It is clear that conditions I–VIII give $s \in S$ and our proof of Lemma 7 is complete.

Now we prove Theorem 2. Let $\langle\langle s_\zeta, q_\zeta\rangle : \zeta<\omega_1\rangle$ be a sequence of elements of P. We show that there exist $\zeta<\eta<\omega_1$ such that $\langle s_\zeta, q_\zeta\rangle$ and $\langle s_\eta, q_\eta\rangle$ are compatible. By the Δ-lemma we may assume that

$$\operatorname{supp}(q_\zeta) = y \cup w_\zeta \quad \text{for every } \zeta<\omega_1$$

where

$$w_\zeta \cap w_\eta = 0 \quad \text{for every } \zeta<\eta<\omega_1 .$$

Let $P'_y = \{\langle s, q \restriction y\rangle : \langle s, q\rangle \in P\}$ be a poset with the ordering relation

$$\langle s, q\rangle \leqslant \langle s', q'\rangle \quad \text{iff} \quad s\supset s' \ \& \ \forall \langle\alpha, n\rangle \in y\,[q(\alpha, n)\leqslant q'(\alpha, n)] .$$

Clearly P_y and P'_y are isomorphic.

Let us consider a set $\{\langle s_\zeta, q_\zeta \restriction y\rangle : \zeta<\omega_1\}$ of elements of P'_y. By Lemma 7 there exist $\zeta<\eta<\omega_1$ and $\langle s, q\rangle \in P'_y$ such that $\langle s, q\rangle \leqslant \langle s_\zeta, q_\zeta \restriction y\rangle$ and $\langle s, q\rangle \leqslant s_\eta, q_\eta \restriction y\rangle$.

Let $q' \in Q$ be defined by

$$q'(\alpha, n) = \begin{cases} q(\alpha, n) & \text{for } \langle\alpha, n\rangle \in y ,\\ q_\zeta(\alpha, n) & \text{for } \langle\alpha, n\rangle \in w_\zeta ,\\ q_\eta(\alpha, n) & \text{for } \langle\alpha, n\rangle \in w_\eta ,\\ \langle 0, 0\rangle & \text{otherwise.} \end{cases}$$

It is easy to see that $\langle s, q'\rangle \in P$ and $\langle s, q'\rangle \leqslant \langle s_\zeta, q_\zeta\rangle$ and $\langle s, q'\rangle \leqslant \langle s_\eta, q_\eta\rangle$.

This completes the proof.

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Accepté par la Rédaction le 22. 9. 1980

Topological games and products, II

by

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Abstract. The purpose of this paper is to study the topological games (in the sense of R. Telgársky) of product spaces: Assume that Player *I* has winning strategies in the given topological games of X and Y. Then we consider the conditions of a product space $X\times Y$ under which he has a winning strategy in a certain topological game of $X\times Y$. Moreover, we can apply the results obtained from this kind of argument to the product theorem in dimension theory.

Introduction. R. Telgársky [14] introduced and studied the topological game $G(\boldsymbol{K}, X)$. In our previous paper [19], we have used it to study the covering properties of product spaces. In the present paper, we shall study the topological game on product spaces. If the above $\boldsymbol{K}$ is the class of all one-point spaces, then the game $G(\boldsymbol{K}, X)$ is often abbreviated by $G(X)$, which is called the point-open game. R. Telgársky [15] stated the following: If Player *I* has winning strategies in $G(X)$ and $G(Y)$, then he has a winning strategy in $G(X\times Y)$. This gives the positive answer to [14, Question 14.1]. In this connection, we raise the following natural question: Assume that Player *I* has winning strategies in $G(\boldsymbol{K}_1, X)$ and $G(\boldsymbol{K}_2, Y)$. What is a topological game of $X\times Y$ which is interesting to investigate? What is a condition on $X\times Y$ under which he has a winning strategy in such a game? In § 2 and § 3, we discuss this question. In § 4, using the result of § 2, we give a product theorem in dimension theory.

Each space considered here is assumed to be a Hausdorff space. N denotes the set of all natural numbers and $\mathfrak{m}$ denotes an infinite cardinal number. For a space or a set X, by $\chi(X)$ we mean the character of X and by $|X|$ the cardinality of X. For a collection $\mathfrak{F}$ of subsets of X, $\bigcup\mathfrak{F}$ denotes $\bigcup\{F: F\in\mathfrak{F}\}$.

§ 1. Topological games. R. Telgársky [15] has introduced an equivalent form of the game $G(\boldsymbol{K}, X)$ defined in [14]. The new form of the game we use below.

Let $\boldsymbol{L}$ be a class of spaces and let X be a space. We define the topological game $G(\boldsymbol{L}, X)$ as follows: There are two players; Player *I* and Player *II*. Player *I* chooses a closed set E_1 of X with $E_1 \in \boldsymbol{L}$, and after that Player *II* chooses an open set U_1 of X with $E_1 \subset U_1$. Again Player *I* chooses a closed set E_2 of X with $E_2 \in \boldsymbol{L}$ and Player *II* chooses an open set U_2 of X with $E_2 \subset U_2$, and so on. Here, the infinite sequence $\langle E_1, U_1, E_2, U_2, \ldots\rangle$ is a *play* of $G(\boldsymbol{L}, X)$. Player I wins the play $\langle E_1, U_1, E_2, U_2, \ldots\rangle$ if $\{U_n\colon n \in N\}$ covers X, otherwise Player II wins.

A finite sequence $\langle E_1, U_1, \ldots, E_n, U_n\rangle$ of subsets in X is said to be *admissible* for $G(\boldsymbol{L}, X)$ if the infinite sequence $\langle E_1, U_1, \ldots, E_n, U_n, \emptyset, \emptyset, \ldots\rangle$ is a play of $G(\boldsymbol{L}, X)$.

A function s is said to be a *strategy* for Player I in $G(\boldsymbol{L}, X)$ if the domain of s consists of the void sequence $\emptyset$ and the finite sequences $\langle U_1, \ldots, U_n\rangle$ of open sets in X and if $s(\emptyset)$ and $s(U_1, \ldots, U_n)$ are closed in X and belong to $\boldsymbol{L}$.

A strategy s for Player I in $G(\boldsymbol{L}, X)$ is said to be *winning* if he wins each play $\langle E_1, U_1, E_2, U_2, \ldots\rangle$ in $G(\boldsymbol{L}, X)$ such that $E_1 = s(\emptyset)$ and $E_{n+1} = s(U_1, \ldots, U_n)$ for each $n \in N$. In this paper, we shall not deal with a winning strategy of Player II in $G(\boldsymbol{L}, X)$, but only the one of Player I.

Here, note that the class $\boldsymbol{L}$ is not assumed to be hereditary with respect to closed sets. Indeed, we deal with a class which is not hereditary with respect to closed sets, as below. In such a case, the original topological game in [14] seems to be inappropriate. However, it is stated in [15] that the both topological games are equivalent if the class $\boldsymbol{L}$ is hereditary with respect to closed sets. The following classes are frequently used in this paper and some other classes will be defined later when they will be necessary.

For a class $\boldsymbol{L}$ of spaces, we denote by $\boldsymbol{DL}$ ($\boldsymbol{FL}$) the class of all spaces which have a discrete (finite) closed cover consisting of members of $\boldsymbol{L}$. We denote by $\boldsymbol{C}$ ($\boldsymbol{C}_{\mathfrak{m}}$) the class of all ($\mathfrak{m}$-)compact spaces.

Throughout this paper, let $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ be the arbitrary classes of spaces which are hereditary with respect to closed sets. We define

$$\boldsymbol{K}_1 \times \boldsymbol{K}_2 = \{X \times Y\colon X \in \boldsymbol{K}_1 \text{ and } Y \in \boldsymbol{K}_2\}.$$

It is clear that the class $\boldsymbol{K}_1 \times \boldsymbol{K}_2$ of product spaces is not hereditary with respect to closed sets in general.

§ 2. D-products. In this section, we first find a condition for a product space $X \times Y$ under which Player I has a winning strategy in $G(\boldsymbol{D}(\boldsymbol{K}_1 \times \boldsymbol{K}_2), X \times Y)$ if he has winning strategies in $G(\boldsymbol{K}_1, X)$ and $G(\boldsymbol{K}_2, Y)$.

DEFINITION. A product space $X \times Y$ is said to be a *D-product* if for each closed set M of $X \times Y$ and each open set O of $X \times Y$ with $M \subset O$ there is a σ-discrete collection $\mathfrak{F}$ by closed rectangles in $X \times Y$ such that $M \subset \bigcup \mathfrak{F} \subset O$.

For a closed rectangle R in $X \times Y$, R' and R'' denote the projections of R into X and Y respectively. So, R is a closed (open) rectangle in $X \times Y$ iff R' and R'' are closed (open) in X and Y respectively such that $R = R' \times R''$.

Our main theorem is as follows.

THEOREM 2.1. *Let X and Y be spaces such that $X \times Y$ is a D-product. If Player I has winning strategies in $G(\boldsymbol{K}_1, X)$ and $G(\boldsymbol{K}_2, Y)$, then he has a winning strategy in $G(\boldsymbol{D}(\boldsymbol{K}_1 \times \boldsymbol{K}_2), X \times Y)$.*

Before its proof, we prepare some notations and a lemma. N^* denotes the set

consisting of the void sequence $\emptyset$ and all finite sequences consisting of natural numbers. For each $e \in N^* \setminus \{\emptyset\}$, e' (e'') denotes the subsequence of e consisting of all odd (even) numbers. Moreover, let $e = (n_1, \ldots, n_k)$. We use the following notations; $\sum e = n_1 + \ldots + n_k$, $|e| = k$ $(|\emptyset| = 0)$, $(e, n) = (n_1, \ldots, n_k, n)$ and $e_{-j} = (n_1, \ldots, n_{k-j})$ for $0 \leqslant j \leqslant k-1$ $(e_{-k} = \emptyset)$. For a finite sequence $S\langle R\rangle = \langle R_1, \ldots, R_k\rangle$ of sets, $\langle S\langle R\rangle, A, B\rangle$ denotes the finite sequence $\langle R_1, \ldots, R_k, A, B\rangle$.

LEMMA 2.1. *For a class $\boldsymbol{L}$ of spaces, Player I has a winning strategy in $G(\boldsymbol{L}, X)$ iff he has a winning strategy in $G(\boldsymbol{FL}, X)$.*

This is essentially given by R. Telgársky [14, Theorem 4.1].

The proof of Theorem 2.1. Let s and t be winning strategies for Player I in $G(\boldsymbol{K}_1, X)$ and $G(\boldsymbol{K}_2, Y)$ respectively. It is sufficient from Lemma 2.1 that we construct his winning strategy p in $G(\boldsymbol{L}, X \times Y)$, where $\boldsymbol{L} = \boldsymbol{F}(\boldsymbol{D}(\boldsymbol{K}_1 \times \boldsymbol{K}_2))$.

Now, assume that we have already constructed an admissible sequence $\langle P_1, O_1, \ldots, P_m, O_m\rangle$ in $G(\boldsymbol{L}, X \times Y)$ such that $P_1 = p(\emptyset)$ and $P_{i+1} = p(O_1, \ldots, O_i)$ for $1 \leqslant i \leqslant m-1$ and such that there is a family

$$\{\langle \mathfrak{R}(e), \varphi_e\rangle\colon e \in N^* \text{ and } \textstyle\sum e_{-1} \leqslant m-1\}$$

of the pairs of collections $\mathfrak{R}(e)$ by closed rectangles in $X \times Y$ and the functions φ_e of $\mathfrak{R}(e)$ onto $\mathfrak{R}(e_{-1})$, where $\mathfrak{R}(\emptyset) = \{X \times Y\}$, satisfying the following conditions (1)–(4): For each $e \in N^*$ with $\sum e_{-1} \leqslant m-1$,

(1) $\mathfrak{R}(e)$ is discrete in $X \times Y$,

(2) if $(x, y) \in R \in \mathfrak{R}(e_{-1})$ and $(x, y) \notin O_{(\Sigma e_{-1})+1}$, then there are some $n \in N$ and $Q \in \mathfrak{R}(e_{-1}, n)$ such that $(x, y) \in Q$ and $\varphi_{(e_{-1}, n)}(Q) = R$.

Let $\varphi_e^0 = 1_{\mathfrak{R}(e)}$ and $\varphi_e^k = \varphi_{e_{1-k}} \circ \ldots \circ \varphi_{e_{-1}} \circ \varphi_e$ for $1 \leqslant k \leqslant |e|-1$. For each $e \in N^*$ with $\sum e_{-1} \leqslant m-1$ and each $R \in \mathfrak{R}(e)$,

(3) let $U_k = X \setminus (\varphi_e^{|e|-i_k}(R))'$ for $1 \leqslant k \leqslant q$, where $e' = (i_1, \ldots, i_q)$, and if we put $E_1 = s(\emptyset)$ and $E_{k+1} = s(U_1, \ldots, U_k)$ for $1 \leqslant k \leqslant q-1$, then the finite sequence $\langle E_1, U_1, \ldots, E_q, U_q\rangle$ is admissible for $G(\boldsymbol{K}_1, X)$,

(4) let $V_k = Y \setminus (\varphi_e^{|e|-j_k}(R))''$ for $1 \leqslant k \leqslant r$, where $e'' = (j_1, \ldots, j_r)$, and if we put $F_1 = t(\emptyset)$ and $F_{k+1} = t(V_1, \ldots, V_k)$ for $1 \leqslant k \leqslant r-1$, then the finite sequence $\langle F_1, V_1, \ldots, F_r, V_r\rangle$ is admissible for $G(\boldsymbol{K}_2, Y)$.

In the above (3) and (4), let $S\langle R\rangle = \langle E_1, U_1, \ldots, E_q, U_q\rangle$, $\tilde{S}\langle R\rangle = \langle U_1, \ldots, U_q\rangle$, $T\langle R\rangle = \langle F_1, V_1, \ldots, F_r, V_r\rangle$ and $\tilde{T}\langle R\rangle = \langle V_1, \ldots, V_r\rangle$.

We pick an $e \in N^*$ with $\sum e = m$. Let $|e'| = q$ and $|e''| = r$. From $\sum e_{-1} < m$, $\mathfrak{R}(e)$ has been already constructed. So we also pick an $R \in \mathfrak{R}(e)$. From the assumptions (3) and (4), we can put $E_{q+1} = s(\tilde{S}\langle R\rangle)$ and $F_{r+1} = t(\tilde{T}\langle R\rangle)$. Moreover, we put $P_{m+1}(R) = (E_{q+1} \times F_{r+1}) \cap R$. Then $P_{m+1}(R)$ belongs to $\boldsymbol{K}_1 \times \boldsymbol{K}_2$ and $P_{m+1}(R) \subset R$. Note that $P_{m+1}(R)$ may be empty. Here, we set

$$P_{m+1} = p(O_1, \ldots, O_m) = \bigcup \{P_{m+1}(R)\colon R \in \mathfrak{R}(e) \text{ and } \textstyle\sum e = m\}.$$

It follows from the assumption (1) that P_{m+1} belongs to L. Next, Player II chooses an arbitrary open set O_{m+1} of $X\times Y$ with $P_{m+1}\subset O_{m+1}$. Again, let $R\in\mathfrak{R}(e)$. Since $X\times Y$ is a D-product, R is too. Hence there is a collection

$$\mathfrak{Q}(R) = \bigcup\{\mathfrak{Q}_n(R)\colon\ n\in N\}$$

of closed rectangles in R such that each $\mathfrak{Q}_n(R)$ is discrete in R and $R\setminus O_{m+1}\subset\bigcup\mathfrak{Q}(R)\subset R\setminus P_{m+1}$. For each $n\in N$, we put

$$\mathfrak{Q}_R(e, 2n-1) = \{Q\in\mathfrak{Q}_n(R)\colon\ Q'\cap s(\tilde{S}\langle R\rangle) = \emptyset\}\,,$$
$$\mathfrak{Q}_R(e, 2n) = \{Q\in\mathfrak{Q}_n(R)\colon\ Q'\cap s(\tilde{S}\langle R\rangle) \neq \emptyset\}\,.$$

Moreover, we here set for each $n\in N$

$$\mathfrak{R}(e, n) = \bigcup\{\mathfrak{Q}_R(e, n)\colon\ R\in\mathfrak{R}(e)\}$$

and define the function $\varphi_{(e,n)}\colon\ \mathfrak{R}(e, n)\to\mathfrak{R}(e)$ such that $\varphi_{(e,n)}(\mathfrak{Q}_R(e, n)) = \{R\}$ for each $R\in\mathfrak{R}(e)$. We show that the pair $\langle\mathfrak{R}(e, n), \varphi_{(e,n)}\rangle$ satisfies the conditions (1)–(4). Since each $\mathfrak{Q}_n(R)$ is discrete in R, it follows from the assumption (1) that each $\mathfrak{R}(e, n)$ is discrete in $X\times Y$. The condition (1) is satisfied. Let $(x, y)\in R\in\mathfrak{R}(e)$ and $(x, y)\notin O_{m+1}$. Then we can take some $n\in N$ and $Q\in\mathfrak{Q}_n(R)$ such that $(x, y)\in Q$. Hence we have either $Q\in\mathfrak{R}(e, 2n-1)$ and $\varphi_{(e,2n-1)}(Q) = R$ or $Q\in\mathfrak{R}(e, 2n)$ and $\varphi_{(e,2n)}(Q) = R$. The condition (2) is satisfied. Finally, we pick a $Q\in\mathfrak{R}(e, n)$. Let $|e'| = q$, $|e''| = r$ and $R = \varphi_{(e,n)}(Q)$.

Let n be an odd number. Since $(e, n)' = (e', n)$, we put $E_{q+1} = s(\tilde{S}\langle R\rangle)$ and $U_{q+1} = X\setminus Q'$. Then we have $S\langle Q\rangle = \langle S\langle R\rangle, E_{q+1}, U_{q+1}\rangle$. In order to show that $S\langle Q\rangle$ is admissible for $G(K_1, X)$, it is sufficient from the assumption (3) to show $E_{q+1}\subset U_{q+1}$. Since n is odd and Q is in $\mathfrak{Q}_R(e, n)$, we have

$$Q'\cap E_{q+1} = \emptyset\,.$$

Hence the condition (3) is satisfied. By $(e, n)'' = e''$, we have $T\langle Q\rangle = T\langle R\rangle$. So the condition (4) is clearly satisfied.

Let n be an even number. By $(e, n)' = e'$, we have $S\langle Q\rangle = S\langle R\rangle$. So the condition (3) is clearly satisfied. Since $(e, n)'' = (e'', n)$, we put $F_{r+1} = t(\tilde{T}\langle R\rangle)$ and $V_{r+1} = Y\setminus Q''$. Then we have $T\langle Q\rangle = \langle T\langle R\rangle, F_{r+1}, V_{r+1}\rangle$. From the same reason as the above, we show $F_{r+1}\subset V_{r+1}$. Assume the contrary. We can choose some $y_0\in Q''\cap F_{r+1}$. Since n is even and Q is in $\mathfrak{Q}_R(e, n)$, we can also choose some $x_0\in Q'\cap s(\tilde{S}\langle R\rangle)$. Thus we have

$$(x_0, y_0)\in Q\cap s(\tilde{S}\langle R\rangle)\times F_{r+1}\subset P_{m+1}(R)\subset P_{m+1}\,.$$

On the other hand, we have $(x_0, y_0)\in Q\subset R\setminus P_{m+1}$. This is a contradiction. Hence the condition (4) is satisfied.

If $R = X\times Y\in\mathfrak{R}(\emptyset)$, then we set $P_1 = p(\emptyset) = s(\emptyset)\times t(\emptyset)$ and a simple version of the above argument with no inductive assumptions gives the desired family $\{\mathfrak{R}(n)\colon\ n\in N\}$. The constructions by induction have been completed with the facts

mentioned above. Since the choice of Player II is free, we have defined a strategy p of Player I in $G(\boldsymbol{L}, X\times Y)$.

The obtained infinite sequence $\langle P_1, O_1, P_2, O_2, \ldots\rangle$ such that $P_1 = p(\emptyset)$ and $P_{m+1} = p(O_1, \ldots, O_m)$ for each $m\in N$ is a play of $G(\boldsymbol{L}, X)$. We show that $\{O_m\colon\ m\in N\}$ covers $X\times Y$. Assume $(x, y)\notin O_m$ for each $m\in N$. By the condition (2) and $X\times Y\in\mathfrak{R}(\emptyset)$, we can inductively choose some infinite sequence $c = (n_1, n_2, \ldots)$ consisting of natural numbers and some infinite sequence $\langle R_1, R_2, \ldots\rangle$ by closed rectangles in $X\times Y$ such that

$$(x, y)\in R_k\in\mathfrak{R}(n_1, \ldots, n_k)\quad\text{ and }\quad\varphi_{(n_1,\ldots,n_{k+1})}(R_{k+1}) = R_k$$

for each $k\in N$. Then we have $x\in\bigcap\{R_k'\colon\ k\in N\}$ and $y\in\bigcap\{R_k''\colon\ k\in N\}$. Clearly, c contains infinitely many odd numbers or infinitely many even ones.

Let $(n_{i_1}, n_{i_2}, \ldots)$ be the infinite subsequence of c consisting of all odd numbers. We put $U_q = X\setminus R_{i_q}'$, $E_1 = s(\emptyset)$ and $E_{q+1} = s(U_1, \ldots, U_q)$ for each $q\in N$. Then it follows from the condition (3) that the infinite sequence $\langle E_1, U_1, E_2, U_2, \ldots\rangle$ is a play of $G(K_1, X)$. Since s is a winning strategy in $G(K_1, X)$, $\{U_q\colon\ q\in N\}$ covers X. Hence we have $\bigcap\{R_k'\colon\ k\in N\} = \emptyset$. This is a contradiction.

Let $(n_{j_1}, n_{j_2}, \ldots)$ be the infinite subsequence of c consisting of all even numbers. Then, by the condition (4), the infinite sequence $\langle F_1, V_1, F_2, V_2, \ldots\rangle$ defined by $V_r = Y\setminus R_{j_r}''$, $F_1 = t(\emptyset)$ and $F_{r+1} = t(V_1, \ldots, V_r)$ for each $r\in N$ is a play of $G(K_2, Y)$. Hence we have $\bigcap\{R_k''\colon\ k\in N\} = \emptyset$, which is a contradiction.

Thus $\{O_m\colon\ m\in N\}$ covers $X\times Y$. This implies that p is a winning strategy of Player I in $G(\boldsymbol{L}, X\times Y)$. The proof of Theorem 2.1 is complete.

Now we consider what kind of product spaces are D-products.

THEOREM 2.2. *Let X be a collectionwise normal space and Y a subparacompact space with $\chi(Y)\leqslant\mathfrak{m}$. If Player I has a winning strategy in $G(\boldsymbol{DC}_{\mathfrak{m}}, X)$, then every open cover of $X\times Y$ with power $\leqslant\mathfrak{m}$ has a σ-discrete refinement by closed rectangles in $X\times Y$.*

Proof. We shall proceed the proof, using the same technique as that of [19, Theorem 2.1]. Let s be a winning strategy of Player I in $G(\boldsymbol{DC}_{\mathfrak{m}}, X)$. Let $\mathfrak{G}$ be an arbitrary open cover of $X\times Y$ with $|\mathfrak{G}|\leqslant\mathfrak{m}$.

First, we construct a sequence $\{\mathfrak{F}_n\colon\ n\geqslant 0\}$ of collections of closed rectangles in $X\times Y$ and a sequence $\{\langle\mathfrak{R}_n, \varphi_n\rangle\colon\ n\geqslant 0\}$ of the pairs of collections $\mathfrak{R}_n$ by closed rectangles in $X\times Y$ and the functions φ_n of $\mathfrak{R}_n$ onto $\mathfrak{R}_{n-1}$, satisfying the following conditions (1)–(5):

(1) $\mathfrak{F}_n$ is σ-discrete in $X\times Y$.

(2) $\mathfrak{R}_n$ is σ-discrete in $X\times Y$.

(3) Each $F\in\mathfrak{F}_n$ is contained in some $G\in\mathfrak{G}$.

(4) If $(x, y)\in R_{n-1}\in\mathfrak{R}_{n-1}$ and $(x, y)\notin\bigcup\mathfrak{F}_n$, then there is some $R_n\in\mathfrak{R}_n$ such that $(x, y)\in R_n$ and $\varphi_n(R_n) = R_{n-1}$.

(5) For an $R\in\mathfrak{R}_n$, let $U_n = X\setminus R'$ and $U_k = X\setminus(\varphi_{k+1}\circ\ldots\circ\varphi_n(R))'$ for $1\leqslant k\leqslant n-1$. If we put $E_1 = s(\emptyset)$ and $E_{k+1} = s(U_1, \ldots, U_k)$ for $1\leqslant k\leqslant n-1$, then the finite sequence $\langle E_1, U_1, \ldots, E_n, U_n\rangle$ is admissible for $G(\boldsymbol{DC}_{\mathfrak{m}}, X)$.

Let $\mathfrak{F}_0 = \{\emptyset\}$ and $\mathfrak{R}_0 = \{X\times Y\}$. Assume that we have already constructed the above $\{\mathfrak{F}_i: i\leqslant n\}$ and $\{\langle \mathfrak{R}_i, \varphi_i\rangle: i\leqslant n\}$. We pick an $R\in \mathfrak{R}_n$. Let

$$\langle E_1, U_1, \ldots, E_n, U_n\rangle$$

be the admissible sequence in $G(\boldsymbol{DC}_{\mathfrak{m}}, X)$ which is described for this R in the assumption (5). So there is a discrete collection $\{C_\alpha: \alpha\in \Omega(R)\}$ by $\mathfrak{m}$-compact closed sets in R' such that

$$s(U_1, \ldots, U_n)\cap R' = \bigcup\{C_\alpha: \alpha\in\Omega(R)\}.$$

We can take a discrete collection $\{W_\alpha: \alpha\in\Omega(R)\}$ of open sets in R' such that $C_\alpha\subset W_\alpha$ for each $\alpha\in\Omega(R)$. Since C_α is $\mathfrak{m}$-compact, $|\mathfrak{G}|\leqslant\mathfrak{m}$, $\chi(Y)\leqslant\mathfrak{m}$ and R'' is subparacompact, there is a collection

$$\mathfrak{F}_{n+1}^{\alpha} = \{\mathrm{Cl}\, U_\lambda^{\alpha,i}\times H_\lambda: i = 1, \ldots, k_\lambda \text{ and } \lambda\in\Lambda(\alpha)\}$$

by closed rectangles in R, satisfying the following:

(i) Each $U_\lambda^{\alpha,i}$ is open in R'.

(ii) $C_\alpha\subset\bigcup\{U_\lambda^{\alpha,i}: i = 1, \ldots, k_\lambda\}\subset W_\alpha$.

(iii) Each $\mathrm{Cl}\, U_\lambda^{\alpha,i}\times H_\lambda$ is contained in some $G\in\mathfrak{G}$.

(iv) $\{H_\lambda: \lambda\in\Lambda(\alpha)\}$ is a σ-discrete closed cover of R''.

Then $\mathfrak{F}_{n+1}(R) = \bigcup\{\mathfrak{F}_{n+1}^{\alpha}: \alpha\in\Omega(R)\}$ is σ-discrete in $X\times Y$. We put for each $\lambda\in\Lambda(\alpha)$

$$R_\lambda^\alpha = (\mathrm{Cl}\, W_\alpha\setminus\bigcup\{U_\lambda^{\alpha,i}: 1\leqslant i\leqslant k_\lambda\})\times H_\lambda.$$

Next we put $\tilde{R} = (R'\setminus\bigcup\{W_\alpha: \alpha\in\Omega(R)\})\times R''$. Moreover, let us put

$$\mathfrak{R}_{n+1}(R) = \{\tilde{R}\}\cup\{R_\lambda^\alpha: \lambda\in\Lambda(\alpha) \text{ and } \alpha\in\Omega(R)\}.$$

Then $\mathfrak{R}_{n+1}(R)$ is also a σ-discrete collection by closed rectangles in R. Here we set $\mathfrak{F}_{n+1} = \bigcup\{\mathfrak{F}_{n+1}(R): R\in\mathfrak{R}_n\}$ and $\mathfrak{R}_{n+1} = \bigcup\{\mathfrak{R}_{n+1}(R): R\in\mathfrak{R}_n\}$. The function $\varphi_{n+1}: \mathfrak{R}_{n+1}\to\mathfrak{R}_n$ is defined as $\varphi_{n+1}(\mathfrak{R}_{n+1}(R)) = \{R\}$ for each $R\in\mathfrak{R}_n$. From the assumption (1), $\mathfrak{F}_{n+1}$ and $\mathfrak{R}_{n+1}$ are σ-discrete in $X\times Y$. The conditions (1) and (2) are satisfied. By (iii), the condition (3) is also satisfied. The verification that the conditions (4) and (5) are satisfied is analogous to that of the cases of (1.5_n) and (1.6_n) respectively in the proof of [19, Theorem 2.1]. Thus the above constructions by induction are completed.

Let $\mathfrak{F} = \bigcup\{\mathfrak{F}_n: n\in N\}$. We can show similarly to our previous proof that $\mathfrak{F}$ is a cover of $X\times Y$. Hence $\mathfrak{F}$ is a σ-discrete refinement of $\mathfrak{G}$ by closed rectangles in $X\times Y$. The proof is complete.

As the immediate consequences of Theorem 2.2, we have

COROLLARY 2.1. *Let X be a collectionwise normal space and Y a subparacompact space with $\chi(Y)\leqslant\mathfrak{m}$. If Player I has a winning strategy in $G(\boldsymbol{DC}_{\mathfrak{m}}, X)$, then $X\times Y$ is a D-product.*

COROLLARY 2.2. *Let X be a paracompact space and Y a subparacompact space. If Player I has a winning strategy in $G(\boldsymbol{DC}, X)$, then $X\times Y$ is subparacompact.*

Let $\boldsymbol{PC}_{\mathfrak{m}}$ be the class of all product spaces with the first factor being $\mathfrak{m}$-compact. The following corollary is an immediate consequence of Theorem 2.1 and Corollary 2.1.

COROLLARY 2.3. *Let X be a collectionwise normal space and Y a subparacompact space with $\chi(Y)\leqslant\mathfrak{m}$. If Player I has a winning strategy in $G(\boldsymbol{DC}_{\mathfrak{m}}, X)$, then he has a winning strategy in $G(\boldsymbol{D}(\boldsymbol{PC}_{\mathfrak{m}}), X\times Y)$.*

Moreover, let us give some other typical examples of D-product.

PROPOSITION 2.1. *Let X and Y be spaces such that $X\times Y$ is normal. If Y is a σ-space or has a σ-discrete cover by compact sets, then $X\times Y$ is a D-product.*

Proof. Let M be a closed set of $X\times Y$ and O an open set of $X\times Y$ with $M\subset O$. First, let Y be a σ-space. So Y has a σ-discrete closed net $\mathfrak{F} = \{F_\lambda: \lambda\in\Lambda\}$. Since $X\times Y$ is normal, we can take an open set W of $X\times Y$ such that $M\subset W\subset\mathrm{Cl}\, W\subset O$. For each $\lambda\in\Lambda$, we put

$$U_\lambda = \bigcup\{U: U \text{ is open in } X \text{ with } U\times F_\lambda\subset W\}.$$

Since $\mathfrak{F}$ is a net of Y, we have $\bigcup\{U_\lambda\times F_\lambda: \lambda\in\Lambda\} = W$. Hence $\{\mathrm{Cl}\, U_\lambda\times F_\lambda: \lambda\in\Lambda\}$ is a σ-discrete collection by closed rectangles in $X\times Y$ such that

$$M\subset W\subset\bigcup\{\mathrm{Cl}\, U_\lambda\times F_\lambda: \lambda\in\Lambda\}\subset\mathrm{Cl}\, W\subset O.$$

Secondly, let Y have a σ-discrete cover by compact sets. It is easily seen that we can assume, without loss of generality, Y to be compact. We take a binary cozero cover $\{G, H\}$ of $X\times Y$ such that $G\subset O$ and $H\subset X\setminus M$. By the lemma of J. Terasawa described in [2], there is a normal cover $\mathfrak{U} = \{U_\lambda: \lambda\in\Lambda\}$ of X and a family $\{\mathfrak{V}_\lambda: \lambda\in\Lambda\}$ of the finite open covers of Y such that

$$\{U_\lambda\times\mathrm{Cl}\, V: V\in\mathfrak{V}_\lambda \text{ and } \lambda\in\Lambda\}$$

is a refinement of $\{G, H\}$. Since $\mathfrak{U}$ is normal, there is a σ-discrete closed cover $\mathfrak{E} = \{E_\lambda: \lambda\in\Lambda\}$ of X such that $E_\lambda\subset U_\lambda$ for each $\lambda\in\Lambda$. Then

$$\{E_\lambda\times\mathrm{Cl}\, V: V\in\mathfrak{V}_\lambda \text{ and } \lambda\in\Lambda\}$$

is a σ-discrete closed refinement of $\{G, H\}$ by closed rectangles in $X\times Y$. This implies $X\times Y$ is a D-product. The proof is complete.

PROPOSITION 2.2. *Let X be a subparacompact P-space and Y a regular strong Σ-space* (cf. [7]). *Then every open cover of $X\times Y$ has a σ-discrete refinement by closed rectangles in $X\times Y$. Therefore $X\times Y$ is a D-product.*

Proof. We modify the proof of [7, Theorem 4.1]. Let $\mathfrak{G}$ be an open cover of $X\times Y$. Let $\mathfrak{F}_i = \{F(\alpha_1, \ldots, \alpha_i): \alpha_1, \ldots, \alpha_i\in\Omega\}$, $i\in N$, be a spectral Σ-net of Y (cf. [7, Definition 1.5]). Let $\mathfrak{W}(\alpha_1, \ldots, \alpha_i) = \{U_\lambda\times V_\lambda: \lambda\in\Lambda(\alpha_1, \ldots, \alpha_i)\}$ be the maximal collection by open rectangles in $X\times Y$, satisfying

(i) $F(\alpha_1, \ldots, \alpha_i)\subset V_\lambda$,

(ii) V_λ is a finite union of open sets $V_{\lambda 1}, \ldots, V_{\lambda n(\lambda)}$ of Y such that each $U_\lambda\times\mathrm{Cl}\, V_{\lambda j}$ is contained in some $G\in\mathfrak{G}$, for each $\lambda\in\Lambda(\alpha_1, \ldots, \alpha_i)$.

Put $U(\alpha_1, \ldots, \alpha_i) = \bigcup \{U_\lambda\colon \lambda \in \Lambda(\alpha_1, \ldots, \alpha_i)\}$. Then we have $U(\alpha_1, \ldots, \alpha_i) \subset U(\alpha_1, \ldots, \alpha_i, \alpha_{i+1})$ for each $\alpha_1, \ldots, \alpha_{i+1} \in \Omega$. So there is a collection $\{C(\alpha_1, \ldots, \alpha_i)\colon \alpha_1, \ldots, \alpha_i \in \Omega\}$ of closed sets of X such that $C(\alpha_1, \ldots, \alpha_i) \subset U(\alpha_1, \ldots, \alpha_i)$ for each $\alpha_1, \ldots, \alpha_i \in \Omega$ and

$$\bigcup \{U(\alpha_1, \ldots, \alpha_i)\colon i \in N\} = X$$

implies

$$\bigcup \{C(\alpha_1, \ldots, \alpha_i)\colon i \in N\} = X.$$

Since $\{U_\lambda\colon \lambda \in \Lambda(\alpha_1, \ldots, \alpha_i)\} \cup \{X \setminus C(\alpha_1, \ldots, \alpha_i)\}$ is an open cover of X, we can take a σ-discrete collection $\{E_\lambda\colon \lambda \in \Lambda(\alpha_1, \ldots, \alpha_i)\}$ of closed sets of X such that $C(\alpha_1, \ldots, \alpha_i) \subset \bigcup \{E_\lambda\colon \lambda \in \Lambda(\alpha_1, \ldots, \alpha_i)\}$ and $E_\lambda \subset U_\lambda$ for each $\lambda \in \Lambda(\alpha_1, \ldots, \alpha_i)$. Since each $\mathfrak{F}_i$ is locally finite in Y and Y is subparacompact (cf. [6, Theorem 1]), there is a σ-discrete closed cover $\mathfrak{H}_i$ of Y such that each $H \in \mathfrak{H}_i$ intersects at most finite many $F \in \mathfrak{F}_i$. Let us put for each $i \in N$

$$\mathfrak{R}_i = \{E_\lambda \times (F(\alpha_1, \ldots, \alpha_i) \cap \operatorname{Cl} V_{\lambda j} \cap H)\colon H \in \mathfrak{H}_i,$$
$$j = 1, \ldots, n(\lambda),\ \lambda \in \Lambda(\alpha_1, \ldots, \alpha_i) \text{ and } \alpha_1, \ldots, \alpha_i \in \Omega\}.$$

Then we can see that each $\mathfrak{R}_i$ is σ-discrete in $X \times Y$. So it is verified that $\bigcup \{\mathfrak{R}_i\colon i \in N\}$ is a σ-discrete refinement of $\mathfrak{G}$ by closed rectangles in $X \times Y$. The proof is complete.

Remark. Under the assumption of Proposition 2.2, D. J. Lutzer [5] and T. Mizokami [6] essentially showed that every open cover of $X \times Y$ has a σ-locally finite refinement by closed rectangles in $X \times Y$. It can be also shown that the product of a metacompact P-space and a metacompact Σ-space is metacompact. This is an improvement of [6, Theorem 3]. Indeed, from [6, Lemma 6], the proof is obtained by the modification of the above one.

PROPOSITION 2.3. *If a product space $X \times Y$ is a paracompact (normal) D-product, then every (finite) open cover of $X \times Y$ has a σ-discrete refinement by closed rectangles in $X \times Y$.*

The proof is left to the reader as an exercise.

§ 3. C-products. In this section, modifying the concept of D-product, we give a condition for a product space $X \times Y$ under which Player I has a winning strategy in $G(K_1 \times K_2, X \times Y)$ if he has winning strategies in $G(K_1, X)$ and $G(K_2, Y)$.

DEFINITION. A product space $X \times Y$ is said to be a *C-product* if for each closed set M of $X \times Y$ and each open set of $X \times Y$ with $M \subset O$ there is a countable collection $\mathfrak{F}$ by closed rectangles in $X \times Y$ such that $M \subset \bigcup \mathfrak{F} \subset O$.

As an analogy of Theorem 2.1, we obtain the following:

THEOREM 3.1. *Let X and Y be spaces such that $X \times Y$ is a C-product. If Player I has winning strategies in $G(K_1, X)$ and $G(K_2, Y)$, then he has a winning strategy in $G(K_1 \times K_2, X \times Y)$.*

The proof is quite parallel to that of Theorem 2.1. So the detail of it is left to the reader.

PROPOSITION 3.1. *A product space which is regular and has the Lindelöf property is a C-product.*

The proof is almost obvious.

Remark. From Theorem 3.1, Proposition 3.1 and [14, Corollary 14.14], we can obtain the result of R. Telgársky [15] concerning the point-open games of product spaces, which is mentioned in our Introduction.

EXAMPLE 3.1. Let S be the Sorgenfrey line. Its product $S^2 = S \times S$ is a D-product which is not a C-product.

Let M be a closed set of S^2 and O an open set of S^2 with $M \subset O$. Let $\mathfrak{W} = \{W_\lambda\colon \lambda \in \Lambda\}$ be a collection by open rectangles in S^2 such that $M \subset \bigcup \mathfrak{W} \subset O$ and $W_\lambda = [x_\lambda, x'_\lambda) \times [y_\lambda, y'_\lambda)$ for each $\lambda \in \Lambda$. Put $W^1_\lambda = W_\lambda \setminus \{x_\lambda\} \times S$ and $W^2_\lambda = W_\lambda \setminus S \times \{y_\lambda\}$. Each W^1_λ is open in $\boldsymbol{R} \times S$ and each W^2_λ is so in $S \times \boldsymbol{R}$, where $\boldsymbol{R}$ is the real-line with the usual topology. Put

$$G = \bigcup \{W^1_\lambda\colon \lambda \in \Lambda\} \quad \text{and} \quad H = \bigcup \{W^2_\lambda\colon \lambda \in \Lambda\}.$$

Moreover, put $D = M \setminus (G \cup H)$. Since $M \subset \bigcup \mathfrak{W}$ and $|W_\lambda \cap D| \leqslant 1$ for each $\lambda \in \Lambda$, D is a discrete closed set of S^2. Since $\boldsymbol{R} \times S$ is perfectly normal and G is open in $\boldsymbol{R} \times S$, G is an F_σ-set of $\boldsymbol{R} \times S$. So, put $G = \bigcup \{E_n\colon n \in N\}$, where each E_n is closed in $\boldsymbol{R} \times S$. Similarly, put $H = \bigcup \{F_n\colon n \in N\}$, where each F_n is closed in $S \times \boldsymbol{R}$. Since $\boldsymbol{R} \times S$ is a D-product, there is a σ-discrete collection $\mathfrak{E}_n$ by closed rectangles in $\boldsymbol{R} \times S$ such that $E_n \subset \bigcup \mathfrak{E}_n \subset G$ for each $n \in N$. Then each $\mathfrak{E}_n$ is also a σ-discrete (in S^2) collection by closed rectangles in S^2. Similarly, there is a σ-discrete (in S^2) collection $\mathfrak{F}_n$ by closed rectangles in S^2 such that $F_n \subset \bigcup \mathfrak{F}_n \subset H$ for each $n \in N$. Here we set

$$\mathfrak{A} = \{\{p\}\colon p \in D\} \cup (\bigcup \{\mathfrak{E}_n, \mathfrak{F}_n\colon n \in N\}).$$

Then $\mathfrak{A}$ is a σ-discrete collection by closed rectangles in S^2 such that $M \subset \bigcup \mathfrak{A} \subset O$. Hence S^2 is a D-product.

Next, let $M = \{(x, y) \in S^2\colon x+y \geqslant 0\}$. Then M is a clopen set of S^2. Let $\Delta = \{(x, y) \in S^2\colon x+y = 0\}$. Let $\mathfrak{F}$ be an arbitrary collection by closed rectangles in S^2 such that $M = \bigcup \mathfrak{F}$. For each $(x, y) \in \Delta$, we choose an $F_{x,y} \in \mathfrak{F}$ containing (x, y). Note that $F_{x,y} \subset [x, \infty) \times [y, \infty)$ for each $(x, y) \in \Delta$. So $F_{x,y}$ and $F_{x',y'}$ are distinct for each distinct $(x, y), (x', y') \in \Delta$. Hence we have $|\mathfrak{F}| \geqslant c$. Thus there is no countable collection by closed rectangles in S^2 whose union is M. This implies S^2 is not a C-product.

Remark. Besides, it is verified that S^2 is not a F-product (in the sense of J. Nagata [8]).

A product $X \times Y$ of Tychonoff spaces X and Y is said to be *rectangular* [10] if every finite cozero cover of $X \times Y$ has a σ-locally finite refinement by cozero rectangles in $X \times Y$.

EXAMPLE 3.2. There are Tychonoff spaces X and Y such that $X \times Y$ is a C-product which is not rectangular.

Let $X = N \cup \{p\}$ and $Y = \beta N \setminus \{p\}$, where $p \in \beta N \setminus N$. Since X is countable, $X \times Y$ is a C-product. It is pointed out in [4] that $X \times Y$ is not rectangular.

§ 4. Closure-preserving collections by $\mathfrak{m}$-compact sets. In this section, we mainly deal with spaces which have a σ-closure-preserving closed cover by $\mathfrak{m}$-compact sets.

LEMMA 4.1. *Let X be a space which has a closure-preserving closed cover $\mathfrak{F}$ by $\mathfrak{m}$-compact sets. Then to each closed set E of X one can assign a discrete collection $\mathfrak{D}(E)$ by $\mathfrak{m}$-compact closed subsets of E, satisfying the following conditions:*

(1) Each $D \in \mathfrak{D}(E)$ is contained in some $F \in \mathfrak{F}$.

(2) If $\langle E_1, E_2, \ldots\rangle$ is a decreasing sequence of closed sets of X such that $E_1 \cap (\bigcup \mathfrak{D}(X)) = \emptyset$ and $E_{n+1} \cap (\bigcup \mathfrak{D}(E_n)) = \emptyset$ for each $n \in N$, then

$$\bigcap \{E_n \colon n \in N\} = \emptyset .$$

This is essentially proved in [16, Lemma 5] and yields the following result which is stated in [19, Proposition 1.1].

PROPOSITION 4.1. *If a space X has a σ-closure-preserving closed cover by $\mathfrak{m}$-compact sets, then Player I has a winning strategy in $G(\boldsymbol{DC}_{\mathfrak{m}}, X)$.*

Let $\mathrm{Dim}_n = \{Z\colon Z$ is a space with $\dim Z \leqslant n\}$ and $\mathrm{Ind}_n = \{Z\colon Z$ is a normal space with $\mathrm{Ind}\, Z \leqslant n\}$.

LEMMA 4.2. *Let X be a normal space. If Player I has a winning strategy in $G(\mathrm{Dim}_n, X)$, then $\dim X \leqslant n$.*

LEMMA 4.3. *Let X be a totally normal space. If Player I has a winning strategy in $G(\mathrm{Ind}_n, X)$, then $\mathrm{Ind}\, X \leqslant n$.*

Lemma 4.2 is proved in [16] and Lemma 4.3 is in [18]. Now, using these lemmas, we obtain the following product theorem in dimension theory as an application of the result in § 2.

THEOREM 4.1. *Let X be a collectionwise normal space which has a σ-closure-preserving closed cover by $\mathfrak{m}$-compact sets and Y a subparacompact space with $\chi(Y) \leqslant \mathfrak{m}$. Assuming either X or Y is non-empty, we have the following:*

(1) If $X \times Y$ is normal, then $\dim(X \times Y) \leqslant \dim X + \dim Y$.

(2) If $X \times Y$ is totally normal, then $\mathrm{Ind}(X \times Y) \leqslant \mathrm{Ind}\, X + \mathrm{Ind}\, Y$.

Proof. Let $X \times Y$ be a normal space with $\dim X \leqslant m$ and $\dim Y \leqslant n$. Let $A \times B$ be a product space such that A is $\mathfrak{m}$-compact and $\chi(B) \leqslant \mathfrak{m}$. Since the projection of $A \times B$ onto B is a closed map, $A \times B$ is rectangular (cf. [10, Proposition 1]). It follows from the product theorem of B. A. Pasynkov [10] that $\dim(A \times B) \leqslant \dim A + \dim B$ holds. So, for each closed rectangle R in $X \times Y$ with $R \in \boldsymbol{PC}_{\mathfrak{m}}$, we have

$$\dim R \leqslant \dim R' + \dim R'' \leqslant m+n .$$

Hence for each closed sets P of $X \times Y$ with $P \in \boldsymbol{D}(\boldsymbol{PC}_{\mathfrak{m}})$ we have $\dim P \leqslant m+n$. This implies from Corollary 2.3 and Proposition 4.1 that Player I has a winning strategy in $G(\mathrm{Dim}_{m+n}, X \times Y)$. Since $X \times Y$ is normal, it follows from Lemma 4.2 that $\dim(X \times Y) \leqslant m+n$ holds.

Let $X \times Y$ be a totally normal space with $\mathrm{Ind}\, X \leqslant m$ and $\mathrm{Ind}\, Y \leqslant n$. Let R be a closed rectangle in $X \times Y$ with $R \in \boldsymbol{PC}_{\mathfrak{m}}$. Since R is totally normal and rectangular (equivalently, F-product), it follows from the product theorem of J. Nagata [8] that $\mathrm{Ind}\, R \leqslant m+n$ holds. So Player I has a winning strategy in $G(\mathrm{Ind}_{m+n}, X \times Y)$ similar to the above case. Since $X \times Y$ is totally normal, it follows from Lemma 4.3 that $\mathrm{Ind}(X \times Y) \leqslant m+n$ holds. The proof is complete.

Comparing Theorem 4.1 with [19, Theorem 2.1], we need to pay attention to the examples below.

EXAMPLE 4.1. There are a paracompact space X with a closure-preserving cover by compact sets and a subparacompact space Y such that $X \times Y$ is normal and non-paracompact.

Let Y be a subparacompact and normal space which is not paracompact. As such a space Y, we may consider the space described in [1, Example H]. Let $\mathfrak{n}$ be a regular cardinal number with $\mathfrak{n} > |Y|$. Let ω_α be the initial ordinal of $\mathfrak{n}$. Note $\omega_\alpha > \omega_1$. Let $X = [0, \omega_\alpha]$. The topology for X is defined as follows; the neighborhood base at ω_α is $\{(x, \omega_\alpha]\colon x < \omega_\alpha\}$ and each x, $x < \omega_\alpha$, is an isolated point. The space X is a paracompact space with a closure-preserving cover by two-point sets. We show that $X \times Y$ is normal. Let H be a closed set of $X \times Y$ with $H \cap \{\omega_\alpha\} \times Y = \emptyset$. For each $y \in Y$, Let $H_y = \{x \in X\colon (x, y) \in H \cap X \times \{y\}\}$. We can choose some $x(y) < \omega_\alpha$ such that $H_y \cap (x(y), \omega_\alpha] = \emptyset$ for each $y \in Y$. Let $x_0 = \sup\{x(y)\colon y \in Y\}$. Since $|Y| < \mathfrak{n}$ and $\mathfrak{n}$ is regular, we have $x_0 < \omega_\alpha$. Let $U = [0, x_0] \times Y$ and $V = (x_0, \omega_\alpha] \times Y$. Then U and V are disjoint open sets of $X \times Y$ such that $H \subset U$ and $\{\omega_\alpha\} \times Y \subset V$. Hence it follows from [12, Lemma 2.9a] that $X \times Y$ is normal.

EXAMPLE 4.2. There are a paracompact space X with a closure-preserving cover by compact sets and a subparacompact Tychonoff space Y such that $X \times Y$ is not rectangular.

Let X be the space $[0, \omega_\alpha]$ which is described in Example 4.1. Then X is the desired space with $|X| > \aleph_1$. Here, note that X is not locally compact. For the space X, it follows from [9, Theorem 5.1] that there is a subparacompact Tychonoff space Y such that $w(Y) \leqslant \mathfrak{n}$ and $\upsilon(X \times Y) \neq \upsilon X \times \upsilon Y (= X \times \upsilon Y)$. So we have $\mu(X \times Y) \neq \mu X \times \mu Y$. Hence it follows from [4, Theorem 4] that $X \times Y$ is not rectangular.

In connection with Theorem 4.1, we remain the following two problems:

PROBLEM 4.1. Assume that X is a paracompact space which has a closure-preserving cover by compact sets and Y is a subparacompact normal space. Is $\dim(X \times Y) \leqslant \dim X + \dim Y$?

PROBLEM 4.2. Are there a paracompact space X which has a closure-preserving cover by compact sets and a subparacompact space Y such that $X \times Y$ is normal and but not rectangular?

Remark. An affirmative answer of Problem 4.2 would give a class of product spaces which essentially contains at least one non-rectangular product, however, in which the product theorem for covering dimension holds.

The following result is a generalization of [11, Theorem 1]. Besides, the proof seems to be simpler.

THEOREM 4.2. *Let $\mathfrak{F}$ be a closure-preserving collection by $\mathfrak{m}$-compact closed sets of a space X, and let $\mathfrak{U}$ be an open cover of X with $|\mathfrak{U}|\leqslant\mathfrak{m}$. Then $\mathfrak{U}$ has an open refinement $\mathfrak{V}$ such that $\{V\in\mathfrak{V}\colon V\cap F\neq\emptyset\}$ is finite for each $F\in\mathfrak{F}$.*

Proof. We may assume without loss of generality that $\mathfrak{F}$ is a closure-preserving cover by $\mathfrak{m}$-compact closed sets of X. For each closed set E of X, let $\mathfrak{D}(E)$ be the discrete collection described in Lemma 4.1. We construct a sequence $\{\mathfrak{G}_n\colon n\geqslant 0\}$ of collections of open sets in X, satisfying the following conditions:

(1) $\mathfrak{G}_n$ covers $\bigcup\mathfrak{D}(E_n)$, where $E_n = X\setminus\bigcup\{\bigcup\mathfrak{G}_i\colon i=1,\ldots,n-1\}$.

(2) Each $G\in\mathfrak{G}_n$ is contained in some $U\in\mathfrak{U}$.

(3) $\{G\in\mathfrak{G}_n\colon G\cap F\neq\emptyset\}$ is finite for each $F\in\mathfrak{F}$.

Let $\mathfrak{G}_0 = \{\emptyset\}$. Assume that the collections $\{\mathfrak{G}_i\colon i\leqslant n-1\}$ have been constructed. Let $E_n = X\setminus\bigcup\{\bigcup\mathfrak{G}_i\colon i\leqslant n-1\}$. Since each $D\in\mathfrak{D}(E_n)$ is $\mathfrak{m}$-compact and $|\mathfrak{U}|\leqslant\mathfrak{m}$, there is a finite subcollection $\mathfrak{U}(D)$ of $\mathfrak{U}$ covering D. For each $U\in\mathfrak{U}(D)$, we put

$$G(D,U) = U\setminus\bigcup\{F\in\mathfrak{F}\colon F\cap D=\emptyset\},$$

which is an open set of X. Here we set

$$\mathfrak{G}_n = \{G(D,U)\colon U\in\mathfrak{U}(D) \text{ and } D\in\mathfrak{D}(E_n)\}.$$

It is clear that $\mathfrak{G}_n$ satisfies (1) and (2). We pick an $F\in\mathfrak{F}$. Let

$$\mathfrak{D}(E_n)_F = \{D\in\mathfrak{D}(E_n)\colon D\cap F\neq\emptyset\}.$$

Since $\mathfrak{D}(E_n)$ is discrete and F is countably compact, $\mathfrak{D}(E_n)_F$ is finite. Pick a $G\in\mathfrak{G}_n$ with $G\cap F\neq\emptyset$. Then we can easily verify $G = G(D,U)$ for some $D\in\mathfrak{D}(E_n)_F$ and $U\in\mathfrak{U}(D)$. Hence $\{G\in\mathfrak{G}_n\colon G\cap F\neq\emptyset\}$ is finite, so that $\mathfrak{G}_n$ satisfies (3). Thus $\mathfrak{G}_n$ satisfies the conditions (1)–(3). It follows from Lemma 4.1 and (1) that $\bigcup\{\mathfrak{G}_n\colon n\in N\}$ is a cover of X. For each $n\in N$, let $L_n = \bigcup\{F\in\mathfrak{F}\colon F\subset\bigcup\{\bigcup\mathfrak{G}_i\colon i\leqslant n-1\}\}$. For each $G\in\mathfrak{G}_n$, let $V_G = G\setminus L_n$. Here we set $\mathfrak{V}_n = \{V_G\colon G\in\mathfrak{G}_n\}$ and $\mathfrak{V} = \bigcup\{\mathfrak{V}_n\colon n\in N\}$. One can show similarly to the proof of [11, Theorem 1] that $\mathfrak{V}$ is the desired open refinement of $\mathfrak{U}$. The proof is complete.

§ 5. Total paracompactness. A space X is said to be *totally paracompact* if each open basis of X contains a locally finite cover of X. The following result is a generalization of [17, Theorem 1] and gives an affirmative answer to [13, Problem 5.11].

THEOREM 5.1. *Let X be a paracompact space. If Player I has a winning strategy in $G(\boldsymbol{DC}, X)$, then X is totally paracompact.*

Proof. Let $\mathfrak{B}$ be an open basis of X. Let s be a winning strategy of Player I in $G(\boldsymbol{DC}, X)$. Since $\boldsymbol{DC}$ is hereditary with respect to closed sets, we may assume without loss of generality that each $s(U_1,\ldots,U_n)$ is disjoint from $U_1\cup\ldots\cup U_n$. Let $E_1 = s(\emptyset)$. Since the closed set E_1 of X belongs to $\boldsymbol{DC}$, we can choose a subcollection $\mathfrak{B}_1$ of $\mathfrak{B}$ such that $\mathfrak{B}_1$ is locally finite in X and $E_1\subset\bigcup\mathfrak{B}_1$. Take a cozero-set U_1 of X with $E_1\subset U_1\subset\bigcup\mathfrak{B}_1$. Let $\{U_1^n\colon n\in N\}$ be a sequence of open sets of X such that $U_1 = \bigcup\{U_1^n\colon n\in N\}$ and $\operatorname{Cl}U_1^n\subset U_1^{n+1}$ for each $n\in N$. Let $E_2 = s(U_1)$

$\in\boldsymbol{DC}$. We can also choose a subcollection $\mathfrak{B}_2$ of $\mathfrak{B}$ such that $\mathfrak{B}_2$ is locally finite in X and $E_2\subset\bigcup\mathfrak{B}_2\subset X\setminus\operatorname{Cl}U_1^1$. Take a cozero-set U_2 of X with $E_2\subset U_2\subset\bigcup\mathfrak{B}_2$. Let $\{U_2^n\colon n\in N\}$ be a sequence of open sets of X such that $U_2 = \bigcup\{U_2^n\colon n\in N\}$ and $\operatorname{Cl}U_2^n\subset U_2^{n+1}$ for each $n\in N$. Let $E_3 = s(U_1, U_2)$. Moreover, we can choose a subcollection $\mathfrak{B}_3$ of $\mathfrak{B}$ such that $\mathfrak{B}_3$ is locally finite in X and $E_3\subset\bigcup\mathfrak{B}_3\subset X\setminus\operatorname{Cl}(U_1^2\cup U_2^2)$. Continuing in that manner, we get the three sequences $\{\mathfrak{B}_n\colon n\in N\}$, $\{U_n\colon n\in N\}$ and $\{U_n^m\colon n,m\in N\}$, satisfying the following conditions (1)–(4):

(1) $\mathfrak{B}_n$ is a locally finite subcollection of $\mathfrak{B}$.

(2) U_n is a cozero-set of X such that $s(\emptyset)\subset U_1\subset\bigcup\mathfrak{B}_1$ and $s(U_1,\ldots,U_{n-1})\subset U_n\subset\bigcup\mathfrak{B}_n$.

(3) Each U_n^m is an open set of X such that $U_n = \bigcup\{U_n^m\colon m\in N\}$ and $\operatorname{Cl}U_n^m\subset U_n^{m+1}$ for each $m\in N$.

(4) $\bigcup\mathfrak{B}_n\subset X\setminus\operatorname{Cl}(U_1^{n-1}\cup\ldots\cup U_{n-1}^{n-1})$.

Here we set $\mathfrak{A} = \bigcup\{\mathfrak{B}_n\colon n\in N\}$. Then $\mathfrak{A}$ is a subcollection of $\mathfrak{B}$. It follows from (2) that $\{U_n\colon n\in N\}$ covers X, so that $\mathfrak{A}$ is a cover of X. Let $x\in X$. We can choose some $i,j\in N$ such that $x\in U_i^j$. Let $n_0 = \max\{i,j\}$. By (3) and (4), we have for each $n>n_0$

$$\big(\bigcup\mathfrak{B}_n\big)\cap U_i^j\subset\big(\bigcup\mathfrak{B}_n\big)\cap U_i^{n-1}\subset\big(\bigcup\mathfrak{B}_n\big)\cap\big(\bigcup\{U_k^{n-1}\colon 1\leqslant k\leqslant n-1\}\big) = \emptyset.$$

Since, by (1), $\bigcup\{\mathfrak{B}_n\colon n\leqslant n_0\}$ is locally finite in X, $\mathfrak{A}$ is locally finite at x. Hence $\mathfrak{A}$ is a locally finite subcover of $\mathfrak{B}$. The proof is complete.

Finally, let us remark that we cannot expand the arguments in § 2 and § 3 to infinite products. It is clear that Player I has a winning strategy in $G(N)$. However, he has no winning strategy in $G(N^\omega)$, where N^ω denotes the product of countable many copies of N. Indeed, this fact is derived from

EXAMPLE 5.1. Player I has no winning strategy in $G(\boldsymbol{DC}, N^\omega)$.

Since N^ω is the irrational space, it is a metric space. But it is known that N^ω is not totally paracompact (cf. [3, Theorem 1]). So Example 5.1 follows from Theorem 5.1.

Added in proof.

1. Recently, the author has proved that we can replace, in Theorem 2.2, the collectionwise normality of X with the condition X is subparacompact and regular. In fact, the proof is obtained by the modification of that of Theorem 2.2. So, we can also replace the collectionwise normality of X in Corollaries 2.1, 2.3 and Theorem 4.1 and the paracompactness of X in Corollary 2.2 with the condition X is subparacompact and regular, respectively. Here, note that $\mathfrak{m}$ is meaningless.

2. After this version, H. Ohta has kindly informed the author that he gave an affirmative answer to Problem 4.2.

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Accepté par la Rédaction le 23. 9. 1980

Periodic points of symmetric product mappings *

by

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Abstract. We study questions concerning periodic points of symmetric product mappings. By means of the Maxwell trace homomorphism we define the notion of the mth Lefschetz number of a symmetric product mapping of a compact polyhedron. We show that the nonvanishing of this mth Lefschetz number is a sufficient condition for the existence of a periodic point of period $\leqslant m$. Next we relate that mth Lefschetz numbers of a symmetric product mapping to a certain characteristic function and obtain further results concerning periodic points.

1. Introduction. Let X be a topological space and X^n the cartesian product. Let G be a group of permutations of the numbers $1, 2, \ldots, n$. The orbit space of X^n under the action of G (with the identification topology) is called the *n-th symmetric product of X* and is denoted by X^n/G. A continuous map of the form $f\colon X \to X^n/G$ is called a *symmetric product mapping*.

For symmetric product mappings of compact polyhedra, C. N. Maxwell defined the notion of a Lefschetz number [4]. He showed that a nonzero Lefschetz number implies the existence of a fixed point. This extension of the Lefschetz fixed point theorem also holds for symmetric product mappings of metric ANR's provided the mapping is compact [3].

In the present paper we define the notion of a periodic point of period $\leqslant m$ for $f\colon X \to X^n/G$. In the case that X is a compact polyhedron, we define the mth Lefschetz number of f by appealing to the simplicial machinery developed in Maxwell [4]. We show that the nonvanishing of this number is a sufficient condition for the existence of periodic points of period $\leqslant m$. Further results concerning periodic points are obtained upon relating the mth Lefschetz numbers to a certain characteristic function. For instance, we show that if X is a compact polyhedron such that $H_i(X) = 0$ for i odd, where H is the homology functor with coefficients in the rational field, then any map $f\colon X \to X^n/G$ has a periodic point of period $\leqslant$ the Euler characteristic of X.

* This paper is based in part on the author's dissertation at Indiana University under the direction of Professor Jan Jaworowski.