppose θ is regular and $\theta > \omega$, and suppose that there is a maximal θ-independent family $\mathcal{S} \subset \mathcal{P}(\varkappa)$ with $|\mathcal{S}| \geqslant \theta$. Then*

a) $2^{<\theta} = \theta$ *and*

b) *there is a λ with*

$$\sup\{(2^{\alpha})^{+}\colon \alpha < \theta\} \leqslant \lambda \leqslant \min(\varkappa, 2^{\theta})$$

such that there is a non-trivial θ^{+}-saturated λ-complete ideal over λ. ■

We refer the reader to the text [J] for a general discussion of saturated ideals. Observe that in (b), $\lambda \geqslant \theta$, so the ideal is λ^+-saturated, which yields an inner model with a measurable cardinal (see Theorem 86 of [J] or Theorem 11.12 of [K1]). If θ is not strongly inaccessible then $\lambda \geqslant \theta^+$, so, by Ulam, λ is weakly inaccessible, and, by Solovay, λ is also weakly Mahlo, weakly hyper-Mahlo, and so forth (see § 35 of [J]). However, if θ is strongly inaccessible, it is consistent that $\lambda = \theta = \varkappa$; see the end of § 2.

Setting $\theta = \omega_1$, we have all the results claimed in the Abstract except for the consistency of such a family. This is given by

THEOREM 2. *If* ZFC *plus the existence of a measurable cardinal is consistent, so is* ZFC *plus the existence of a maximal σ-independent family* $\mathcal{S} \subset \mathcal{P}(2^{\omega_1})$. ■

The fact that a maximal σ-independent family $\mathcal{S} \subset \mathcal{P}(\varkappa)$ yields $\varkappa > 2^{\omega}$ and an inner model with a measurable cardinal was first proved by Baumgartner; we comment further on his method at the end of § 1.

Observe that if $\mathcal{S} \subset \mathcal{P}(\varkappa)$ is maximal θ-independent with $|\mathcal{S}| \geqslant \theta$, and $\varkappa' \geqslant \varkappa$, then $\mathcal{S}$ is still maximal θ-independent when viewed as a family of subsets of $\varkappa'$. This triviality may be avoided by calling $\mathcal{S} \subset \mathcal{P}(\varkappa)$ *uniform* iff each

$$\bigcap \{A\colon A \in \mathcal{S}_0\} \cap \bigcap \{\varkappa \setminus A\colon A \in \mathcal{S}_1\}$$

has size $\varkappa$. In our proof of Theorem 2 we shall produce a uniform family. In general, if we fix θ and let $\varkappa$ be least such that there is maximal θ-independent $\mathcal{S} \subset \mathcal{P}(\varkappa)$ with $|\mathcal{S}| \geqslant \theta$, then any such $\mathcal{S}$ is uniform; this will be immediate from Lemma 1.1.

§ 1. Proof of Theorem 1. It will be convenient to borrow some of the terminology of forcing. Let

$$\mathrm{Fn}(I, J, \theta) = \{p\colon p \text{ is a function and } \mathrm{dom}(p) \subset I \text{ and } \mathrm{ran}(p) \subset J \text{ and } |p| < \theta\},$$

and define $p \leqslant q$ iff $q \subset p$. This partial order has a largest element, $\mathbb{1} = \emptyset$. See VII, § 6 of [K2] or § 19 of [J] for more on this partial order.

Now, fix $\mathcal{S} \subset \mathcal{P}(\varkappa)$, and set $\boldsymbol{P} = \mathrm{Fn}(\mathcal{S}, 2, \theta)$. Define, for $p \in \boldsymbol{P}$,

$$\varphi(p) = \bigcap \{A\colon A \in \mathrm{dom}(p) \text{ and } p(A) = 1\} \cap \bigcap \{\varkappa \setminus A\colon A \in \mathrm{dom}(p) \text{ and } p(A) = 0\}.$$

We adopt the convention here that $\bigcap \emptyset = \varkappa$, so that $\varphi(\mathbb{1}) = \varkappa$. Clearly $p \leqslant q \Rightarrow \varphi(p) \subset \varphi(q)$. Also, $\mathcal{S}$ is θ-independent iff $\varphi(p) \neq \emptyset$ for all $p \in \boldsymbol{P}$, and in that case φ is an isomorphism from $(\boldsymbol{P}, \leqslant)$ into $(\mathcal{P}(\varkappa), \subset)$. Observe also that when $\mathcal{S}$ is θ-independent, p and q are compatible iff $\varphi(p) \cap \varphi(q) \neq \emptyset$.

If $\mathcal{S}$ is θ-independent, $\mathcal{S}$ is maximal iff

$$\forall X \subset \varkappa \, \exists p \in \boldsymbol{P} (\varphi(p) \subset X \text{ or } \varphi(p) \subset \varkappa \setminus X).$$

Following S. Glazer, we call $\mathcal{S}$ *globally maximal* iff

$$\forall q \in \boldsymbol{P} \, \forall X \subset \varphi(q) \, \exists p \leqslant q (\varphi(p) \subset X \text{ or } \varphi(p) \subset \varphi(q) \setminus X).$$

1.1. LEMMA (Glazer). *Suppose $\mathcal{S} \subset \mathcal{P}(\varkappa)$ is a maximal θ-independent family with $|\mathcal{S}| \geqslant \theta$. Then there is a $\varkappa' \leqslant \varkappa$ and an $\mathcal{S}' \subset \mathcal{P}(\varkappa')$ which is globally maximally θ-independent such that $|\mathcal{S}'| \geqslant \theta$.*

Proof. For each $r \in \boldsymbol{P}$, let

$$\mathcal{S}_r = \{A \cap \varphi(r)\colon A \in \mathcal{S} \setminus \mathrm{dom}(r)\}.$$

It is sufficient to show that for some r, $\mathcal{S}_r \subset \mathcal{P}(\varphi(r))$ is globally maximal (then $\varkappa' = |\varphi(r)|$). Let D be the set of all $q \in \boldsymbol{P}$ such that for some $X_q \subset \varphi(q)$,

$$\neg \exists p \leqslant q (\varphi(p) \subset X_q \text{ or } \varphi(p) \subset \varphi(q) \setminus X_q).$$

If no $\mathcal{S}_r$ is globally maximal, then D is dense in $\boldsymbol{P}$. In that case, let $A \subset D$ be a maximal antichain in $\boldsymbol{P}$, and let $X = \bigcup_{q \in A} X_q$. Now, fixing p such that

$$\varphi(p) \subset X \quad \text{or} \quad \varphi(p) \subset \varkappa \setminus X,$$

we see that p is incompatible with every element of A, a contradiction. ■

We now may, and shall, assume that $\mathcal{S} \subset \mathcal{P}(\varkappa)$ is globally maximal in our proof of Theorem 1, since if not, we could replace it with $\mathcal{S}' \subset \mathcal{P}(\varkappa')$. Assume also that $|\mathcal{S}| \geqslant \theta$ and that θ is regular.

Now, let

$$\mathcal{F} = \{X \subset \varkappa\colon \neg \exists p \in \boldsymbol{P} (\varphi(p) \subset X)\}.$$

By global maximality, $\mathcal{F}$ is an ideal. Since $|\mathcal{S}| \geqslant \theta$, $\mathcal{F}$ is non-principal and contains all singletons.

1.2. LEMMA. *$\mathcal{F}$ is $(2^{\alpha})^+$-complete for all $\alpha < \theta$.*

Proof. If not, there is an $X \notin \mathcal{F}$ and $Y_f \in \mathcal{F}$ for $f \in {}^{\alpha}2$ such that the Y_f are disjoint and $X \subset \bigcup_f Y_f$. Now inductively define a $g\colon \alpha \to 2$ and a decreasing chain $\langle p_\xi\colon \xi \leqslant \alpha \rangle$ in $\boldsymbol{P}$ so that

1) $\varphi(p_0) \subset X$,

2) $p_\xi = \bigcup \{p_\eta\colon \eta < \xi\}$ if ξ is a limit,

3) $p_{\xi+1} \leqslant p_\xi$ and $\varphi(p_{\xi+1}) \subset \bigcup \{Y_f\colon f(\xi) = g(\xi)\}$.

In (2), $p_\xi \in \boldsymbol{P}$ since θ is regular. In (3), $g(\xi) \in \{0, 1\}$ and $p_{\xi+1}$ may be chosen by global maximality. But now,

$$\varphi(p_\alpha) \subset \bigcap_{\xi < \alpha} \bigcup \{Y_f\colon f(\xi) = g(\xi)\} = Y_g,$$

contradicting $Y_g \in \mathcal{F}$. ■

1.3. LEMMA. *$\mathcal{F}$ is $(2^{<\theta})^+$-saturated.*

Proof. If not, then there are $X_\gamma \subset \varkappa$ for $\gamma < (2^{<\theta})^+$ with each $X_\gamma \notin \mathcal{F}$ but $X_\gamma \cap X_\delta \in \mathcal{F}$ whenever $\gamma \neq \delta$. Fix p_γ with $\varphi(p_\gamma) \subset X_\gamma$. But then $\{p_\gamma\colon \gamma < (2^{<\theta})^+\}$ contradicts the $(2^{<\theta})^+$ chain condition of $\boldsymbol{P}$ (see [J] Lemma 19.8 or [K2] Lemma V, 6.10). ■

1.4. LEMMA. $2^{<\theta} = \theta$.

Proof. Fix $\alpha < \theta$, and we produce a map from θ onto ${}^{\alpha}2$. Fix distinct $A_{\varrho\xi} \in \mathcal{S}$ $(\varrho < \theta, \xi < \alpha)$. For each $\delta \in \varkappa$, define $\psi_\delta\colon \theta \to {}^{\alpha}2$ so that $\psi_\delta(\varrho)(\xi) = 1$ iff $\delta \in A_{\varrho\xi}$. For each $f \in {}^{\alpha}2$, let R_f be $\{\delta\colon f \notin \mathrm{ran}(\psi_\delta)\}$. $R_f \in \mathcal{F}$, since if $\varphi(p) \subset R_f$, we could fix ϱ with $\mathrm{dom}(p)$ disjoint from $\{A_{\varrho\xi}: \xi<\alpha\}$, and let $q\leqslant p$ with $q(A_{\varrho\xi})=f(\xi)$ for each $\xi<\alpha$; but then

$$\varphi(q)\subset\{\delta: \psi_\delta(\varrho)=f\}\subset\varkappa\setminus R_f,$$

a contradiction. Since $\mathscr{F}$ is $(2^\alpha)^+$-complete, fix $\delta\in\varkappa\setminus\bigcup\{R_f: f\in{}^\alpha 2\}$. Then ψ_δ maps θ onto ${}^\alpha 2$. ∎

Thus, $\mathscr{F}$ is θ^+-saturated. Since $\mathscr{F}$ is non-principal, we may, following Ulam, let λ be the least cardinal such that $\mathscr{F}$ is not λ-complete; then there is a non-trivial λ-complete θ^+-saturated ideal $\mathscr{F}$ over λ. By 1.2, $\lambda\geqslant\sup\{(2^\alpha)^+: \alpha<\theta\}$, and clearly $\lambda\leqslant\varkappa$. Finally, $\lambda\leqslant 2^\theta$ follows from the Ulam–Tarski tree argument. Or, to see directly that $\lambda\leqslant 2^\theta$, let $\mathscr{S}_0\subset\mathscr{S}$ with $|\mathscr{S}_0|=\theta$, and define, for $f\colon \mathscr{S}_0\to 2$,

$$\varphi(f)=\bigcap\{A: A\in\mathscr{S}_0 \text{ and } f(A)=1\}\cap\bigcap\{\varkappa\setminus A: A\in\mathscr{S}_0 \text{ and } f(A)=0\}.$$

Then

$$\varkappa=\bigcup\{\varphi(f): f\in{}^{\mathscr{S}_0}2\},$$

and each $\varphi(f)\in\mathscr{F}$, so $\mathscr{F}$ is not $(2^\theta)^+$-complete.

This concludes the proof of Theorem 1. Of course, the proofs of Lemmas 1.2 and 1.4 are familiar arguments. 1.2 is a modification of the proof that measurable cardinals are strongly inaccessible. 1.4 is a modification of the proof that $2^{<\theta}=\theta$ in the generic extension by $\boldsymbol{P}$; in fact, we could have proved 1.4 by simply quoting this fact about $\boldsymbol{P}$ and applying Solovay's Boolean ultrapower technique.

An earlier line of argument, due to Baumgartner, produced a weaker version of Theorem 1. He observed that a globally maximal σ-independent family $\mathscr{S}\subset\mathscr{P}(\varkappa)$ gives a strategy for Non-empty in the double cut-and-choose game; by results of Galvin and Solovay, this yields $\varkappa>2^\omega$ and an inner model with a measurable cardinal.

§ 2. Proof of Theorem 2. If $\boldsymbol{P}$ is any partial order, let $\mathscr{B}(\boldsymbol{P})$ be the (unique) complete Boolean algebra into which P is densely embedded (see [K2], II § 3 or [J] § 17). The argument of § 1 shows that if $\mathscr{S}\subset\mathscr{P}(\varkappa)$ is a globally maximal θ-independent family, then φ defines an isomorphism from $\mathscr{B}(\mathrm{Fn}(\mathscr{S},2,\theta))$ onto $\mathscr{P}(\varkappa)/\mathscr{F}$. Conversely, we may establish the consistency of such families by working in a model where there is such an isomorphism. Specifically,

2.1. LEMMA. *Suppose θ is regular, $2^{<\theta}=\theta$, $\theta\leqslant\lambda$ and $\mathscr{F}$ is a θ^+-saturated λ-complete ideal over λ with $\mathscr{B}(\mathrm{Fn}(2^\lambda,2,\theta))$ isomorphic to $\mathscr{P}(\lambda)/\mathscr{F}$. Then there is a maximal θ-independent $\mathscr{S}\subset\mathscr{P}(\lambda)$.*

Proof. Let ψ be the isomorphism. For $\delta<2^\lambda$, let $[A_\delta]=\psi(\{\langle\delta,1\rangle\})$; here, $A_\delta\subset\lambda$ and $[A_\delta]\in\mathscr{P}(\lambda)/\mathscr{F}$ is its equivalence class. Let $\mathscr{S}=\{A_\delta: \delta<2^\lambda\}$ and define $\varphi\colon \boldsymbol{P}\to\mathscr{P}(\lambda)$ as in § 1, where $\boldsymbol{P}=\mathrm{Fn}(\mathscr{S},2,0)$. The fact that ψ is an isomorphism implies that $\mathscr{S}$ is θ-independent and

$$\forall X\subset\lambda\,\exists p\in\boldsymbol{P}([\varphi(p)]\leqslant[X] \text{ or } [\varphi(p)]\leqslant[\lambda\setminus X]).$$

This does not quite imply that $\mathscr{S}$ is maximal, since we would need

(*) $$\forall X\subset\lambda\,\exists p\in\boldsymbol{P}(\varphi(p)\subset X \text{ or } \varphi(p)\subset\lambda\setminus X).$$

To fix this, we modify $\mathscr{S}$. Let $\mathscr{F}=\{C_\delta: \delta<2^\lambda\}$, where each $C\in\mathscr{F}$ is listed at least θ times, let $A'_\delta=A_\delta\setminus C_\delta$ and $\mathscr{S}'=\{A_\delta: \delta<2^\lambda\}$, and let $\varphi'\colon \boldsymbol{P}\to\mathscr{P}(\lambda)$ be defined by $\mathscr{S}'$. Then $[A_\delta]=[A'_\delta]$ for each δ. $\mathscr{S}'$ satisfies (*), since if $[\varphi'(p)]\leqslant[X]$, we may fix $\delta\notin\mathrm{dom}(p)$ with $C_\delta=\varphi'(p)\setminus X$; then $\varphi'(p\cup\{\langle A'_\delta,1\rangle\})\subseteq X$. ∎

Now, to prove Theorem 2, we need only produce a model in which the Continuum Hypothesis (CH) holds, and, if $\lambda=2^{\omega_1}$, there is an ω_2-saturated λ-complete ideal $\mathscr{F}$ over λ with $\mathscr{P}(\lambda)/\mathscr{F}$ isomorphic to $\mathscr{B}(\mathrm{Fn}(2^\lambda,2,\omega_1))$. But this has essentially been done by Prikry (see [P] or Exercise 34.5 of [J]). Namely, in the ground model M: assume that λ is measurable and CH holds, let $\boldsymbol{P}=\mathrm{Fn}(\lambda,2,\omega_1)$, and let $\mathscr{U}$ be a normal ultrafilter over λ. Let G be P-generic over M. Then in $M[G]$, CH holds, $2^{\omega_1}=\lambda$, and, if

$$\mathscr{F}=\{X\subset\lambda: \exists Y\in\mathscr{U}(X\cap Y=\emptyset)\},$$

then $\mathscr{F}$ is λ-complete and ω_2-saturated (since $\boldsymbol{P}$ has the ω_2-c.c. in M). Let

$$i\colon (M^\lambda\cap M)/\mathscr{U}\to M^*$$

be the Scott ultrapower embedding, and let $\lambda^*=i(\lambda)$. Then $2^\lambda<\lambda^*<(2^\lambda)^+$, so $\mathrm{Fn}(2^\lambda,2,\omega_1)$ is isomorphic to $\mathrm{Fn}(\lambda^*\setminus\lambda,2,\omega_1)$. An isomorphism

$$\chi\colon \mathscr{P}(\lambda)/\mathscr{F}\to\mathscr{B}(\mathrm{Fn}(\lambda^*\setminus\lambda,2,\omega_1))$$

is implicit in Prikry's work. Namely, in M if τ is a $\boldsymbol{P}$-name and $[\![\tau\subset\check\lambda]\!]=\mathit{1}$, then $[\![\check\lambda\in i(\tau)]\!]\in\mathscr{B}(\mathrm{Fn}(\lambda^*,2,\omega_1))$, and in $M[G]$, we may let

$$\chi(\tau_G)=\bigvee\{q\in\mathrm{Fn}(\lambda^*\setminus\lambda,2,\omega_1): \exists p\in G(p\cup q\leqslant[\![\check\lambda\in i(\tau)]\!]\}.$$

This concludes the proof of Theorem 2. An easy modification of the proof will show that if λ is strongly compact in M, then in $M[G]$ there are uniform maximal σ-independent families on all $\varkappa\geqslant\lambda$ such that $\mathrm{cf}(\varkappa)\geqslant\lambda$.

Finally, we describe how to obtain models of GCH in which there is a strong inaccessible λ with a maximal λ-independent $\mathscr{S}\subset\mathscr{P}(\lambda)$. In M, assume GCH, let $\mathscr{U}$ be a normal ultrafilter on λ, and let $A\in\mathscr{U}$ be a set of inaccessibles. Let $\boldsymbol{P}$ be the reverse Easton extension which adds α^+ generic subsets of α for each $\alpha\in A$. If G is $\boldsymbol{P}$-generic over M, then $M[G]$ still satisfies GCH and there will be a λ^+-saturated ideal $\mathscr{F}$ on λ with $\mathscr{P}(\lambda)/\mathscr{F}$ isomorphic to $\mathscr{B}(\mathrm{Fn}(\lambda^+,2,\lambda))$, so Lemma 2.1 applies.

λ may or may not be measurable in $M[G]$. If M satisfies $V=L[\mathscr{U}]$, then λ will not be, since if it were, A would have normal measure 1 (by uniqueness of $\mathscr{U}$), and an ultrapower argument would produce, in $M[G]$, a subset of λ which is $\mathrm{Fn}(\lambda,2,\lambda)$-generic over $M[G]$. However, if in M, $\mathscr{U}$ and $\mathscr{V}$ are distinct normal ultrafilters and $A\notin\mathscr{V}$, then λ will be measurable in $M[G]$ by virtue of $\mathscr{V}$.

References

[H] F. Hausdorff, *Über zwei Sätze von G. Fichtenholz und L. Kantorovitch*, Studia Math. 6 (1936), pp. 18–19.

[J] T. Jech, *Set Theory*, Academic Press, 1978.

[K1] K. Kunen, *Some applications of iterated ultrapowers in set theory*, Ann. Math. Logic 1 (1970), pp. 179–227.
[K2] — *Set Theory*, North-Holland. 1980.
[P] K. Prikry, *Changing measurable into accessible cardinals*, Dissertationes Math. 68 (1970), pp. 5–52.

DEPARTMENT OF MATHEMATICS
THE UNIVERSITY OF WISCONSIN
Madison, Wisconsin 53706

Accepté par la Rédaction le 27. 10. 1980

Periodic homeomorphisms of chainable continua

by

Wayne Lewis (Lubbock, Tex.)

Abstract. We prove that for every positive integer n there exists a chainable continuum M_n which admits a homeomorphism of period n. While each M_n can be made to be a pseudo-arc, there also exist for each n non-hereditarily indecomposable chainable continua with homeomorphisms of period n.

Introduction. It is an easy exercise to show that every periodic homeomorphism of an arc must either have period 2 or be the identity. Beverly Brechner [Br–1] has constructed a homeomorphism of the wedge of two pseudo-arcs of period 4. (The pseudo-arc itself has an obvious homeomorphism of period 2.) Until now, all known periodic homeomorphisms of chainable continua had periods 1, 2, or 4.

Michel Smith and Sam Young [SY–1] have shown that if a chainable continuum admits a homeomorphism of period greater than 2, then the continuum must contain an indecomposable continuum. Since the pseudo-arc is hereditarily indecomposable [M–1], [B–1], and in many other ways is at the opposite end of the spectrum from an arc, it would seem a natural place to try to construct a homeomorphism of period greater than 2. It will follow from our results that the pseudo-arc has such homeomorphisms of high period, but our construction is more easily described for non-hereditarily indecomposable continua. (The author earlier [L–1] announced the existence of homeomorphisms of prime period for the pseudo-arc. The results here, in addition to being stronger, have what are hopefully much more readily understandable proofs.)

Throughout, by saying that a homeomorphism h has period n we will mean that n is the smallest positive integer such that h^n is the identity.

Construction of M_n. Let T_n be the continuum resulting from $[0,1]\times\times\{0,1,2,\ldots,n-1\}$ by identifying $\{0\}\times\{0,1,2,\ldots,n-1\}$ to a point. (Thus T_3 is the standard triod.)

Let M be an integer greater than n. Let f_i be a map from T_n onto T_n satisfying the following conditions. Let

$$\delta_i = \frac{1}{2^{2(2^i-1)}M^{(2^i)}}.$$