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On locally expansive selfcoverings of compact metrizable spaces

by

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Abstract. This paper is concerned with characterizing, in terms of certain properties of their compositions, the open local expansions defined on compact locally connected Hausdorff spaces.

1. Introduction. Let (M, ϱ) be a metric space and $f\colon M \to M$ a continuous selfmapping of M. We will call f a *local expansion on* (M, ϱ) [3] (cf. also [2]) if

(A) for each $z \in M$ there is a neighborhood U of z and a number $\lambda_U > 1$ such that

$$\varrho(f(x), f(y)) \geqslant \lambda_U \varrho(x, y) \quad \text{for} \quad x, y \in U. \tag{1}$$

If there exists a number $\lambda > 1$ such that condition (A) holds with $\lambda_U \geqslant \lambda$, we say that f is a *local λ-expansion on* (M, ϱ).

Now let M be a metrizable topological space and $f\colon M \to M$ a continuous selfmapping of M. We will say that f is a *topological local expansion* (resp. *topological local λ-expansion*) *on* M if M admits a metric ϱ compatible with the given topology and such that f is a local expansion (resp. local λ-expansion) on (M, ϱ).

(Note that if M is compact then f is a local expansion on (M, ϱ) iff for some $\lambda > 1$ it is a local λ-expansion on (M, ϱ)).

A sequence A_n, $n = 0, 1, \ldots$, of subsets of a topological space M is said to be *fine* if for each open covering $\mathscr{C}$ of M there exists an integer n such that for $m \geqslant n$, each connected component of A_m is a subset of some member of $\mathscr{C}$.

It is easily shown (cf. [2] or [3]) that if M is compact, locally connected and metrizable and f is an open topological local expansion of M onto itself, then f is a local homeomorphism (and therefore a selfcovering of M) and

(B) for each point z of M there exists a neighborhood U of z such that the sequence $f^{-n}(U)$, $n = 0, 1, \ldots$, is fine.

Since this condition does not involve the metric and has a topological character, it is natural to ask the following question. Let M be a compact, locally connected Hausdorff space and $f\colon M \to M$ a local homeomorphism of M onto itself satisfying the condition (B). Is it possible to find a metric ϱ generating the given topology of M such that the mapping f is a local expansion on (M, ϱ)?

The purpose of this paper is to show that the answer to this question is affirmative as well as to give the construction of the desired metric ϱ.

It should be mentioned that each C^1 expanding endomorphism of a compact differentiable manifold M without boundary is an open topological local expansion on M, namely, it is a local expansion on M with respect to a distance function, coming from a Riemannian metric on M (cf. [5]). In [4] it is shown that any compact flat manifold (i.e., the orbit space R^n/G, where G is a discrete uniform subgroup of the group of isometries of R^n) admits an expanding endomorphism, hence an open topological local expansion.

2. The construction of the metric making f locally expansive. For a subset A of M and a family $\mathscr{C}$ of subsets of M, let $\mathscr{C}(A)$ denote the union of all those $C\in\mathscr{C}$ such that $A\cap C\neq\emptyset$ and let $[\mathscr{C}]^2=\{\mathscr{C}(C)\colon C\in\mathscr{C}\}$.

LEMMA 1. *Let M be a Hausdorff topological space and $f\colon M\to M$ a continuous and open selfmapping of M, and suppose that there exists a sequence $\mathscr{C}_n$, $n=0,1,\ldots$, of open coverings of M such that*

(i) *for each $n=0,1,\ldots$, $[\mathscr{C}_{n+1}]^2$ refines $\mathscr{C}_n$,*

(ii) *for each $x\in M$, $\{C\in\mathscr{C}_n\colon x\in C, n=0,1,\ldots\}$ is a neighborhood base at x,*

(iii) *for each $x\in M$, if $y\in\mathscr{C}_0(\{x\})$ and $f(y)\in\mathscr{C}_n(\{f(x)\})$ then $y\in\mathscr{C}_{n+1}(\{x\})$.*

Then M is metrizable and f is a topological 2-expansion on M.

Proof. First we introduce a function $s\colon M\times M\to[0,1]$ as follows

$$s(x,y)=\begin{cases}0 & \text{if } x=y,\\ 2^{-n(x,y)} & \text{if } x\neq y,\end{cases}$$

where for $x\neq y$,

$$n(x,y)=\min\{m\geqslant 0\colon x\in\mathscr{C}_m(\{y\})\}.$$

A desired metric ϱ can be defined as

$$\varrho(x,y)=\inf\sum_{i=0}^{k-1}s(z_i,z_{i+1}),$$

where the infimum is taken over all possible finite chains of points $z_0,z_1,\ldots,z_k$ of M such that $z_0=x$ and $z_k=y$.

It follows now from (i) and (ii) that ϱ is a metric on M inducing the given topology (see [6]; cf. also [1], Chapter IX, § 1, Theorem 1).

In order to prove that f is a local 2-expansion on (M,ϱ) we have to show that condition (A) holds with $\lambda_U\geqslant 2$.

Now, let $z\in M$. Since f is continuous, there exists a neighborhood W of z such that

$$W\subset V_1,\quad f(W)\subset V_2\quad\text{for some } V_1,V_2\in\mathscr{C}_0. \tag{2}$$

Since f is open, there is a number $r>0$ with

$$K_\varrho(f(z),r)\subset f(W).$$

We define a neighborhood U of z by

$$U=W\cap f^{-1}\big(K_\varrho(f(z),\tfrac{1}{3}r)\big).$$

Let $x,y\in U$ and let $\xi>0$ be a given number. It follows from the definition of U that $\varrho(f(x),f(y))<\frac{2}{3}r$. Thus, there exists a number $\delta\in[0,\xi]$ such that

$$\varrho(f(x),f(y))+2\delta<\tfrac{2}{3}r. \tag{3}$$

From the definition of ϱ, there exists a finite system of elements $z_0,z_1,\ldots,z_k$ of M with $z_0=f(x)$, $z_k=f(y)$ and such that

$$\sum_{i=0}^{k-1}s(z_i,z_{i+1})\leqslant\varrho(f(x),f(y))+2\delta. \tag{4}$$

From (3) and (4) we have

$$\varrho(f(x),z_j)\leqslant\sum_{i=0}^{j-1}s(z_i,z_{i+1})\leqslant\sum_{i=0}^{k-1}s(z_i,z_{i+1})<\tfrac{2}{3}r\quad\text{for}\quad 1\leqslant j\leqslant k.$$

Thus,

$$\varrho(f(z),z_j)\leqslant\varrho(f(z),f(x))+\varrho(f(x),z_j)<\tfrac{1}{3}r+\tfrac{2}{3}r=r,$$

and, by definition of r, we obtain

$$z_j\in K_\varrho(f(z),r)\subset f(W)\quad\text{for}\quad 0\leqslant j\leqslant k.$$

It follows from (2) and (iii) that the map $f\colon W\to f(W)$ is a homeomorphism. Thus, there are elements $\bar{z}_i\in W$, $i=0,1,\ldots,k$ such that $f(\bar{z}_i)=z_i$ and $\bar{z}_0=x$, $\bar{z}_k=y$.

From (iii) we obtain

$$n(z_i,z_{i+1})\leqslant n(\bar{z}_i,\bar{z}_{i+1})-1,$$

for $i=0,1,\ldots,k-1$. Therefore, by the definition of ϱ we have

$$\varrho(x,y)\leqslant\sum_{i=0}^{k-1}s(\bar{z}_i,\bar{z}_{i+1})=\sum_{i=0}^{k-1}2^{-n(\bar{z}_i,\bar{z}_{i+1})}\leqslant\tfrac{1}{2}\sum_{i=0}^{k-1}2^{-n(z_i,z_{i+1})}=\tfrac{1}{2}\sum_{i=0}^{k-1}s(z_i,z_{i+1}),$$

which, together with (4), gives

$$\varrho(x,y)\leqslant\tfrac{1}{2}\varrho(f(x),f(y))+\delta.$$

Since ξ was chosen arbitrary and $\delta\leqslant\xi$, we obtain

$$\varrho(f(x),f(y))\geqslant 2\varrho(x,y)\quad\text{for}\quad x,y\in U,$$

which proves our assertion.

LEMMA 2. *Let M be a metrizable space and $f\colon M\to M$ a continuous selfmapping of M and let $\lambda>1$. If there exists an integer $n\geqslant 1$ such that f^n is a topological local λ-expansion on M, then f is a topological local $\sqrt[n]{\lambda}$-expansion on M.*

Proof. Suppose that for some $n\geqslant 1$, f^n is a topological local λ-expansion on M. Let ϱ be a metric on M compatible with the topology and $\mathscr{C}$ an open covering of M such that if $U\in\mathscr{C}$, then

(5) $$\varrho(f^n(x),f^n(y))\geqslant\lambda\varrho(x,y)\quad \text{for}\quad x,y\in U.$$

Define $\bar\varrho$ as follows

$$\bar\varrho(x,y)=\sum_{i=0}^{n-1}\frac{1}{\alpha^i}\varrho(f^i(x),f^i(y))\quad \text{for}\quad x,y\in M,$$

where $\alpha=\sqrt[n]{\lambda}$. Clearly, $\bar\varrho$ is a metric on M and it is topologically equivalent to ϱ. In order to prove that f is a local α-expansion on $(M,\bar\varrho)$ it is sufficient to show that the inequality (1) holds for $\bar\varrho$ and each $U\in\mathscr{C}$ and $\lambda_U\geqslant\alpha$.

Let $U\in\mathscr{C}$ and $x,y\in U$. From (5) we obtain

$$\begin{aligned}\bar\varrho(f(x),f(y))&=\sum_{i=0}^{n-1}\frac{1}{\alpha^i}\varrho(f^{i+1}(x),f^{i+1}(y))\\ &=\alpha\sum_{i=1}^{n-1}\frac{1}{\alpha^i}\varrho(f^i(x),f^i(y))+\frac{\alpha}{\alpha^n}\varrho(f^n(x),f^n(y))\\ &\geqslant\alpha\sum_{i=1}^{n-1}\frac{1}{\alpha^i}\varrho(f^i(x),f^i(y))+\alpha\varrho(x,y)\\ &=\alpha\sum_{i=0}^{n-1}\frac{1}{\alpha^i}\varrho(f^i(x),f^i(y))=\alpha\bar\varrho(x,y).\end{aligned}$$

This completes the proof.

As a consequence of the above lemmas, we obtain

THEOREM. *Let M be a compact locally connected Hausdorff space and $f\colon M\to M$ a local homeomorphism of M onto itself satisfying the following condition*

(B) *for each point z of M there is a neighborhood U of z such that the sequence $f^{-n}(U)$, $n=0,1,\ldots$, is fine.*

Then, M is metrizable and f is a topological local expansion on M.

Proof. First, we prove that there exists a finite open covering $\mathscr{A}$ of M such that if $U\in\mathscr{A}$, then

(6) *U is connected,*

(7) *for each $n=0,1,\ldots$, U is evenly covered by f^n (i.e., f^n maps each connected component of $f^{-n}(U)$ homeomorphically onto U),*

(8) *the sequence $f^{-n}(U)$, $n=0,1,\ldots$, is fine.*

In order to prove the existence of $\mathscr{A}$, let $\mathscr{C}$ be an open covering of M such that if $U\in\mathscr{C}$, then U is connected and it is evenly covered by f. Let $z\in M$. By (B), there exists a neighborhood U_z of z such that the sequence $f^{-n}(U_z)$, $n=0,1,\ldots$, is fine. Thus, there is an integer $n_z\geqslant 1$ such that, for $m\geqslant n_z$, each connected component of $f^{-m}(U_z)$ is a subset of some member of $\mathscr{C}$, hence is evenly covered by f. Let $V_z\subset U_z$ be an open and connected neighborhood of z which is evenly covered by f^{n_z}. Thus, V_z is evenly covered by f^i for $1\leqslant i\leqslant n_z$ and, inductively on n, we may verify that for each $n=0,1,\ldots$, V_z is evenly covered by f^n. Since M is compact, there exists a finite subcovering $\mathscr{A}\subset\{V_z\colon z\in M\}$. Clearly, $\mathscr{A}$ satisfies conditions (6), (7) and (8).

Now, for each integer $n=0,1,\ldots$, we construct an open covering $\mathscr{A}_n$ of M as follows

$$\mathscr{A}_n=\begin{cases}\mathscr{A} & \text{if } n=0,\\ \bigcup\{c(f^{-1}(U))\colon U\in\mathscr{A}_{n-1}\} & \text{if } n>0,\end{cases}$$

where for each $A\subset M$, $c(A)$ denotes the set of all connected components of A. Let $\mathscr{C}$ be a refinement of $\mathscr{A}_0$ such that $[\mathscr{C}]^2$ refines $\mathscr{A}_0$. Since $\mathscr{A}_0$ is finite, the condition (8) and the construction of the $\mathscr{A}_n$ insure that there is an integer $k_0\geqslant 1$ such that $[\mathscr{A}_{k_0}]^2$ refines $\mathscr{C}$.

Hence,

(9) $$[\mathscr{A}_{k_0}]^2 \text{ refines } \mathscr{A}_0.$$

Since, f^{k_0} is a local homeomorphism, it is open.

It is easily shown by using (6), (7), (8) and (9) that the mapping f^{k_0} together with the sequence

$$\mathscr{C}_n=\mathscr{A}_{(n+1)k_0},\quad n=0,1,\ldots,$$

of open coverings of M satisfy the conditions (i), (ii) and (iii) of Lemma 1. Thus M is metrizable and f^{k_0} is a topological local 2-expansion on M. Therefore, by Lemma 2, f is a topological local $\sqrt[k_0]{2}$-expansion on M. This completes the proof.

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