[3] B. Fitzpatrick Jr. and R. M. Ford, *On the equivalence of small and large inductive dimension in certain metric spaces*, Duke. Math. J. 34 (1967), pp. 33–37.
[4] T. Hoshina and K. Morita, *On rectangular products of topological spaces*, Topology and its Appl. 11 (1980), pp. 47–57.
[5] D. J. Lutzer, *Another property of the Sorgenfrey line*, Compositio Math. 24 (1972), pp. 359–363.
[6] T. Mizokami, *Metacompactness and subparacompactness of product spaces*, J. Math. Soc. Japan 31 (1979), pp. 263–272.
[7] K. Nagami, *Σ-spaces*, Fund. Math. 65 (1969), pp. 169–192.
[8] J. Nagata, *Product theorems in dimension theory, I*, Bull. Acad. Polon. Sci. 15 (1967), pp. 439–448.
[9] H. Ohta, *On Hewitt realcompactifications of products*, Ph. D. Thesis at Univ. of Tsukuba, 1979.
[10] B. A. Pasynkov, *On the dimension of rectangular products*, Soviet Math. Dokl. 16 (1975), pp. 344–347.
[11] H. Potoczny and H. Junnila, *Closure-preserving families and metacompactness*, Proc. Amer. Math. Soc. 53 (1975), pp. 523–529.
[12] M. Starbird, *The normality of products with a compact or a metric factor*, Ph. D. Thesis at Univ. of Wisconsin, 1974.
[13] R. Telgársky, *On topological properties defined by games*, Topics in Topology (Proc. Colloq. Keszthely, 1972), Colloq. Math. Soc. János Bolyai, Vol. 8, North Holland, Amsterdam, 1974, pp. 617–624.
[14] — *Space defined by topological games*, Fund. Math. 88 (1975), pp. 193–223.
[15] — *On point-open games and their generalizations*, Colloquia Mathematica Societatis János Bolyai 23, Budapest, Hungary, 1978, pp. 1167–1172.
[16] — and Y. Yajima, *On order locally finite and closure-preserving covers*, Fund. Math. 109 (1980), pp. 211–216.
[17] Y. Yajima, *Solution of R. Telgársky's problem*, Proc. Japan Acad. 52 (1976), pp. 348–350.
[18] — *On order star-finite and closure-preserving covers*, Proc. Japan Acad. 55 (1979), pp. 19–22.
[19] — *Topological games and products I*, Fund. Math. 113 (1981), pp. 141–153.

DEPARTMENT OF MATHEMATICS
KANAGAWA UNIVERSITY
Yokohama, 221
Japan

Accepté par la Rédaction le 23. 9. 1980

Periodic points of symmetric product mappings *

by

Nancy Rallis (Boston)

Abstract. We study questions concerning periodic points of symmetric product mappings. By means of the Maxwell trace homomorphism we define the notion of the mth Lefschetz number of a symmetric product mapping of a compact polyhedron. We show that the nonvanishing of this mth Lefschetz number is a sufficient condition for the existence of a periodic point of period $\leqslant m$. Next we relate that mth Lefschetz numbers of a symmetric product mapping to a certain characteristic function and obtain further results concerning periodic points.

1. Introduction. Let X be a topological space and X^n the cartesian product. Let G be a group of permutations of the numbers $1, 2, \ldots, n$. The orbit space of X^n under the action of G (with the identification topology) is called the *n-th symmetric product of* X and is denoted by X^n/G. A continuous map of the form $f\colon X \to X^n/G$ is called a *symmetric product mapping*.

For symmetric product mappings of compact polyhedra, C. N. Maxwell defined the notion of a Lefschetz number [4]. He showed that a nonzero Lefschetz number implies the existence of a fixed point. This extension of the Lefschetz fixed point theorem also holds for symmetric product mappings of metric ANR's provided the mapping is compact [3].

In the present paper we define the notion of a periodic point of period $\leqslant m$ for $f\colon X \to X^n/G$. In the case that X is a compact polyhedron, we define the mth Lefschetz number of f by appealing to the simplicial machinery developed in Maxwell [4]. We show that the nonvanishing of this number is a sufficient condition for the existence of periodic points of period $\leqslant m$. Further results concerning periodic points are obtained upon relating the mth Lefschetz numbers to a certain characteristic function. For instance, we show that if X is a compact polyhedron such that $H_i(X) = 0$ for i odd, where H is the homology functor with coefficients in the rational field, then any map $f\colon X \to X^n/G$ has a periodic point of period $\leqslant$ the Euler characteristic of X.

* This paper is based in part on the author's dissertation at Indiana University under the direction of Professor Jan Jaworowski.

2. Preliminaries. The purpose of this section is to fix basic notation and terminology and discuss general concepts which will be used in this paper; additional detail can be found in [3] and [4].

Let $\eta\colon X^n \to X^n/G$ be the identification map and $\pi_i\colon X^n \to X$ the ith projection.

For $x \in X$ and $y \in X^n/G$, x is said to be a *coordinate of* y if $\eta(z) = y$ implies that $x = z_i$ for some i where $z = (z_1, \ldots, z_n) \in X^n$. Let $x \in y$ read x is a coordinate of y.

2.1. Definition. Let $f\colon X \to X^n/G$. A point $x \in X$ is a *periodic point of f with period $\leqslant m$* if there exists points $x_1, \ldots, x_m \in X$ such that: (i) $x = x_1$, (ii) $x_i \in f(x_{i-1})$ for $2 \leqslant i \leqslant m$, and (iii) $x_1 \in f(x_m)$.

If $m = 1$, then the above definition coincides with the definition of a fixed point of a symmetric product mapping, that is, $x \in X$ is a fixed point of $f\colon X \to X^n/G$ if $x \in f(x)$.

Next suppose X is a metric space with metric d. Let $\bar{d}$ be the usual euclidean metric on X^n. A metric $\tilde{d}$ can be defined on X^n/G by

$$\tilde{d}(\eta(z), \eta(z')) = \inf\{\bar{d}(z, gz')\colon g \in G\}$$

where $z, z' \in X^n$. Consider the map $\omega\colon X \times X^n/G \to \boldsymbol{R}^{\geqslant 0}$ defined by

$$\omega(x, \eta(z)) = \inf\{d(x, \pi_i(z))\colon i = 1, 2, \ldots, n\}$$

where $x \in X$ and $z \in X^n$. The map ω is continuous and satisfies the inequalities

$$\omega(x, y) \leqslant \omega(x, y') + \tilde{d}(y, y') \quad \text{and} \quad \omega(x, y) \leqslant d(x, z) + \omega(z, y)$$

where $x, z \in X$ and $y, y' \in X^n/G$. In terms of the map ω we define the following function from $X^m \times (X^n/G)^m$ into $\boldsymbol{R}^{\geqslant 0}$:

$$\Omega\big((y_1, \ldots, y_m), (v_1, \ldots, v_m)\big) = \omega(y_1, v_m) + \sum_{i=2}^{m} \omega(y_i, v_{i-1})$$

where $(y_1, \ldots, y_m) \in X^m$ and $(v_1, \ldots, v_m) \in (X^n/G)^m$. The function Ω is continuous and satisfies the following inequalities

$$\Omega\big((y_1, \ldots, y_m), (v_1, \ldots, v_m)\big) \leqslant \omega(y_1, t_1) + \tilde{d}(t_1, v_m) + \sum_{i=2}^{m} \omega(y_i, t_i) + \tilde{d}(t_i, v_{i-1})$$

and

$$\Omega\big((y_1, \ldots, y_m), (v_1, \ldots, v_m)\big) \leqslant d(y_1, z_1) + \omega(z_1, v_m) + \sum_{i=2}^{m} d(y_i, z_i) + \omega(z_i, v_{i-2})$$

where $(z_1, \ldots, z_m), (y_1, \ldots, y_m) \in X^m$ and $(v_1, \ldots, v_m), (t_1, \ldots, t_m) \in (X^n/G)^m$.

Now suppose that $X = |K|$ is a compact polyhedron. Assume that K is an ordered complex. K determines a triangulation K^n of X^n naturally. Let $\mathrm{Sd}(K^n)$ denote the first barycentric subdivision of K^n and $\varphi\colon \mathrm{Sd}(K^n) \to K^n$ the simplicial map which associates to each barycenter b_t the least vertex of t. A triangulation $K(n, G)$ of X^n/G can be defined in terms of $\mathrm{Sd}(K^n)$ so that the p-simplexes of $K(n, G)$ are in one-to-one correspondence with the equivalence classes of p-simplexes of $\mathrm{Sd}(K^n)$. Let $C(K)$, $C(K^n)$, $C(\mathrm{Sd}(K^n))$ and $C(K(n, G))$ be the integral chain groups defined on the simplexes of the respective complexes. Then there exists a chain map $\mu\colon C(K(n, G)) \to C(K)$ which makes the following diagram commute

$$\begin{array}{ccc} C(K^n) & \xleftarrow{\ \varphi_{\#}\ } & C(\mathrm{Sd}(K^n)) \\ {\scriptstyle \sum_{i=1}^{n} \pi_{i\#}} \big\downarrow & & \big\downarrow {\scriptstyle \eta_{\#}} \\ C(K) & \xleftarrow[\ \mu\ ]{} & C(K(n, G)) \end{array}$$

Specifically, given any generator σ of $C(\mathrm{Sd}(K^n))$ for which $\eta_{\#}(\sigma) = t$ set $\mu(t) = \sum_{i=1}^{n} \pi_{i\#}\varphi_{\#}(\sigma)$. The map μ then induces a homomorphism μ_* from the simplicial homology of X^n/G into that of X. This homomorphism due to Maxwell shall be referred to as the *Maxwell trace homomorphism*. It can be shown that μ_* is natural and hence independent of the triangulation. Further, μ_* satisfies the following condition:

$$(*) \qquad \mu_*\eta_* = \sum_{i=1}^{n} \pi_{i_*}.$$

3. Periodic point theorems. As before, let $X = |K|$ be a compact polyhedron. Throughout the rest of this paper H_* will denote simplicial homology theory with coefficients in the rational field. For $f\colon X \to X^n/G$, the mth Lefschetz number $L^m(f)$ is defined to be the number $\sum_{p=0}^{\infty} (-1)^p \operatorname{trace}(\mu_{*_p} f_{*p})^m$. If $n = 1$, then μ_* is the identity and $L^m(f)$ coincides with the Lefschetz number of f^m, the mth iterate of f. If $m = 1$, then $L(f)$ is the Lefschetz number of f as defined by Maxwell [4].

We now turn to the main theorem of this section in which we show that the nonvanishing of the mth Lefschetz number of a symmetric product mapping is a sufficient condition for the existence of a periodic point. The proof is in the spirit of the proof of the Lefschetz fixed point theorem.

3.1. Theorem. *If X is a compact polyhedron and $f\colon X \to X^n/G$ with $L^m(f) \neq 0$ then f has a periodic point of period $\leqslant m$.*

Proof. Assume, by way of contradiction that f has no periodic point of period $\leqslant m$. Let $\Omega\colon X^m \times (X^n/G)^m \to \boldsymbol{R}^{\geqslant 0}$ be the map defined in section 2. Then $\Omega\big((x_1, \ldots, x_m), (f(x_1), \ldots, f(x_m))\big) > 0$ for all $(x_1, \ldots, x_m) \in X^m$. Further, since X is compact, there exists an $\varepsilon > 0$ such that $\Omega\big((x_1, \ldots, x_n), (f(x_1), \ldots, f(x_m))\big) > \varepsilon$ for all $(x_1, \ldots, x_m) \in X^m$.

Let K be a triangulation of X chosen so fine that $\operatorname{mesh} K^n < \varepsilon/3m$. Then $\operatorname{mesh} K < \varepsilon/3m$ and $\operatorname{mesh} K(n, G) < \varepsilon/3m$. Further there exist a subdivision K' of K and map $h\colon K' \to K(n, G)$ such that h is simplicial and $\tilde{d}(h(x), f(x)) < \varepsilon/3m$ for all $x \in X$.

Consider the following composition of chain maps

$$C(K) \xrightarrow{\nu} C(K') \xrightarrow{h_{\#}} C(K(n, G)) \xrightarrow{\mu} C(K)$$

where ν denotes the subdivision chain map. Let t be the set transformation which associates to each simplex s of K the closed subset

$$t(s) = \bigcup_{i_1, \ldots, i_r, \ldots, i_m = 1}^{n} \pi_{i_m}\varphi\eta^{-1}h(\pi_{i_{m-1}}\varphi\eta^{-1}h)(\ldots(\pi_{i_1}\varphi\eta^{-1}h|s|)\ldots)$$

where φ, η and π_i are the maps described in section 2. Then $|(\mu h_{\#}\nu)^m(s)| \subseteq t(s)$.

It is claimed that for every simplex s of K that $|s| \cap t(s) = \emptyset$. To show this we assume by way of contradiction that there exists a simplex s of K and $x \in X$ such that $x \in |s| \cap t(s)$. So for some $i_1, \ldots, i_m$ where $1 \leqslant i_1, \ldots, i_m \leqslant n$ we have that

$$x \in \pi_{i_m}\varphi\eta^{-1}h(\pi_{i_{m-1}}\varphi\eta^{-1}h(\ldots(\pi_{i_1}\varphi\eta^{-1}h|s|)\ldots).$$

Then there exist

$$z_m \in \varphi\eta^{-1}h(\pi_{i_{m-1}} \ldots (\pi_{i_1}\varphi\eta^{-1}h|s|)\ldots),$$
$$z'_m \in \eta^{-1}h(\pi_{i_{m-1}} \ldots (\pi_{i_1}\varphi\eta^{-1}h|s|)\ldots)$$

and

$$x_1 \in \pi_{i_{m-1}}\varphi\eta^{-1}h(\ldots(\pi_{i_1}\varphi\eta^{-1}h|s|)\ldots)$$

such that $\pi_{i_m}(z_m) = x$, $\varphi(z'_m) = z_m$ and $\eta(z'_m) = h(x_1)$. Continuing in this way, for each integer j, $1 \leqslant j \leqslant m-2$, there exists

$$x_j \in \pi_{i_r}\varphi\eta^{-1}h(\ldots(\pi_{i_1}\varphi\eta^{-1}h|s|)\ldots),$$
$$z_r \in \varphi\eta^{-1}h(\pi_{i_{r-1}} \ldots (\pi_{i_1}\varphi\eta^{-1}h|s|)\ldots)$$

and

$$z'_r \in \eta^{-1}h(\pi_{i_{r-1}} \ldots (\pi_{i_1}\varphi\eta^{-1}h|s|)\ldots)$$

where $j+r = m$ such that $\pi_{i_r}(z_r) = x_j$, $\varphi(z'_r) = z_r$ and $\eta(z'_r) = h(x_{j+1})$. Finally there is $x_m \in |s|$, $z_1 \in \varphi\eta^{-1}h|s|$ and $z'_1 \in \eta^{-1}h|s|$ such that $\pi_{i_1}(z_1) = x_{m-1}$, $\varphi(z'_1) = z_1$ and $\eta(z'_1) = h(x_m)$.

Since x and x_m both belong to $|s|$ we have that

$$\omega(x_m, \eta(z_m)) \leqslant d(x_m, \pi_{i_m}(z_m)) < \varepsilon/3m.$$

Further, from the above discussion it follows that

$$\omega(x_j, \eta(z_r)) \leqslant d(x_j, \pi_{i_r}(z_r)) = 0$$

where $r = m-j$ and $1 \leqslant j \leqslant m-1$. Also for $1 \leqslant r \leqslant m$ we see that

$$\tilde{d}(\eta(z_r), \eta(z'_r)) \leqslant \tilde{d}(z_r, z'_r) < \varepsilon/3m.$$

Finally, for $0 \leqslant j \leqslant m-1$, we have that

$$\tilde{d}(\eta(z'_{m-j}), f(x_{j+1})) = \tilde{d}(h(x_{j+1}), f(x_{j+1})) < \varepsilon/3\mathrm{m}.$$

Therefore, by the inequalities of section 2, we have

$$\begin{aligned} &\Omega((x_m, x_{m-1}, \ldots, x_1), (f(x_m), f(x_{m-1}), \ldots, f(x_1))) \\ &\quad \leqslant \omega(x_m, \eta(z_m)) + \tilde{d}(\eta(z_m), \eta(z'_m)) + \tilde{d}(\eta(z'_m), f(x_1)) \\ &\quad + \sum_{j=1}^{m-1} \omega(x_j, \eta(z_{m-j})) + \tilde{d}(\eta(z_{m-j}), \eta(z'_{m-j})) + \tilde{d}(\eta(z'_{m-j}), f(x_{j+1})) \\ &\quad < \frac{\varepsilon}{3m} + m\frac{\varepsilon}{3m} + m\frac{\varepsilon}{3m} < \varepsilon. \end{aligned}$$

This inequality contradicts the choice of ε. Thus $|s| \cap t(s) = \emptyset$ for all simplexes s of K and the claim is shown.

Since t is the carrier of the chain map $(\mu h_{\#}\nu)^m$ we have that $(\mu h_{\#}\nu)^m$ has zero entries along the diagonal when expressed as a matrix. Hence $\sum_{p=0}^{\infty}(-1)^p$ trace $(\mu_{*_p} f_{*_p})^m = 0$. This contradicts the hypothesis that $L^m(f) \neq 0$. So f has a periodic point of period $\leqslant m$. ■

If $m = 1$, then the above theorem coincides with the Maxwell–Lefschetz fixed point theorem for symmetric product mappings [4].

In [5], we extend Theorem 3.1 to symmetric product mappings of compact ANRs as well as A-ANRs by standard argument involving the domination of these spaces by polyhedra. We generalize Theorem 3.1 further to compact symmetric product mappings of (not necessarily finite) polyhedra and metric ANRs by means of direct limits.

We now consider the following relationship between the mth Lefschetz numbers of a symmetric product mapping and a certain characteristic rational function. The development of this relationship generalizes Halpern [1]. Once again let X be a compact polyhedron. For $f\colon X \to X^n/G$, let $P(\mu_* f_*)$ denote the characteristic polynomial of the transformation $H_*(X) \xrightarrow{f} H_*(X^n/G) \xrightarrow{\mu_*} H_*(X)$. Consider the rational function

$$K(f) = \Pi_i P(\mu_{*2i} f_{*2i}) / \Pi_i P(\mu_{*2i+1} f_{*2i+1}).$$

$K(f)$ has a unique expansion into a formal Laurent series

$$K(f) = \lambda^{\chi}(1 + a_1\lambda^{-1} + a_2\lambda^{-2} + \ldots)$$

where χ is the Euler characteristic of X. From Kelly and Spanier [2], we have that the n tuplets $\{a_1, a_2, \ldots, a_n\}$ and $\{L(f), L^2(f), \ldots, L^n(f)\}$ determine one another. In particular, $a_1 = a_2 = \ldots = a_n = 0$ if and only if $L(f) = L^2(f) = \ldots = L^n(f) = 0$. So by Theorem 3.1 we have the following result.

3.2. THEOREM. *Let X be a compact polyhedron and $f\colon X \to X^n/G$. If f has no periodic point of period $\leqslant m$, where $m = 1, \ldots, p$, then $a_1 = a_2 = \ldots = a_p = 0$.*

Next conditions are found which ensure that for some p, $a_p \neq 0$, and hence the existence of a periodic point.

3.3. LEMMA. *If $B = \{B_{ij}\}$ is an $m\times m$ matrix with rational coefficients such that for each i, $\sum_j B_{ij} = n$, then $\lambda-n$ is a factor of the characteristic polynomial of B.*

The proof is immediate. From this lemma we obtain the following result.

3.4. COROLLARY. *Let X be a compact polyhedron and $f\colon X\to X^n/G$. Then $(\lambda-n)|P(\mu_{*0}f_{*0})$.*

Proof. Suppose X consists of m path components. Then the matrix representing $\mu_{*0}f_{*0}$ is an $m\times m$ diagonal with n's down the diagonal since on a path connected component $\mu_{*0}f_{*0} = \mu_{*0} = \sum_{i=1}^{n}\pi_{i*} = n$. The second equality follows from expression (*) of section 2. ■

3.5. THEOREM. *Let X be a compact polyhedron and $f\colon X\to X^n/G$. If $H_i(X) = 0$, for i odd, then there is some m, $1\leqslant m\leqslant\chi$, such that f has a periodic point of period $\leqslant m$. (χ is the Euler characteristic of X.)*

Proof. Since $H_i(X) = 0$ for odd i, $K(f) = \Pi_i P(\mu_{*2i}f_{*2i}) = \lambda^\chi + a_1\lambda^{\chi-1}+ +\ldots+a_\chi$. But $(\lambda-n)|P(\mu_{*0}f_{*0})$ and so $(\lambda-n)|K(f)$. Hence not all a_i's vanish. So by Theorem 3.2 the desired result follows. ■

3.6. EXAMPLE. Let S^k be the k-sphere with k even. Then any map $f\colon S^k\to(S^k)^n/G$ has a periodic point of period $\leqslant 2$.

We note that Theorem 3.2 and Theorem 3.5 both can be extended to symmetric product mappings of compact ANRs.

References

[1] B. Halpern, *Fixed points for iterates*, Pacific J. Math. 25 (2) (1968), pp. 255–275.

[2] J. L. Kelley and E. Spanier, *Uniqueness of the Euler and Lefschetz functions*, Pacific J. Math. 26 (2) (1968), pp. 317–339.

[3] S. Masih, *Fixed points of symmetric product mappings of polyhedra and metric absolute neighborhood retracts*, Fund. Math. 80 (1973), pp. 149–156.

[4] C. N. Maxwell, *Fixed points of symmetric product mappings*, Proc. Amer. Math. Soc. 8 (1957), pp. 808–815.

[5] N. E. Rallis, *Periodic points and a fixed point index theory for symmetric product mappings*, Ph. D. Dissertation, Indiana University, 1978.

DEPARTMENT OF MATHEMATICS
BOSTON COLLEGE
Chestnut Hill Mass.

Accepté par la Rédaction le 27. 9. 1980

Geordnete Läuchli Kontinuen

by

Norbert Brunner (Baden)

Inhalt. Ein linear geordnetes Kontinuum im geordneten Mostowski Modell ist genau dann ein Dedekind abzählbarer Lusinraum, wenn es stark zusammenhängend ist und die abzählbare Kettenbedingung erfüllt.

1. Einleitung.

1.1. DEFINITION. (X, Z) sei ein *topologischer Raum*: Er ist stark zusammenhängend, wenn jede stetige, reellwertige Funktion konstant ist. Ein stark zusammenhängendes Kontinuum heißt Läuchli Kontinuum.

1.2. BEISPIEL (Existenz). (a) H. Läuchli [5] hat ein Permutationsmodell konstruiert, in dem ein geordnetes AA1-Läuchli Kontinuum existiert (1).

(b) Ist (X, Z) ein Kontinuum und $P(X)$ D-finit, so ist (X, Z) Läuchli'sch.

(c) $X = [a, b]$ sei ein abgeschlossenes Intervall von Atomen des geordneten Mostowski Modells ([8]) mit der Ordnungstopologie Z bzgl. der natürlichen Ordnung $<$ des Mostowski Modells: Ist $a<b$, so ist $(X, Z, <)$ ein nicht-triviales geordnetes Läuchli Kontinuum.

(d) Im zweiten Fraenkel Modell ([7]) gibt es keine nichttrivialen Läuchli Kontinuen ([1], S. 55), weil das Urysohn'sche Lemma gilt.

1.3. BEISPIEL (Anwendung). (a) Der Satz von Gelfand und Kolmogoroff, daß kompakte T_2-Räume homöomorph sind, wenn die Ringe der stetigen, reellwertigen Funktionen isomorph sind, ist in ZF+BPI nicht beweisbar (gemäß 1.2.c., [3] und [9]).

(b) Der Satz von Nakano und Stone, wonach ein kompakter T_2-Raum (X, Z) genau dann extrem unzusammenhängend ist, wenn jede nicht-leere punktweise beschränkte Teilmenge von $C(X)$ ein Supremum besitzt, ist in ZF+BPI nicht beweisbar.

1.4. BEMERKUNG. (a) *Ist (X_1, Z_1) ein Läuchli Kontinuum, (X_2, Z_2) T_2 und ist $f\colon X_1\to X_2$ stetig, so ist $f(X_1)$ ein Läuchli Kontinuum.*

(1) Die Mostowski'sche Konstruktion der Permutationsmodelle wird nach Specker [12] modifiziert, wodurch ZF° Modelle entstehen: a ist ein Urelement, wenn $a = \{a\}$ ist.