On the netweight of subspaces

by

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Abstract. In this paper we give a (consistent) solution to a problem of A. Hajnal and I. Juhász [3], namely we show a model of set theory with $2^\omega > \omega_1$ in which there exists a regular topological space $X$ with an uncountable netweight and such that every subspace of $X$ of power smaller than that of $X$ has a countable netweight.

Introduction. In [3] A. Hajnal and I. Juhász, in connection with a problem of M. G. Tkachenko, showed that it is consistent with set theory to assume that there exists a Hausdorff space $X$ of power $\omega_2$ with the following properties:

1. $nw(X) = \omega_2$,
2. $nw(Y) = \omega$ for every subspace $Y \subseteq X$ of power $\omega_1$.

They suggested the natural problem whether an analogous result for regular spaces could be proved. This paper gives a solution to this problem.

We recall that $nw(X)$ is the netweight of $X$, i.e. the smallest cardinal of a network for $X$.

Throughout the paper we use the standard set-theoretical notation. We use the forcing technique as described e.g. in [1].

The graph topology. Let $[X]^{\leq 2} = \{ \{x, y\} : |x| = 1 \lor |y| = 2\}$.

We say that the function $f : [X]^{\leq 2} \to 2$ is a graph iff $f(\{x\}) = 0$ for every $x \in X$ (the elements $x, y \in X$ are considered to be connected by an edge in the graph iff $f(\{x, y\}) = 0$).

For every $x \in X$ and $i < 2$ we put

$$U_i = \{ y \in X : f(\{x, y\}) = i \},$$

in particular $x \in U_0^x$ for every $x \in X$.

We are going to study the topology $\tau_f$ on $X$ generated by the subbasis

$$\{ U_i : x \in X & i < 2 \}.$$

Clearly the space $X$ with the topology $\tau_f$ is 0-dimensional.

Let $H(X)$ be the set of all functions from finite subsets of $X$ into 2. For $f \in H(X)$ we shall put

$$U_f = \bigcap_{x \in \text{dom}(f)} U_f^x.$$
Hence the family \( \{ U_e : e \in H(X) \} \) is a basis for \( \tau \).

Let \( F \) be a family of sets. We say that the graph \( f : \{X\}^{k+2} \rightarrow \mathbb{R} \) is \( \omega \)-full over \( F \) if for every infinite \( C \in F \cap P(X) \) and every \( e \in H(X) \) there exists \( a \in C \) such that \( f((a, e)) = a(e) \) for each \( e \in \text{dom}(a) \).

Let us note that if \( f : \{X\}^{k+2} \rightarrow \mathbb{R} \) is \( \omega \)-full over \( F \) then for every infinite subfamily \( Y \subseteq X \) such that \( Y \in F \) we have the equivalence:

\[
U_e \cap Y = U_{e'} \cap Y \quad \text{iff} \quad e \equiv e' \quad \text{for every} \quad e, e' \in H(X).
\]

If \( Y \) is a subspace of \( X \) then we put

\[
Q(Y) = \{ (A, e) : A \subseteq Y \text{ and } A \text{ is finite} \}\quad \text{with the ordering relation}
\]

\[\langle A, e \rangle \preceq \langle B, e \rangle \quad \text{iff} \quad A \supseteq B \text{ and } e \equiv e.\]

We say that a subset \( Q \) of a partially ordered set is compatible if every two elements of \( Q \) are compatible.

Now we can formulate the following

**Lemma 1.** If \( f : \{X\}^{k+2} \rightarrow \mathbb{R} \) is \( \omega \)-full over \( F \), then \( Y \) is an infinite subspace of \( X \) and \( Q(Y) \) is a union of a countable family of compatible sets then \( \text{nw}(Y) \leq \omega \).

The proof of the lemma is contained implicitly in [3].

The idea of the proof is to construct a model with a regular topological space having the required properties we first add generically a graph \( f : \{X\}^{k+2} \rightarrow \mathbb{R} \) for a regular cardinal \( \mu \) using finite conditions and then add for each \( \alpha < \mu \) a generic decomposition of \( Q(\alpha) \) into a countable family of compatible sets.

The \( \omega \)-fullness of the graph \( f \) over the family \( \kappa = \{ \alpha : \alpha < \kappa \} \) follows by the genericity of \( f \). It easily gives the regularity of \( (\kappa, \tau) \). By Lemma 1 we also get a countable network for each subspace \( \kappa < \kappa \), where \( \kappa < \kappa \).

Since our forcing is coc, it remains to show that \( \text{nw}(\kappa) = \omega \).

It is also very important to mention that in order to define the poset \( Q(\alpha) \) for \( \alpha < \kappa \) it suffices to know the values of the graph only for pairs \( \{ \alpha, \beta \} \) such that \( \min(\{ \alpha, \beta \}) < \kappa \).

Now we turn to details.

**Construction of the model.** Let \( \kappa > \omega_1 \) be a regular cardinal. We define several posets:

(i) \( S = \{ x \in H(\{X\}^2) : \forall x \in \text{dom}(\alpha) (|\{x\}| = 0) \} \)

ordered by reverse inclusion,

(ii) \( R_\alpha = \{ (A, e) : A \subseteq \alpha \text{ and } A \text{ is finite} \} \quad \text{with the ordering relation} \)

\[\langle A, e \rangle \preceq \langle B, e \rangle \quad \text{iff} \quad A \supseteq B \text{ and } e \equiv e.\]

(iii) \( Q = \{ q : \text{Fnc}(q) \cap \text{dom}(q) = \emptyset \} \)

\[= \{ a \times \omega \cup \{ (a, n) : q(a, n) \in R \} \} \quad \text{with the ordering relation} \]

\[q_1 \leq q_2 \quad \text{iff} \quad q_1(a, n) \leq q_2(a, n) \quad \text{for any} \quad a \in \kappa \text{ and } n < \omega,\]

(iv) \( P = \{ (s, q) \in S \times Q : \forall s < s \forall n < n \forall a \forall b \forall q \langle q(a, n), n \rangle = \langle A, e \rangle \text{ and } (a, e) \in \text{dom}(s) \text{ and } (b, e) \in \text{dom}(s) \} \)

with the ordering relation

\[\langle s_1, q_1 \rangle \preceq \langle s_2, q_2 \rangle \quad \text{iff} \quad s_1 \leq s_2 \text{ and } q_1 \leq q_2.\]

Let us remark that for the forcing \( P \) can be considered as a product forcing, i.e.

\[P = \{ (s, q) \in S \times Q : \forall s < s \forall n < n \forall a \forall b \forall q \langle q(a, n), n \rangle = Q(s, a) \} \]

**Theorem 2.** The forcing \( P \) is c.c.c.

The proof will be postponed until the last section of this paper.

Let \( \alpha < \kappa \). We fix some notation:

\[D_\alpha = \{\alpha, \beta \}^\omega, \quad D_\beta = \{\alpha, \beta \}^{\omega_1}, \quad D_\beta = \{\alpha, \beta \}^{\omega_1} = \{\alpha, \beta \}^{\omega_2} \text{ min}(\{\alpha, \beta \}) < \omega, \]

\[S_\alpha = \{ s : s \in S \mid s \text{ is compatible} \}\quad \text{and} \quad \alpha_s = \{ s : s \in S \mid s \text{ is compatible} \}.\]

The orderings of \( S_\alpha \) and \( S_\beta \) are the reverse inclusion.

Next, let

\[Q_\alpha = \{ q : \alpha \times \omega \cup \{ q(a) : a \in \omega \} \}
\]

both be ordered by

\[q_1 \leq q_2 \quad \text{iff} \quad q_1(\beta, n) \leq q_2(\beta, n) \quad \text{for every} \quad (\beta, n) \in \text{dom}(q_1).\]

It is clear that \( S \subseteq S_\alpha \times S_\beta \) and \( Q \approx Q_\alpha \times Q_\beta \).

Finally, let

\[R_\alpha = \{ (s_1, q_1, s_2, q_2) : s_1 \times Q_2 \times S_\beta \times S_\beta \}
\]

\[= \{ \langle A, e \rangle : (A, e) \in \text{dom}(s_1) \} \quad \text{with the ordering relation} \]

\[\langle s_1, q_1 \rangle \preceq \langle s_2, q_2 \rangle \quad \text{iff} \quad s_1 \leq s_2 
\]

It is easy to see that a mapping \( g_\alpha : P \to R_\alpha \) defined by

\[g_\alpha(s, q) = \{ q(a, n) \times a \times \omega : a \times \omega \}
\]

is an order isomorphism of \( P \) and \( R_\alpha \).
From now on we shall identify \( P \) with \( R_x \).

Let \( M \) be a countable transitive model of set theory and let \( \kappa > \omega \), be a regular cardinal in \( M \). We consider a forcing \( P \) in \( M \) defined for \( x \) and let \( G \) be an \( M \)-generic filter over \( P \).

We define

\[
G_x = \{ \langle s, q \rangle \in S_x \times Q_x : \langle s, q, \langle 0, 0 \rangle \rangle \in G \}
\]

where \( 1 \) is the maximal element of \( Q_x \),

\[
G^* = \{ \langle s, q \rangle \in S^* \times Q^* : \exists \langle s_1, q_1 \rangle \in G \}
\]

Next,

\[
P_x = \{ \langle s, q \rangle \in S_x \times Q_x : \langle s, q, \langle 0, 0 \rangle \rangle \in P \} \in M,
\]

\[
P^* = \{ \langle s, q \rangle \in S^* \times Q^* : \exists \langle s_1, q_1 \rangle \in G \}
\]

are the posets ordered by

\[
\langle s_1, q_1 \rangle \succeq \langle s_2, q_2 \rangle \iff s_1 \subseteq s_2 \text{ and } q_1 \subseteq q_2.
\]

A standard argument shows

**Proposition 3.** \( G_x \) is \( M \)-generic over \( P_x \). \( G^* \) is \( M[G_x] \)-generic over \( P^* \) and \( M[G^*] = M[G_x][G^*] \).

Let \( f = \cup \{ \langle s, q \rangle \in G \} \) and \( X = \langle x, \gamma \rangle \).

**Theorem 4.** \( M[G^*] \) is a ccc extension of \( M \) such that

(1) \( X \) is regular (even hereditarily normal),

(2) \( mw(Y) < \omega \) for every subspace \( Y \) of \( X \) of power smaller than \( \kappa \),

(3) \( mw(X) = \kappa \).

**Proof.** We begin with the following

**Proposition 5.** For every \( x < \omega \) the graph \( f \upharpoonright D^x \) is \( \omega \)-full over \( M[G_x] \). In particular \( f \) is \( \omega \)-full over \( x \).

For the proof it is enough to show that for every infinite \( K \in M[G_x] \cap P(\omega \times \omega) \) and every \( s \in H(\omega \times \omega) \) and every \( e \in H(\omega \times \omega) \) the set

\[
D = \{ \langle s, q \rangle \in P_x : \exists \eta \in K \forall \xi \in dom(\eta) \exists \langle \eta, \xi \rangle = \langle \xi, \eta \rangle \}
\]

is dense in \( P^* \).

(1) By Proposition 5 it follows immediately that for every \( \xi < \eta < \omega \) there exists \( \eta < \omega \) such that \( \xi < \omega \) \( \eta < \omega \) and \( \xi < \omega \) \( \eta < \omega \), i.e. \( X \) is a Hausdorff space. Since \( X \) is 0-dimensional, it is also regular.

(2) From an obvious inequality \( mw(Y) < mw(X) \) for a subspace \( Y \) of \( X \) and from Proposition 5 and Lemma 1 it follows that it is enough to show that for any infinite \( \omega < \omega \) the set \( Q(\omega) \) is a union of a countable family of compatible sets.

We take \( \omega < \omega \) and let

\[
Q_x = \{ q(\omega, n) : \langle s, q \rangle \in G \} \quad \text{for} \quad n < \omega.
\]

We show that \( Q(\omega) = \bigcup \{ Q_x' : n < \omega \} \). If \( \langle s, q \rangle \in Q_x' \) for some \( n < \omega \) then there exists \( \langle s, q \rangle \in G \) such that \( q(\omega, n) = \langle s, q \rangle \). Hence, by the definition of \( f, A \in U_x \), i.e. \( \langle s, q \rangle \in Q(\omega) \) then

\[
\forall q \in A \forall b \exists \eta \in dom(\eta) \forall \langle \xi, \eta \rangle = \langle \eta, \xi \rangle.
\]

Hence, by the finiteness of \( A \times \text{dom}(\omega) \), there exists \( \langle q_x, q_x \rangle \in G \) such that

\[
\forall q \in A \forall b \exists \eta \in dom(\eta) \forall \langle \xi, \eta \rangle = \langle \eta, \xi \rangle.
\]

It is enough to show that the set

\[
\{ \langle s, q \rangle : 1 \leq n < \omega, \langle s, q \rangle \in G \}
\]

is dense below \( \langle q_x, q_x \rangle \).

Let \( \langle s, q \rangle \in P \) and \( \langle s, q \rangle \in \langle q_x, q_x \rangle \). There exists an \( n < \omega \) such that \( \langle s, q \rangle \in \text{supp}(q) \). Let \( q' \in G \) be defined by

\[
q'(\beta, m) = \begin{cases} q(\beta, m) \text{ for } \beta \neq \langle s, q \rangle, \\ q(\beta, m) \text{ for } \beta = \langle s, q \rangle. 
\end{cases}
\]

It is easy to see that \( \langle s, q' \rangle \in P \) and \( \langle s, q' \rangle \in \langle q_x, q_x \rangle \). In order to complete the proof of (2) it is enough to verify that each \( G_x \) is compatible.

Let \( \langle A_x, e_x \rangle, \langle A_y, e_y \rangle \in Q_x \) and \( \langle A_z, e_z \rangle \in Q_x \). Then there exist \( \langle q_x, q_x \rangle, \omega, q_x, q_x \) \( \in G \) such that \( q_x \in \omega \) and \( q_x \notin \text{supp}(q) \) and \( \omega \in \text{supp}(q) \). Then there exists \( \langle A_z, e_z \rangle \) \( \langle A_z, e_z \rangle \in Q_x \) which completes the proof of (2).

Let us note that the space fulfilling condition (3) (where \( x \) is a power of \( X \)) is hereditarily Lindelöf. Hence (see [2]) \( X \) is hereditarily normal.

(3) To the contrary, let us assume that \( \text{supp}(X) < \omega \). Then there exists a network

\[
F_x : \langle \gamma \rangle \quad \text{where} \quad \gamma < \omega.
\]

By the regularity of \( X \) we can assume that all \( F_x \) are closed for \( \gamma < \omega \). Hence, by hereditary Lindelöfness, we can assume that

\[
F_x = \omega \times U_f \quad \text{for any} \quad \gamma < \omega.
\]

Let \( E : \gamma \times \omega \to H(\omega) \) be a mapping defined by

\[
E(\gamma, n) = q_f \quad \text{for any} \quad \gamma < \omega \text{ and } n < \omega.
\]

A standard argument shows that there exists an \( x < \omega \) such that

(i) \( \langle s, q \rangle < \omega \).

(ii) \( \langle s, q \rangle < \omega \text{ and } \langle s, q \rangle < \omega \).

(iii) \( \langle s, q \rangle < \omega \text{ and } \langle s, q \rangle < \omega \).

We can also assume that

\[
\bigcup \{ \text{dom}(e) : \langle \gamma \rangle < \gamma \} < \omega.
\]

\[
\bigcup \{ \text{dom}(e) : \langle \gamma \rangle < \gamma \} < \omega.
\]

\[
\bigcup \{ \text{dom}(e) : \langle \gamma \rangle < \gamma \} < \omega.
\]
Since \( f \downarrow D_\alpha \in M[G_\beta] \) and the fact that for the definition of \( U_\alpha \), where \( \alpha \in H(\alpha) \), the knowledge of \( f \downarrow D_\alpha \) is sufficient, we have
\[
F_\beta \in M[G_\beta] \quad \text{for each } \beta \in \gamma.
\]

Let \( \beta \geq \alpha \). We show that
\[
\forall \gamma \in \gamma [\beta \in \gamma \iff F_\beta \in \gamma',]
\]
which contradicts the assumption that \( \{ F_\beta : \beta \in \gamma \} \) is a network.

Let \( \zeta \prec \gamma \). If \( F_\beta \nless \gamma \) then, by (ii), \( \beta \notin F_\gamma \). If \( F_\beta \nless \gamma \) then \( F_\beta \nless \gamma \in M[G_\beta] \) is an infinite subset of \( \gamma \times \alpha \). Hence, by Proposition 5, there exists an \( \eta \notin F_\beta \) such that \( f(\eta, \beta) = 1 \). So \( \eta \notin \gamma' \), i.e., \( F_\beta \nless \gamma' \).

This completes the proof of Theorem 4.

Proof of Theorem 2. Let \( \gamma = \{ \langle \alpha_0, m_0 \rangle, \ldots, \langle \alpha_n, m_n \rangle \} \) be a subset of \( \times \times \alpha \). We define the posets:
\[
\Omega_r = \{ q : \text{Foc}(q) \cup \text{dom}(q) = n \cup \forall \gamma \prec \gamma \left[ q(i) \in B_\alpha \right] \}
\]
with ordering relation
\[
q_1 \leq q_2 \iff q_1(i) \leq q_2(i) \quad \text{for every } i < n,
\]
\[
P_r = \{ \langle s, q \rangle : s \in \Omega_r \}, \quad \forall \gamma \pi \gamma \forall \forall \beta \gamma \langle q(i) = \langle \langle \alpha, b \rangle \subseteq \text{dom}(s) \Rightarrow \langle \langle \alpha, b \rangle, s(b) \in s' \gamma (a = b \in \text{dom}(s)) = 0 \rangle \}
\]
with the ordering relation
\[
\langle s_1, q_1 \rangle \leq \langle s_2, q_2 \rangle \iff s_1 \subseteq s_2 \text{ and } q_1 \leq q_2.
\]

We shall repeatedly use the following simple combinatorial

**Proposition 6.** If \( B \) is finite, \( C \) is countable and \( h_i : B \to C \) for \( \zeta < \omega_1 \), then \( C \) has no uncountable subset \( K \) of \( \omega_1 \), such that \( h_i = h_j \) for every \( \zeta, \epsilon \in K \).

**Lemma 7.** \( P_r \) is \( \mathfrak{c} \mathcal{c} \).

**Proof.** Let \( \langle \langle s, q \rangle, \langle \alpha, b \rangle \rangle \in \gamma \) be a sequence of elements of \( P_r \) and let
\[
a(i) = \langle \alpha, b \rangle \quad \text{for each } i \in \gamma \text{ and } \zeta < \omega_1.
\]

We shall show that there exist \( \zeta < \omega_1 \) such that \( \langle s_1, q_1 \rangle \) and \( \langle s_2, q_2 \rangle \) are compatible.

Without limiting generality we may assume that for every \( \zeta < \omega_1 \)
\[
\begin{align*}
\text{dom}(a) = [d_\zeta]^{<\zeta} & \quad \text{for a certain finite } d_\zeta \subseteq \gamma, \quad (1) \\
\bigcup \text{dom}(a) = \bigcup \mathcal{A}_\zeta = d_\zeta & \quad (2) \\
\text{by the } A\text{-lemma we may assume that} & \quad (3) \\
d_\zeta = a \cup b & \quad \text{for any } \zeta < \omega_1,
\end{align*}
\]
where
\[
a_\zeta \cap a_\zeta = 0 \quad \text{for any } \zeta < \eta < \omega_1.
\]

By applying Proposition 6 to the functions \( x_i \downarrow \gamma \) \( [x_i]^{<\zeta} \), we can assume that \( x_i \downarrow \gamma = x_i \downarrow \gamma \) \( [x_i]^{<\zeta} \) for every \( \zeta < \omega_1 \). So
\[
a_i \cup a_i \subseteq S \quad \text{for every } \zeta < \omega_1.
\]

By applying Proposition 6 to the functions \( h_i : B \to \mathcal{C} \) defined by \( h_i(z) = \langle \alpha, b \rangle \) for \( i \in \gamma \), we can assume that for any \( i \in \gamma \) there exists a \( r_i \) such that \( A_i = r_i \) for any \( \zeta < \omega_1 \). Hence we may assume that
\[
\exists \zeta < \omega_1 \quad A_i = \langle \langle \alpha, b \rangle, \langle \epsilon, a \rangle \rangle \quad \text{for any } \zeta < \omega_1 \text{ and } i < n,
\]
and by the same argument
\[
\exists \zeta < \omega_1 \quad A_i = \langle \langle \alpha, b \rangle, \langle \epsilon, a \rangle \rangle \quad \text{for any } \zeta < \omega_1 \text{ and } i < n.
\]

By applying the same argument to the functions
\[
h_i : \gamma \times \gamma \to 2
\]
defined by
\[
h_i(i, j) = e_i(b_j)
\]
for each \( i < \gamma \) and \( j < r_i \), we may assume that
\[
e_i(b_j) = e_j(b_i)
\]
for every \( \zeta, j < \omega_1 \).

The same argument applied to the functions
\[
h_i : \gamma \to P(\gamma \times \gamma)
\]
defined by
\[
h_i(z) = \langle \langle \alpha, b \rangle, \langle \epsilon, a \rangle \rangle \quad \text{for any } \alpha \in b
\]
allows us to assume that
\[
\begin{align*}
\text{for every } \alpha & \in b \quad (10) \\
\text{if } a = a_i & \text{ then } a = a_i \quad \text{for every } \eta < \omega_1 \\
\text{and similarly} & \quad (11) \\
\text{for every } \alpha & \in b \quad (11) \\
\text{if } a = a_i & \text{ then } a = a_i \quad \text{for every } \eta < \omega_1.
\end{align*}
\]

Finally, by applying Proposition 6 to suitable functions we may assume that
\[
\begin{align*}
\text{if } a_i & = a_i \quad (12) \\
\text{then } & a_i = a_i \quad \text{for every } \eta < \omega_1.
\end{align*}
\]
Let \( \{a_i, \delta_i\} = \{a_i^+, \delta_i^+\} \). If \( a_i = a_i^+ \) and \( \delta_i = \delta_i^+ \), then
\[
s((a_i^+, \delta_i^+)) = s((a_i, \delta_i)) = s((a_i, \delta_i^+)) = s((a_i, \delta_i^+)) = s((a_i^+, \delta_i^+))
\]
Similarly, we show that \( s((a_i, \delta_i^+)) = 0 \), i.e., \( s(i) \) is a function.

Moreover, if \( a_i = a_i^+ \) then we also have \( s((a_i, \delta_i^+)) = 0 \), i.e., \( a_i \in S \).

III. \( a_i \in S \) for any \( i \in n \).

The proof is similar.

IV. \( a_i \cup a_j \cup a_k \) is a function for any \( i < n \).

Let \( \{a_i^+, \delta_i^+\} \in \text{dom}(a_i \cup a_j \cup a_k) \). If \( \{a_j^+, \delta_j^+\} \in \text{dom}(a_j \cup a_k) \) then \( \delta_j^+ \in \overline{a}_j \) and hence, by (11), \( \delta_j^+ = \delta_j^+ \). So, by (9)
\[
s((a_j^+, \delta_j^+)) = s((a_j^+, \delta_j^+)) = s((a_j^+, \delta_j^+)) = s((a_j^+, \delta_j^+)) = s((a_j^+, \delta_j^+)) = s((a_j^+, \delta_j^+))
\]
Similarly, we show that \( s((a_j, \delta_j^+)) = 0 \), i.e., \( s(i) \) is a function.

Moreover, if \( a_i = a_i^+ \) then we also have \( s((a_i, \delta_i^+)) = 0 \), i.e., \( a_i \in S \).

V. \( a_i \cup a_j \cup a_k \) is a function for any \( i < n \).

The proof is similar.

VI. \( a_i \cup a_j \cup a_k \) is a function for any \( i < j < n \).

Let \( \{a_j^+, \delta_j^+\} \in \text{dom}(a_j \cup a_k) \). If \( \{a_k^+, \delta_k^+\} \in \text{dom}(a_k \cup a_j) \) then, by (13), \( a_k = a_k^+ \) and \( \delta_k = \delta_k^+ \) then \( s((a_k^+, \delta_k^+)) = s((a_k^+, \delta_k^+)) = s((a_k^+, \delta_k^+)) = s((a_k^+, \delta_k^+)) = s((a_k^+, \delta_k^+)) = s((a_k^+, \delta_k^+)) \).

If \( a_k = a_k^+ \) and \( \delta_k = \delta_k^+ \) then \( a_k = a_k^+ \) and hence, by (10), \( \delta_k = \delta_k^+ \). So, by (9)
\[
s((a_k^+, \delta_k^+)) = s((a_k^+, \delta_k^+)) = s((a_k^+, \delta_k^+)) = s((a_k^+, \delta_k^+)) = s((a_k^+, \delta_k^+))
\]
Similarly, we show that \( s((a_j, \delta_j^+)) = 0 \), i.e., \( s(i) \) is a function.

Moreover, if \( a_i = a_i^+ \) then we also have \( s((a_i, \delta_i^+)) = 0 \), i.e., \( a_i \in S \).

VII. \( s_i \cup s_j \cup s_k \) is a function for any \( i, j < n \).

The proof is similar.

VIII. \( s_i \cup s_j \cup s_k \) is a function for any \( i < j, k < n \).

Let \( \{a_i^+, \delta_i^+\} \in \text{dom}(a_i \cup a_j \cup a_k) \). If \( \{a_j^+, \delta_j^+\} \in \text{dom}(a_j \cup a_k) \) then \( \delta_j^+ \in \overline{a}_j \) and hence, by (11), \( \delta_j^+ = \delta_j^+ \). So, by (9)
\[
s((a_j^+, \delta_j^+)) = s((a_j^+, \delta_j^+)) = s((a_j^+, \delta_j^+)) = s((a_j^+, \delta_j^+)) = s((a_j^+, \delta_j^+)) = s((a_j^+, \delta_j^+))
\]
Similarly, we show that \( s((a_k^+, \delta_k^+)) = 0 \), i.e., \( s(i) \) is a function.

Moreover, if \( a_i = a_i^+ \) then we also have \( s((a_i, \delta_i^+)) = 0 \), i.e., \( a_i \in S \).
It is clear that conditions I-VIII give \( s \in S \) and our proof of Lemma 7 is complete. Now we prove Theorem 2. Let \( \langle \xi, \zeta \rangle : \zeta < \alpha_1 \) be a sequence of elements of \( P \). We show that there exist \( \zeta < \eta < \alpha_1 \) such that \( \langle \xi, \eta \rangle \) and \( \langle \zeta, \eta \rangle \) are compatible. By the \( A \)-lemma we may assume that

\[
\text{supp}(\xi) = y \cup w \quad \text{for every } \xi < \omega_1 \text{ where}
\]

\[w \cap w = \emptyset \quad \text{for every } \zeta < \eta < \alpha_1.
\]

Let \( P'_y = \{ \langle x, y \rangle : \langle x, y \rangle \in P \} \) be a poset with the ordering relation

\[
\langle x, y \rangle \preceq \langle x', y' \rangle \iff x \preceq x' \land y = y' \lor (y = 0, 0).
\]

Clearly \( P_y \) and \( P'_y \) are isomorphic.

Let us consider a set \( \{ \langle x, y \rangle : \zeta < \omega_1 \} \) of elements of \( P' \). By Lemma 7 there exist \( \zeta < \eta < \alpha_1 \) and \( \langle x, y \rangle \in P' \) such that \( \langle x, y \rangle \not\preceq \langle x', y' \rangle \) and \( \langle x, y \rangle \not\preceq \langle x', y' \rangle \).

Let \( q' \in Q \) be defined by

\[
q'(x, n) = \begin{cases} q(x, n) & \text{for } (x, n) \in y, \\ q(x, n) & \text{for } (x, n) \in w, \\ q(x, n) & \text{for } (x, n) \in w, \\ 0 & \text{otherwise} \end{cases}
\]

It is easy to see that \( \langle x, q' \rangle \in P \) and \( \langle x, q' \rangle \not\preceq \langle x', q' \rangle \) and \( \langle x, q' \rangle \not\preceq \langle x', q' \rangle \).

This completes the proof.

References


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Topological games and products, II

by

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Abstract. The purpose of this paper is to study the topological games (in the sense of R. Telgarsky) of product spaces: Assume that Player I has winning strategies in the given topological games of \( X \) and \( Y \). Then we consider the conditions of a product space \( X \times Y \) under which he has a winning strategy in a certain topological game of \( X \times Y \). Moreover, we can apply the results obtained from this kind of argument to the product theorem in dimension theory.

Introduction. R. Telgarsky [14] introduced and studied the topological game \( G(K, X) \). In our previous paper [19], we have used it to study the covering properties of product spaces. In the present paper, we shall study the topological game on product spaces. If the above \( K \) is the class of all one-point spaces, then the game \( G(K, X) \) is often abbreviated by \( G(X) \), which is called the point-open game. R. Telgarsky [15] stated the following: If Player I has winning strategies in \( G(X) \) and \( G(Y) \), then he has a winning strategy in \( G(X \times Y) \). This gives the positive answer to [14, Question 14.1]. In this connection, we raise the following natural question: Assume that Player I has winning strategies in \( G(K, X) \) and \( G(K, Y) \). What is a topological game of \( X \times Y \) which is interesting to investigate? What is a condition on \( X \times Y \) under which he has a winning strategy in such a game? In §2 and §3, we discuss this question. In §4, using the result of §2, we give a product theorem in dimension theory.

Each space considered here is assumed to be a Hausdorff space. \( N \) denotes the set of all natural numbers and \( \aleph \) denotes an infinite cardinal number. For a space or a set \( X \), by \( \gamma(X) \) we mean the character of \( X \) and by \( |X| \) the cardinality of \( X \). For a collection \( \mathcal{F} \) of subsets of \( X \), \( |\mathcal{F}| \) denotes \( \bigcup \{ F : F \in \mathcal{F} \} \).

§1. Topological games. R. Telgarsky [15] has introduced an equivalent form of the game \( G(K, X) \) defined in [14]. The new form of the game we use below.

Let \( L \) be a class of spaces and let \( X \) be a space. We define the topological game \( G(L, X) \) as follows: There are two players; Player I and Player II. Player I chooses a closed set \( E_1 \) of \( X \) with \( E_1 \subseteq L \), and after that Player II chooses an open set \( U_1 \) of \( X \) with \( E_1 \subseteq U_1 \). Again Player I chooses a closed set \( E_2 \) of \( X \) with \( E_2 \subseteq L \) and Player II chooses an open set \( U_2 \) of \( X \) with \( E_2 \subseteq U_2 \), and so on. Here, the infinite