Topological games and products, III

by

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Abstract. The main purpose of this paper is to study the three covering properties; metacompactness, subparacompactness and submetacompactness, of product spaces. Every product space dealt here has at least one factor in which Player I has a winning strategy for a certain topological game (in the sense of R. Telgársky).

Introduction. R. Telgársky [9] introduced and studied the concept of the topological game $G(K, X)$. Making use of it, we studied the covering properties; strongly rectangular etc., of product spaces in [14], and studied the topological game itself of product spaces in [15]. This paper is a continuation of [14] and [15]. It has 5 sections. Topological games are dealt with in all sections, and product spaces are dealt with in Sections 2, 3 and 4. In Section 1, we restate the topological game in [9] and prepare the common notations and the results quoted in some sections. In Section 2, we study metacompactness of product spaces. The main theorem of this section gives an affirmative answer to H. Junnila's question in [3]. Moreover, we discuss normality of product spaces with a metric factor. In Section 3, we study subparacompactness of product spaces. The result here gives a product theorem in dimension theory. In Section 4, we study submetacompactness (i.e., θ-refinability) of product spaces. For that, the concept called strong submetacompactness is used. In Section 5, we discuss in connection with a certain question in [9]. For that, we introduce the concept of outer-almost $\aleph_0$-expandability.

§ 1. Preliminaries. In this paper, by a space we mean a topological space and no separation axioms are assumed. However, regular spaces and normal spaces considered here are always assumed to be T_1. For a space X, 2^X denotes the collection of all closed sets in X. For a collection $\mathcal{U}$ of subsets in X, $\bigcup \mathcal{U}$ denotes $\bigcup \{U\colon U \in \mathcal{U}\}$. For two collections $\mathcal{U}$ and $\mathcal{O}$ of subsets in X, $\mathcal{U} \prec \mathcal{O}$ implies that each member of $\mathcal{U}$ is contained in some member of $\mathcal{O}$. For a cover $\mathcal{O}$ of X, $\mathcal{U}$ is a refinement of $\mathcal{O}$ if $\mathcal{U}$ is a cover of X such that $\mathcal{U} \prec \mathcal{O}$. The set of all natural numbers is denoted by N and natural numbers are denoted by n, m, i, j, k, etc. However, we omit N without confusion; e.g., $N \cup \{0\}$ is denoted by $\{n\colon n \geqslant 0\}$.

Now, we restate the topological game in [9]. Here, it contains a few notations which are different from the previous one. Let K be a non-void class of spaces which are hereditary with respect to closed sets. Let X be a space. The topological game $G(K, X)$ is defined as follows: There are two players I and II. They alternatively choose consecutive terms of a sequence $\langle E_1, F_1, E_2, F_2, \ldots\rangle$ of closed sets in X, where Player I first choose E_1. When each player chooses his term, he knows K, X and their previous choices.

A sequence $\langle E_1, F_1, E_2, F_2, \ldots\rangle$ of closed sets in X is a play of $G(K, X)$ if and only if for each $n \in N$

(1) E_n is the choice of Player I,
(2) F_n is the choice of Player II,
(3) $E_n \in K$,
(4) $E_{n+1} \subset F_n$,
(5) $F_{n+1} \subset F_n$,
(6) $E_n \cap F_n = \emptyset$.

Player I wins this play if $\bigcap\{F_n: n \in N\} = \emptyset$. Otherwise, Player II wins it.

A finite sequence $\langle E_1, F_1, \ldots, E_n, F_n\rangle$ of closed sets in X is said to be *admissible* for $G(K, X)$ if each E_i and F_i satisfy the above conditions (1)–(6).

A function s is said to be a *strategy* for Player I in $G(K, X)$ if the domain of s consists of all the finite sequences $\langle F_0, F_1, \ldots, F_n\rangle$ of closed sets in X such that $F_0 = X$ and $\langle E_1, F_1, \ldots, E_n, F_n\rangle$ is admissible for $G(K, X)$, where

$$E_i = s(F_0, \ldots, F_{i-1})$$

for $1 \leqslant i \leqslant n$, and if each $s(F_0, \ldots, F_n)$ belongs to $2^X \cap K$ and is contained in F_n.

Let s be a strategy of Player I in $G(K, X)$. A finite sequence $\langle F_1, \ldots, F_n\rangle$ of closed sets in X is said to be an *admissible choice of Player II for s in $G(K, X)$* if the sequence $\langle E_1, F_1, \ldots, E_n, F_n\rangle$ such that $E_i = s(F_0, F_1, \ldots, F_{i-1})$ for $1 \leqslant i \leqslant n$, where $F_0 = X$, is admissible for $G(K, X)$.

A strategy s of Player I in $G(K, X)$ is said to be *winning* if he wins each play $\langle E_1, F_1, E_2, F_2, \ldots\rangle$ such that $E_n = s(F_0, \ldots, F_{n-1})$ for each $n \in N$.

Let s be a winning strategy of Player I in $G(K, X)$. It should be noted that each infinite sequence $\langle F_1, F_2, \ldots\rangle$, such that $\langle F_1, \ldots, F_n\rangle$ is an admissible choice of Player II for s in $G(K, X)$ for each $n \in N$, has the empty intersection.

According to [11], $I(K, X)$ denotes the following statement: Player I has a winning strategy in $G(K, X)$.

In Sections 2, 3 and 4, we shall take for K the following class of spaces:

$\boldsymbol{DC}$ — the class of all spaces which can be decomposed into a discrete collection by compact closed sets. The quite important roles are played by the statement $I(\boldsymbol{DC}, X)$: Player I has a winning strategy in $G(\boldsymbol{DC}, X)$.

Next, we refer P-spaces which were introduced by K. Morita [5]. In the same paper, he showed that X is a normal P-space if and only if $X \times Y$ is normal for any metric space Y. In Sections 2 and 4, P-spaces are required and we make use of the following fact of R. Telgársky [10].

PROPOSITION 1.1. *X is a P-space if and only if there exists a function*

$$F\colon \bigcup\{(2^X)^n: n \in N\} \to 2^X$$

such that

(a) *for each $(E_0, \ldots, E_n) \in (2^X)^{n+1}$ and $n \geqslant 0$,*

$$F(E_0, \ldots, E_n) \cap \bigcap\{E_i: i \leqslant n\} = \emptyset$$

and

(b) *for each $(E_0, E_1, \ldots) \in (2^X)^N$ with $\bigcap\{E_n: n \geqslant 0\} = \emptyset$,*

$$\bigcup\{F(E_0, \ldots, E_n): n \geqslant 0\} = X.$$

Concerning both $I(\boldsymbol{DC}, X)$ and P-spaces, we have the following result.

PROPOSITION 1.2. *If a space X has a σ-closure-preserving closed cover by compact sets, then $I(\boldsymbol{DC}, X)$ and X is a P-space.*

The first half of Proposition 1.2 is given in [9, Corollary 10.2], which is essentially due to H. B. Potoczny [8]. The latter half of it follows from [13, Theorem 2].

A subset of a product space $X \times Y$ of the form $A \times B$ is called a *rectangle*. For a rectangle E in $X \times Y$, E' and E'' denote the projections of E into X and Y, respectively. So we have $E = E' \times E''$. A rectangle E in $X \times Y$ is said to be an *open* (*closed* and *cozero*) *rectangle* if E' and E'' are open (closed and cozero, respectively) in X and Y, respectively. Moreover, a rectangle E in $X \times Y$ is said to be a *closed × open rectangle* if E' is closed in X and E'' is open in Y.

§ 2. Metacompactness of product spaces.

This section contains several results concerning metacompactness of product spaces and concerning P-spaces. First, we give the following main theorem.

THEOREM 2.1. *Let X be a regular metacompact P-space and Y a metacompact space. If $I(\boldsymbol{DC}, X)$, then the product space $X \times Y$ is metacompact.*

Proof. Let s be a winning strategy of Player I in $G(\boldsymbol{DC}, X)$ and

$$F\colon \bigcup\{(2^X)^n: n \in N\} \to 2^X$$

a function described in Proposition 1.1. Their existences are assured by the assumptions. Let $\mathcal{O}$ be any monotone open cover of $X \times Y$. It suffices to show that $\mathcal{O}$ has a point-finite open refinement (cf. [3, Theorem 3.2]).

First, we shall construct three sequences $\{\mathcal{G}_n: n \geqslant 0\}$, $\{\mathcal{R}_n: n \geqslant 0\}$ and $\{\mathcal{H}_n: n \geqslant 0\}$ of collections by rectangles in $X \times Y$ and construct two functions $\varphi_n\colon \mathcal{R}_n \to \mathcal{R}_{n-1}$ and $\psi_n\colon \mathcal{H}_n \to \mathcal{H}_{n-1}$ for each $n \in N$, satisfying the following conditions (1.1)–(1.11) for each $n \in N$:

(1.1) $\mathcal{G}_0 = \{\emptyset\}$ and $\mathcal{R}_0 = \{X \times Y\} = \{H(X \times Y)\} = \mathcal{H}_0$.

(1.2) $\mathcal{G}_n$ is a point-finite collection by open rectangles.

(1.3) $\mathcal{R}_n$ is a collection by closed × open rectangles.

(1.4) $\mathcal{H}_n = \{H(R): R \in \mathcal{R}_n\}$ and it is a point-finite collection by open rectangles such that $R \subset H(R)$ for each $R \in \mathcal{R}_n$.

(1.5) $\mathscr{G}_n \prec \mathcal{O}$.

(1.6) $\bigcup \mathscr{G}_n \subset \bigcup \mathscr{H}_{n-1}$.

(1.7) $H(\varphi_n(R)) = \psi_n(H(R))$ for each $R \in \mathscr{R}_n$.

(1.8) $H(R) \subset \psi_n(H(R))$ for each $R \in \mathscr{R}_n$.

(1.9) If $p \in R_{n-1} \in \mathscr{R}_{n-1}$ and $p \notin \bigcup \mathscr{G}_n$, then there exists some $R_n \in \mathscr{R}_n$ such that $p \in R_n$ and $\varphi_n(R_n) = R_{n-1}$.

(1.10) For each $(R_1, \ldots, R_n) \in \prod_{i=1}^{n} \mathscr{R}_i$ such that $\varphi_i(R_i) = R_{i-1}$ for $2 \leqslant i \leqslant n$, the finite sequence $\langle R_1', \ldots, R_n' \rangle$ is an admissible choice of Player II for s in $G(\boldsymbol{DC}, X)$.

(1.11) For each $(R_0, \ldots, R_n) \in \prod_{i=0}^{n} \mathscr{R}_i$ such that $\varphi_i(R_i) = R_{i-1}$ for $1 \leqslant i \leqslant n$, $F(R_0', \ldots, R_n') \cap H(R_n)' = \varnothing$.

Let $\mathscr{G}_0 = \{\varnothing\}$, $\mathscr{R}_0 = \{X \times Y\}$, $\mathscr{H}_0 = \{H(X \times Y)\}$ and $H(X \times Y) = X \times Y$. Assume that $\{\mathscr{G}_i\colon i \leqslant n\}$, $\{\mathscr{R}_i\colon i \leqslant n\}$, $\{\mathscr{H}_i\colon i \leqslant n\}$, $\{\varphi_i\colon i \leqslant n\}$ and $\{\psi_i\colon i \leqslant n\}$ satisfying the above conditions have been already constructed. Now, fix an $R \in \mathscr{R}_n$. We take $(R_0, R_1, \ldots, R_n) \in \prod_{i=0}^{n} \mathscr{R}_i$ such that $R_n = R$ and $\varphi_i(R_i) = R_{i-1}$ for $1 \leqslant i \leqslant n$. It follows from the assumption (1.10) that there exists a discrete collection $\{C_\alpha\colon \alpha \in \Omega(R)\}$ by compact sets in R' such that

$$s(R_0', R_1', \ldots, R_n') = \bigcup \{C_\alpha\colon \alpha \in \Omega(R)\}.$$

Since X is regular metacompact, there exist two point-finite collections $\{V_\alpha\colon \alpha \in \Omega(R)\}$ and $\{W_\alpha\colon \alpha \in \Omega(R)\}$ of open sets in X such that

$$C_\alpha \subset W_\alpha \subset \operatorname{Cl} W_\alpha \subset V_\alpha \subset H(R') \setminus \bigcup \{C_\beta\colon \beta \in \Omega(R) \setminus \{\alpha\}\}$$

for each $\alpha \in \Omega(R)$. Here, by metacompactness of Y, for each $\alpha \in \Omega(R)$ we can choose a collection $\mathscr{G}_\alpha(R) = \{G_\lambda\colon \lambda \in \Lambda(\alpha)\}$ by open rectangles such that

(i) $C_\alpha \subset G_\lambda' \subset W_\alpha$ for each $\lambda \in \Lambda(\alpha)$,

(ii) $\{G_\lambda''\colon \lambda \in \Lambda(\alpha)\}$ is a point-finite collection of open sets in Y and its union is R'',

(iii) $\mathscr{G}_\alpha(R) \prec \mathcal{O}$.

We put

$$\tilde{R} = (R' \setminus \bigcup \{W_\alpha\colon \alpha \in \Omega(R)\}) \times R''$$

and

$$R(\alpha, \lambda) = (\operatorname{Cl} W_\alpha \cap R' \setminus G_\lambda') \times G_\lambda'' \quad \text{for each } \lambda \in \Lambda(\alpha) \text{ and } \alpha \in \Omega(R).$$

Moreover, we put

$$H(\tilde{R}) = (H(R)' \setminus F(R_0', \ldots, R_n', \tilde{R}')) \times H(R)''$$

and

$$H(R(\alpha, \lambda)) = (V_\alpha \setminus F(R_0', \ldots, R_n', R(\alpha, \lambda)')) \times G_\lambda''.$$

Here, running $R \in \mathscr{R}_n$, we set

$$\mathscr{G}_{n+1} = \bigcup \{\mathscr{G}_\alpha(R)\colon \alpha \in \Omega(R) \text{ and } R \in \mathscr{R}_n\}$$

and

$$\mathscr{R}_{n+1} = \{\tilde{R}\colon R \in \mathscr{R}_n\} \cup \{R(\alpha, \lambda)\colon \lambda \in \Lambda(\alpha), \alpha \in \Omega(R) \text{ and } R \in \mathscr{R}_n\}.$$

Moreover, we set $\mathscr{H}_{n+1} = \{H(Q)\colon Q \in \mathscr{R}_{n+1}\}$. The function $\varphi_{n+1}\colon \mathscr{R}_{n+1} \to \mathscr{R}_n$ is defined by

$$\varphi_{n+1}(\tilde{R}) = R \quad \text{and} \quad \varphi_{n+1}(R(\alpha, \lambda)) = R$$

for each $\lambda \in \Lambda(\alpha)$, $\alpha \in \Omega(R)$ and $R \in \mathscr{R}_n$. Next, the function $\psi_{n+1}\colon \mathscr{H}_{n+1} \to \mathscr{H}_n$ is defined by $\psi_{n+1}(H(Q)) = H(\varphi_{n+1}(Q))$ for each $Q \in \mathscr{R}_{n+1}$. It is obvious from the constructions that the conditions (1.1), (1.3), (1.5)–(1.8) and (1.11) are satisfied. It is easily checked from the constructions and the inductive assumptions that the conditions (1.2) and (1.4) are satisfied. The verifications that the conditions (1.9) and (1.10) are satisfied are similar to that of the conditions (1.5_{n+1}) and (1.6_{n+1}) in the proof of [14, Theorem 2.1], respectively. From the facts mentioned above, the desired constructions have been completed by induction.

Now, we set $\mathscr{G} = \bigcup \{\mathscr{G}_n\colon n \geqslant 0\}$. By (1.5), $\mathscr{G}$ is a collection of open sets in $X \times Y$ such that $\mathscr{G} \prec \mathcal{O}$. We shall show that $\mathscr{G}$ is a point-finite cover of $X \times Y$. For each $m, n \in N$ with $m < n$, we put $\varphi_m^n = \varphi_{m+1} \circ \ldots \circ \varphi_n$, $\psi_m^n = \psi_{m+1} \circ \ldots \circ \psi_n$, $\varphi_n^n = 1_{\mathscr{R}_n}$ and $\psi_n^n = 1_{\mathscr{H}_n}$. Then $\{\mathscr{R}_n, \varphi_m^n\}$ and $\{\mathscr{H}_n, \psi_m^n\}$ are inverse systems. From (1.10), note the following fact.

Claim 1. *For each* $\langle R_n\colon n \geqslant 0\rangle \in \varprojlim \{\mathscr{R}_n, \varphi_m^n\}$, *we have* $\bigcap \{R_n'\colon n \geqslant 0\} = \varnothing$.

It follows from (1.1), (1.9) and Claim 1 that $\mathscr{G}$ is a cover of $X \times Y$. The similar arguments are found in the proof of [14, Theorem 2.1]. The detail can be seen in it.

Claim 2. *For each* $\langle H(R_n)\colon n \geqslant 0\rangle \in \varprojlim \{\mathscr{H}_n, \psi_m^n\}$, *we have*

$$\bigcap \{H(R_n)'\colon n \geqslant 0\} = \varnothing.$$

Proof. By (1.7), we have $\langle R_n\colon n \geqslant 0\rangle \in \varprojlim \{\mathscr{R}_n, \varphi_m^n\}$. Moreover, by Claim 1, we have $\bigcap \{R_n'\colon n \geqslant 0\} = \varnothing$. So $\{F(R_0', \ldots, R_n')\colon n \geqslant 0\}$ covers X. Hence it follows from (1.11) that $\{H(R_n)'\colon n \geqslant 0\}$ has the empty intersection.

Claim 3. $\bigcap \{\bigcup \mathscr{H}_n\colon n \geqslant 0\} = \varnothing$.

This follows from (1.4), (1.8) and Claim 2. The proof is similar to that of Claim 2 in the proof of [14, Theorem 2.1].

Let p be any point of $X \times Y$. By Claim 3, we can take some $n_0 \in N$ such that $p \notin \bigcup \mathscr{H}_{n_0}$. It follows from (1.8) that we have $\mathscr{H}_{n+1} \prec \mathscr{H}_n$ for each $n \geqslant 0$. Hence $p \notin \bigcup \{\bigcup \mathscr{H}_n\colon n \geqslant n_0\}$. Moreover, by (1.6), we have $p \notin \bigcup \{\bigcup \mathscr{G}_n\colon n > n_0\}$. Therefore, it follows from (1.2) that $\mathscr{G}$ is point-finite at p. Thus, we have shown that $\mathscr{G}$ is a point-finite open refinement of $\mathcal{O}$. The proof of Theorem 2.1 is complete.

R. Telgársky [9] proved the following: If a paracompact T_2-space X has a σ-closure-preserving cover by compact sets, then $X \times Y$ is paracompact for every paracompact T_2-space Y. H. Junnila has stated in [3, p. 234] that it is not known whether the above result remains valid if "paracompact" is replaced by "regular metacompact". From Proposition 1.2 and Theorem 2.1, we obtain the following affirmative answer to this question.

COROLLARY 2.1. *If a regular metacompact space X has a σ-closure-preserving cover by compact sets, then $X\times Y$ is metacompact for every metacompact space Y.*

Before the next argument, we need some notations. N^* denotes the set of the void sequence $\varnothing$ and all finite sequences consisting of natural numbers. Let $e=(n_1,\ldots,n_k)\in N^*$. We use the following notations; $|e|=k$ ($|\varnothing|=0$), $e_{-j}=(n_1,\ldots,n_{k-j})$ for $0\leqslant j\leqslant k-1$ ($e_{-k}=\varnothing$), $e\oplus n=(n_1,\ldots,n_k,n)$ for each $n\in N$, $\Sigma e=n_1+\ldots+n_k$ and $l(e)=n_k$. The notations of Σe and $l(e)$ are not used in this section but in the proof of Theorem 5.1 below.

Concerning Proposition 1.2 and Theorem 2.1, it is a natural question to ask whether each metacompact space X such that $I(\boldsymbol{DC},X)$ is a P-space. The following result gives an affirmative answer to this question under the assumption of X being normal.

THEOREM 2.2. *Let X be a normal submetacompact (i.e., θ-refinable) space. If $I(\boldsymbol{DC},X)$, then X is a P-space.*

Proof. Let Y be any metric space. It suffices to show from [5, Theorem 4.1] that $X\times Y$ is normal. We can consider Y as a subspace of the Baire space indexed by a set Λ such that the cardinality of Λ is equal to the weight of Y which is infinite (cf. [5, Theorem 2.1]). For each $i\in N$ we put

$$\mathscr{V}_i=\{V(\lambda_1,\ldots,\lambda_i)\colon \lambda_1,\ldots,\lambda_i\in\Lambda\},$$

where $V(\lambda_1,\ldots,\lambda_i)=\{(\mu_j)\in Y\colon \mu_1=\lambda_1,\ldots,\mu_i=\lambda_i\}$ for each $\lambda_1,\ldots,\lambda_i\in\Lambda$ and put $\mathscr{V}=\bigcup\{\mathscr{V}_i\colon i\in N\}$. Here, note that each $\mathscr{V}_i$ is a discrete cover of Y consisting of closed-open sets and $\mathscr{V}$ is an open basis of Y.

Let $\mathcal{O}=\{O_1,O_2\}$ be any binary open cover of $X\times Y$. It suffices to show that $\mathcal{O}$ is a normal cover. Let s be a winning strategy of Player I in $G(\boldsymbol{DC},X)$.

First, we shall construct two families $\{\mathscr{G}(e)\colon e\in N^*\}$ and $\{\mathscr{R}(e)\colon e\in N^*\}$ of collections by rectangles in $X\times Y$ and a function $\varphi_e\colon \mathscr{R}(e)\to\mathscr{R}(e_{-1})$ for each $e\in N^*\setminus\{\varnothing\}$, satisfying the following conditions (2.1)–(2.7) for each $e\in N^*$:

(2.1) $\mathscr{R}(\varnothing)=\{X\times Y\}$.

(2.2) $\mathscr{G}(e)$ is a σ-discrete collection by cozero rectangles.

(2.3) $\mathscr{R}(e)$ is a collection by closed rectangles such that $R''\in\mathscr{V}$ for each $R\in\mathscr{R}(e)$.

(2.4) For each $V\in\mathscr{V}$, $\{R\in\mathscr{R}(e)\colon R''=V\}$ consists of at most finite members.

(2.5) $\mathscr{G}(e)\prec\mathcal{O}$.

(2.6) If $p\in R\in\mathscr{R}(e)$ and $p\notin\bigcup\mathscr{G}(e)$, then there exist some $n\in N$ and $Q\in\mathscr{R}(e\oplus n)$ such that $p\in Q$ and $\varphi_{e\oplus n}(Q)=R$.

(2.7) For each $(R_1,\ldots,R_{|e|})\in\prod_{i=1}^{|e|}\mathscr{R}(e_{i-|e|})$ such that $\varphi_{e_{i-|e|}}(R_i)=R_{i-1}$ for $2\leqslant i\leqslant|e|$, the finite sequence $\langle R'_1,\ldots,R'_{|e|}\rangle$ is an admissible choice of Player II for s in $G(\boldsymbol{DC},X)$.

Let $\mathscr{R}(\varnothing)=\{X\times Y\}$. Assume that $\{\mathscr{G}(e)\colon e\in N^*$ with $|e|\leqslant m-1\}$, $\{\mathscr{R}(e)\colon e\in N^*$ with $|e|\leqslant m\}$ and $\{\varphi_e\colon e\in N^*\setminus\{\varnothing\}$ with $|e|\leqslant m\}$ satisfying the above conditions have been already constructed. Let us take any $e\in N^*$ with $|e|=m$. Here we construct $\mathscr{G}(e)$ and $\{\mathscr{R}(e\oplus n),\ \varphi_{e\oplus n}\colon n\in N\}$ satisfying the conditions (2.1)–(2.7). Now, fix an $R\in\mathscr{R}(e)$. We take $(R_0,R_1,\ldots,R_m)\in\prod_{i=0}^{m}\mathscr{R}(e_{i-m})$ such that $R_m=R$ and $\varphi_{e_{i-m}}(R_i)=R_{i-1}$ for $1\leqslant i\leqslant m$. It follows from the assumption (2.7) that there exists a discrete collection $\{C_\alpha\colon \alpha\in\Omega(R)\}$ by compact sets in R' such that

$$s(R'_0,R'_1,\ldots,R'_m)=\bigcup\{C_\alpha\colon \alpha\in\Omega(R)\}.$$

For each $\alpha\in\Omega(R)$, we can choose a collection

$$\{G_k(\lambda_1,\ldots,\lambda_i;\alpha)\colon (\lambda_1,\ldots,\lambda_i)\in\Lambda(\alpha)\text{ and }k=1,2\}$$

of open sets in X, where $\Lambda(\alpha)\subset\bigcup\{\Lambda^n\colon n\in N\}$, such that

(i) $C_\alpha\subset G_1(\lambda_1,\ldots,\lambda_i;\alpha)\cup G_2(\lambda_1,\ldots,\lambda_i;\alpha)$,

(ii) $G_k(\lambda_1,\ldots,\lambda_i;\alpha)\times V(\lambda_1,\ldots,\lambda_i)\subset O_k$ for $k=1,2$,

(iii) $\{V(\lambda_1,\ldots,\lambda_i)\colon (\lambda_1,\ldots,\lambda_i)\in\Lambda(\alpha)\}$ is discrete in Y and its union is R''.

Moreover, for each $\alpha\in\Omega(R)$ we can choose a collection

$$\{C_k(\lambda_1,\ldots,\lambda_i;\alpha)\colon (\lambda_1,\ldots,\lambda_i)\in\Lambda(\alpha)\text{ and }k=1,2\}$$

of compact sets in X such that

(iv) $C_1(\lambda_1,\ldots,\lambda_i;\alpha)\cup C_2(\lambda_1,\ldots,\lambda_i;\alpha)=C_\alpha$,

(v) $C_k(\lambda_1,\ldots,\lambda_i;\alpha)\subset G_k(\lambda_1,\ldots,\lambda_i;\alpha)$ for $k=1,2$.

We put $\Lambda(R)=\bigcup\{\Lambda(\alpha)\colon \alpha\in\Omega(R)\}$. Since X is normal, for each $(\lambda_1,\ldots,\lambda_i)\in\Lambda(R)$ and $k=1,2$, there exists a cozero-set $U_k(\lambda_1,\ldots,\lambda_i)$ of X such that

(vi) $\bigcup\{C_k(\lambda_1,\ldots,\lambda_i;\alpha)\colon \alpha\in\Omega(R)$ such that $(\lambda_1,\ldots,\lambda_i)\in\Lambda(\alpha)\}\subset U_k(\lambda_1,\ldots,\lambda_i)\subset\bigcup\{G_k(\lambda_1,\ldots,\lambda_i;\alpha)\colon \alpha\in\Omega(R)$ such that $(\lambda_1,\ldots,\lambda_i)\in\Lambda(\alpha)\}$. So, we put

$$\mathscr{G}(e,R)=\{U_k(\lambda_1,\ldots,\lambda_i)\times V(\lambda_1,\ldots,\lambda_i)\colon (\lambda_1,\ldots,\lambda_i)\in\Lambda(R)\text{ and }k=1,2\}.$$

Since X is θ-expandable (cf. [4, Theorem 2.3]), there exists a sequence

$$\{\mathscr{W}_n(R)=\{W_{\alpha,n}\colon \alpha\in\Omega(R)\}\colon n\in N\}$$

of collections of open sets in X, such that for each $\alpha\in\Omega(R)$ and $n\in N$ C_α is contained in $W_{\alpha,n}$ and for each $x\in X$ one can choose some $n\in N$ such that $\mathscr{W}_n(R)$ is point-finite at x. For each $(\lambda_1,\ldots,\lambda_i)\in\Lambda(R)$ and $n\in N$ we put

$$R_n(\lambda_1,\ldots,\lambda_i)=R'_n(\lambda_1,\ldots,\lambda_i)\times R''_n(\lambda_1,\ldots,\lambda_i),$$

where

$$R'_n(\lambda_1,\ldots,\lambda_i)=R'\setminus\big(\bigcup\{U_k(\lambda_1,\ldots,\lambda_j)\colon (\lambda_1,\ldots,\lambda_j)\in\Lambda(R),\ j\leqslant i\text{ and }k=1,2\}\cup\bigcup\{W_{\alpha,n}\colon \alpha\in\Omega(R)\text{ such that }(\lambda_1,\ldots,\lambda_j)\notin\Lambda(\alpha)\text{ for each }j\leqslant i\}\big)$$ and

$$R''_n(\lambda_1,\ldots,\lambda_i)=V(\lambda_1,\ldots,\lambda_i).$$

Moreover, let us put

$$\mathscr{R}(e\oplus n, R) = \{R_n(\lambda_1, \ldots, \lambda_i) \colon (\lambda_1, \ldots, \lambda_i) \in \Lambda(R)\}\,.$$

Here, running $R \in \mathscr{R}(e)$, we set

$$\mathscr{G}(e) = \bigcup\{\mathscr{G}(e, R) \colon R \in \mathscr{R}(e)\} \quad \text{and} \quad \mathscr{R}(e\oplus n) = \bigcup\{\mathscr{R}(e\oplus n, R) \colon R \in \mathscr{R}(e)\}$$

for each $n \in N$. The function $\varphi_{e\oplus n} \colon \mathscr{R}(e\oplus n) \to \mathscr{R}(e)$ is defined by $\varphi_{e\oplus n}(Q) = R$ for each $Q \in \mathscr{R}(e\oplus n, R)$ and $R \in \mathscr{R}(e)$. Now, we verify that the above constructions satisfy the conditions (2.1)–(2.7). It is obvious from the constructions that the conditions (2.1), (2.3) and (2.5) are satisfied. It follows from the assumption (2.4) that $\{R'' \colon R \in \mathscr{R}(e)\}$ is σ-discrete in Y, where note that $R_1'' = R_2''$ can occur even if $R_1 \neq R_2$. Since each $V(\lambda_1, \ldots, \lambda_i)$, $(\lambda_1, \ldots, \lambda_i) \in \Lambda(R)$, is contained in R'', $\{V(\lambda_1, \ldots, \lambda_i) \colon (\lambda_1, \ldots, \lambda_i) \in \Lambda(R) \text{ and } R \in \mathscr{R}(e)\}$ is σ-discrete in Y, where it also allows the repetition. Hence, $\{G'' \colon G \in \mathscr{G}(e)\}$ is so. Thus $\mathscr{G}(e)$ is σ-discrete in $X \times Y$. Since each member of $\mathscr{G}(e)$ is a cozero rectangle, the condition (2.2) is satisfied. Pick any $V \in \mathscr{V}$ and $n \in N$. Since $\{W \in \mathscr{V} \colon V \subset W\}$ is finite, it follows from the assumption (2.4) that $\{R \in \mathscr{R}(e) \colon V \subset R''\}$ is also finite. For each $R \in \mathscr{R}(e)$, $\{Q \in \mathscr{R}(e\oplus n, R) \colon Q'' = V\}$ consists of at most one member. For each $Q \in \mathscr{R}(e\oplus n, R)$ such that $V \not\subset R''$, we have $Q'' \neq V$. Hence, $\{Q \in \mathscr{R}(e\oplus n) \colon Q'' = V\}$ is finite. The condition (2.4) is satisfied. Assume $p = (x, y) \in R \in \mathscr{R}(e)$ and $p \notin \bigcup \mathscr{G}(e)$. Let $y = (\lambda_1, \lambda_2, \ldots) \in Y$. We can choose some $n \in N$ such that $\mathscr{W}_n(R)$ is point-finite at x. In case of $x \in \bigcup \mathscr{W}_n(R)$: Let $\{\alpha \in \Omega(R) \colon x \in W_{\alpha,n}\} = \{\alpha_1, \ldots, \alpha_l\}$. For each $j \leqslant l$, it follows from (iii) and $y \in R''$ that there exists some $i_j \in N$ such that $(\lambda_1, \ldots, \lambda_{i_j}) \in \Lambda(\alpha_j)$. Let us put $i_0 = \max\{i_j \colon j \leqslant l\}$. In case of $x \notin \bigcup \mathscr{W}_n(R)$: Take any $\alpha_0 \in \Omega(R)$. As the above, we can take some $i_0 \in N$ such that $(\lambda_1, \ldots, \lambda_{i_0}) \in \Lambda(\alpha_0)$. In both cases, note that we have $(\lambda_1, \ldots, \lambda_{i_0}) \in \Lambda(R)$. So $R_n(\lambda_1, \ldots, \lambda_{i_0}) \in \mathscr{R}(e\oplus n)$ and $\varphi_{e\oplus n}(R_n(\lambda_1, \ldots, \lambda_{i_0})) = R$ are clear. We show $p \in R_n(\lambda_1, \ldots, \lambda_{i_0})$. It suffices to show $x \in R_n'(\lambda_1, \ldots, \lambda_{i_0})$. Assume $x \in W_{\alpha,n}$ and $\alpha \in \Omega(R)$. Since α coincides with some α_k, where $k \leqslant l$, we have $(\lambda_1, \ldots, \lambda_{i_k}) \in \Lambda(\alpha)$ and $i_k \leqslant i_0$. Hence we obtain

$$x \notin \bigcup\{W_{\beta,n} \colon \beta \in \Omega(R) \text{ such that } (\lambda_1, \ldots, \lambda_j) \notin \Lambda(\beta) \text{ for each } j \leqslant i_0\}\,.$$

Assume $x \in U_k(\lambda_1, \ldots, \lambda_j)$, where $(\lambda_1, \ldots, \lambda_j) \in \Lambda(R)$, $j \leqslant i_0$ and $k = 1, 2$. Then we have

$$p \in U_k(\lambda_1, \ldots, \lambda_j) \times V(\lambda_1, \ldots, \lambda_j) \in \mathscr{G}(e, R) \subset \mathscr{G}(e)\,,$$

which contradicts to $p \notin \bigcup \mathscr{G}(e)$. Hence we obtain

$$x \notin \bigcup\{U_k(\lambda_1, \ldots, \lambda_j) \colon (\lambda_1, \ldots, \lambda_j) \in \Lambda(R), j \leqslant i_0 \text{ and } k = 1, 2\}\,.$$

Thus it follows from $x \in R'$ that x is in $R_n'(\lambda_1, \ldots, \lambda_{i_0})$. The condition (2.6) is satisfied. Pick any $R \in \mathscr{R}(e)$ and $n \in N$. Let $(\lambda_1, \ldots, \lambda_i) \in \Lambda(R)$ and $\alpha \in \Omega(R)$. In case of $(\lambda_1, \ldots, \lambda_j) \notin \Lambda(\alpha)$ for each $j \leqslant i$: Since $R_n'(\lambda_1, \ldots, \lambda_i)$ is disjoint from $W_{\alpha,n}$, it has no intersection with C_α. In case of $(\lambda_1, \ldots, \lambda_j) \in \Lambda(\alpha)$ for some $j \leqslant i$: From (iv) and (vi), the union of $U_1(\lambda_1, \ldots, \lambda_j)$ and $U_2(\lambda_1, \ldots, \lambda_j)$ contains C_α. Moreover, it is disjoint

from $R_n'(\lambda_1, \ldots, \lambda_i)$. So $R_n'(\lambda_1, \ldots, \lambda_i)$ and C_α have no intersection. Hence we have

$$R_n'(\lambda_1, \ldots, \lambda_i) \cap \bigcup\{C_\alpha \colon \alpha \in \Omega(R)\} = \emptyset$$

for each $(\lambda_1, \ldots, \lambda_i) \in \Lambda(R)$. One can see from the assumption (2.7) that the condition (2.7) is satisfied. From the facts mentioned above, the desired constructions are completed by induction.

Now, we set $\mathscr{G} = \bigcup\{\mathscr{G}(e) \colon e \in N^*\}$. Assume $p \in X \times Y \setminus \bigcup \mathscr{G}$. From (2.1) and (2.6), we can inductively choose some infinite sequences $(n_1, n_2, \ldots) \in N^N$ and $\langle R_1, R_2, \ldots\rangle$ such that $p \in R_k \in \mathscr{R}(n_1, \ldots, n_k)$ and $\varphi_{(n_1,\ldots,n_k)}(R_k) = R_{k-1}$ for each $k \in N$. Then $\bigcap\{R_k' \colon k \in N\} \neq \emptyset$ follows. However, from (2.7) and the definition of s, we have $\bigcap\{R_k' \colon k \in N\} = \emptyset$. This is a contradiction. Hence $\mathscr{G}$ is a cover of $X \times Y$. It follows from (2.2) and (2.5) that $\mathscr{G}$ is a σ-discrete cozero-set refinement of $\mathscr{O}$. Since $\mathscr{G}$ is a normal cover of $X \times Y$ (cf. [5, Theorem 1.2]), $\mathscr{O}$ is so. The proof of Theorem 2.2 is complete.

COROLLARY 2.2. *Each normal subparacompact C-scattered space is a P-space.*

COROLLARY 2.3. *Let X be a normal metacompact space and Y a metacompact space. If $I(\boldsymbol{DC}, X)$, then the product space $X \times Y$ is metacompact.*

Corollary 2.2 follows from [9, Theorem 9.7] and Theorem 2.2. Corollary 2.3 follows from Theorems 2.1 and 2.2.

§ 3. Subparacompactness of product spaces.

We introduced the concept of D-product in [15], which is restated here.

DEFINITION. A product space $X \times Y$ is said to be a *D-product* if for each closed set M and open set O in $X \times Y$ with $M \subset O$ there exists a σ-discrete collection $\mathscr{F}$ by closed rectangles such that $M \subset \bigcup \mathscr{F} \subset O$.

The following result has been stated in "Added in proof" of [15]. However, for the accuracy, we restate it with the proof.

THEOREM 3.1. *Let X be a regular subparacompact space and Y a subparacompact space. If $I(\boldsymbol{DC}, X)$, then the product space $X \times Y$ is subparacompact and is a D-product.*

Proof. The proof is obtained by modifying that of [15, Theorem 2.2]. Let $\mathscr{O}$ be any open cover of $X \times Y$. It suffices to show that $\mathscr{O}$ has a σ-discrete refinement by closed rectangles. Let s be a winning strategy of Player I in $G(\boldsymbol{DC}, X)$.

First, we shall construct two sequences $\{\mathscr{F}_n \colon n \geqslant 0\}$ and $\{\mathscr{R}_n \colon n \geqslant 0\}$ of collections by closed rectangles and a function $\varphi_n \colon \mathscr{R}_n \to \mathscr{R}_{n-1}$ for each $n \in N$, satisfying the following conditions (3.1)–(3.6) for each $n \in N$:

(3.1) $\mathscr{F}_0 = \{\emptyset\}$ and $\mathscr{R}_0 = \{X \times Y\}$.

(3.2) $\mathscr{F}_n$ is σ-discrete.

(3.3) $\mathscr{R}_n$ is σ-discrete.

(3.4) $\mathscr{F}_n \prec \mathscr{O}$.

(3.5) If $p \in R_{n-1} \in \mathscr{R}_{n-1}$ and $p \notin \bigcup \mathscr{F}_n$, then there exists some $R_n \in \mathscr{R}_n$ such that $p \in R_n$ and $\varphi_n(R_n) = R_{n-1}$.

(3.6) For each $(R_1, \ldots, R_n) \in \prod_{i=1}^{n} \mathscr{R}_i$ such that $\varphi_i(R_i) = R_{i-1}$ for $2 \leqslant i \leqslant n$, the finite sequence $\langle R_1', \ldots, R_n' \rangle$ is an admissible choice of Player II for s in $G(\mathbf{DC}, X)$.

Let $\mathscr{F}_0 = \{\emptyset\}$ and $\mathscr{R}_0 = \{X \times Y\}$. Assume that $\{\mathscr{F}_i: i \leqslant n\}$, $\{\mathscr{R}_i: i \leqslant n\}$ and $\{\varphi_i: i \leqslant n\}$ satisfying the above conditions have been already constructed. Now, fix an $R \in \mathscr{R}_n$. We take $(R_0, R_1, \ldots, R_n) \in \prod_{i=0}^{n} \mathscr{R}_i$ such that $R_n = R$ and $\varphi_i(R_i) = R_{i-1}$ for $1 \leqslant i \leqslant n$. It follows from the assumption (3.6) that there exists a discrete collection $\{C_\alpha: \alpha \in \Omega(R)\}$ by compact sets in R' whose union is $s(R_0', R_1', \ldots, R_n')$. Since R' is subparacompact and $\{R' \setminus \bigcup \{C_\beta: \beta \in \Omega(R) \setminus \{\alpha\}\}: \alpha \in \Omega(R)\}$ is an open cover of R', there exists a closed cover $\{F_{\alpha,k}: \alpha \in \Omega(R) \text{ and } k \in N\}$ of R' such that $\{F_{\alpha,k}: \alpha \in \Omega(R)\}$ is discrete in X for each $k \in N$ and $F_{\alpha,k} \subset R' \setminus \bigcup \{C_\beta: \beta \in \Omega(R) \setminus \{\alpha\}\}$ for each $\alpha \in \Omega(R)$ and $k \in N$. By subparacompactness of R'', for each $\alpha \in \Omega(R)$ we can choose a collection

$$\mathscr{F}_\alpha^k(R) = \{\operatorname{Cl} U_{\lambda,i}^k \times H_\lambda: i \leqslant m_\lambda \text{ and } \lambda \in \Lambda_k(\alpha)\}$$

by closed rectangles such that

(i) $U_{\lambda,i}^k$ is open in $F_{\alpha,k}$,

(ii) $C_\alpha \cap F_{\alpha,k} \subset \bigcup \{U_{\lambda,i}^k: i \leqslant m_\lambda\}$,

(iii) $\mathscr{F}_\alpha^k(R) \prec \mathcal{O}$,

(iv) $\{H_\lambda: \lambda \in \Lambda_k(\alpha)\}$ is σ-discrete in Y and its union is R''.

For each $\lambda \in \Lambda_k(\alpha)$, $\alpha \in \Omega(R)$ and $k \in N$ we put

$$R_k(\alpha, \lambda) = (F_{\alpha,k} \setminus \bigcup \{U_{\lambda,i}^k: i \leqslant m_\lambda\}) \times H_\lambda .$$

Here, running $R \in \mathscr{R}_n$, we set

$$\mathscr{F}_{n+1} = \bigcup \{\mathscr{F}_\alpha^k(R): \alpha \in \Omega(R),\ R \in \mathscr{R}_n \text{ and } k \in N\},$$
$$\mathscr{R}_{n+1} = \{R_k(\alpha, \lambda): \lambda \in \Lambda_k(\alpha),\ \alpha \in \Omega(R),\ R \in \mathscr{R}_n \text{ and } k \in N\} .$$

Moreover, $\varphi_{n+1}: \mathscr{R}_{n+1} \to \mathscr{R}_n$ is defined by $\varphi_{n+1}(R_k(\alpha, \lambda)) = R$ for each $\lambda \in \Lambda_k(\alpha)$, $\alpha \in \Omega(R)$, $R \in \mathscr{R}_n$ and $k \in N$. We can similarly check that $\mathscr{F}_{n+1}$, $\mathscr{R}_{n+1}$ and φ_{n+1} satisfy the conditions (3.1)–(3.6).

Now, we set $\mathscr{F} = \bigcup \{\mathscr{F}_n: n \geqslant 0\}$. Then it follows similarly from (3.1), (3.5) and (3.6) that $\mathscr{F}$ is a cover of $X \times Y$. Hence, by (3.2) and (3.4), $\mathscr{F}$ is a σ-discrete closed refinement of $\mathcal{O}$. Moreover, each member of $\mathscr{F}$ is a closed rectangle. The proof is complete.

THEOREM 3.2. *Let $X \times Y$ be a D-product such that either X or Y is non-empty and let $I(\mathbf{DC}, X)$. Then we have the following:*

(a) *If $X \times Y$ is normal, then* $\dim(X \times Y) \leqslant \dim X + \dim Y$.

(b) *If $X \times Y$ is totally normal, then* $\operatorname{Ind}(X \times Y) \leqslant \operatorname{Ind} X + \operatorname{Ind} Y$.

This result has been essentially proved in [15]. In fact, using [15, Theorem 2.1], the proof is quite similar to that of [15, Theorem 4.1].

From Theorems 3.1 and 3.2, it follows

COROLLARY 3.1. *Let X be a subparacompact space which has a σ-closure-preserving cover by compact sets and Y a subparacompact space. Assuming that either X or Y is non-empty, we have the following:*

(a) *If $X \times Y$ is normal, then* $\dim(X \times Y) \leqslant \dim X + \dim Y$.

(b) *If $X \times Y$ is totally normal, then* $\operatorname{Ind}(X \times Y) \leqslant \operatorname{Ind} X + \operatorname{Ind} Y$.

Remark. H. Ohta [6] has shown the existences of two spaces X and Y described in Corollary 3.1 such that $X \times Y$ is perfectly normal but not rectangular. So, our Corollary 3.1 cannot be reduced to Pasynkov's product theorems [7, Theorems 1 and 2].

§ 4. Submetacompactness of product spaces. We use the name "submetacompact" instead of "θ-refinable" as in [3]. A θ-sequence is defined in, for example, [3, Definition 1.2]. Moreover, we need the following definition.

DEFINITION. A sequence $\{\mathscr{V}_n: n \in N\}$ of covers of X is said to be a *strong θ-sequence* if for each $x \in X$ there exists some $n_x \in N$ such that $\mathscr{V}_n$ is point finite at x for each $n \geqslant n_x$.

A space X is said to be *strongly submetacompact* if every open cover of X has a strong θ-sequence of open refinements.

It seems that this definition was first introduced by Y. Uemura [12], and he showed in it that each product space of a locally compact submetacompact space and a strongly submetacompact space is submetacompact. Note that a locally compact submetacompact space has a σ-closure-preserving closed cover by compact sets (cf. [2, Corollary 3.17]). So, we shall extend this result in terms of a topological game.

Before the arguments, we need the following notations. For each $n \in N$, $M(2 \times n)$ denotes the set of all $2 \times n$ matrices consisting of natural numbers, and $M(2 \times 0)$ denotes $\{\emptyset\}$. Let $M^* = \bigcup \{M(2 \times n): n \geqslant 0\}$. When

$$A = \begin{pmatrix} k_1 \ldots k_n \\ m_1 \ldots m_n \end{pmatrix} \in M(2 \times n) ,$$

we use the following notation;

$$A_{-j} = \begin{pmatrix} k_1 \ldots k_{n-j} \\ m_1 \ldots m_{n-j} \end{pmatrix} \quad \text{for} \quad 0 \leqslant j \leqslant n-1 \quad (A_{-n} = \emptyset) .$$

For a subset E of a space X and a collection $\mathscr{U}$ of subsets in X, $\mathscr{U}|E$ denotes $\{U \cap E: U \in \mathscr{U}\}$.

THEOREM 4.1. *Let X be a regular submetacompact P-space and Y a strongly submetacompact space. If $I(\mathbf{DC}, X)$, then the product space $X \times Y$ is submetacompact.*

Proof. Let s be a winning strategy of Player I in $G(\mathbf{DC}, X)$ and

$$F: \bigcup \{(2^X)^n: n \in N\} \to 2^X$$

a function described in Proposition 1.1. Let $\mathcal{O}$ be any monotone open cover of $X \times Y$. It suffices to show that $\mathcal{O}$ has a θ-sequence of open refinements (cf. [3, Theorem 1.6]).

First, we shall construct three families $\{\mathcal{G}(A)\colon A\in M^*\}$, $\{\mathcal{R}(A)\colon A\in M^*\}$ and $\{\mathcal{H}(A)\colon A\in M^*\}$ of collections by rectangles in $X\times Y$ and construct two functions $\varphi_A\colon \mathcal{R}(A)\to\mathcal{R}(A_{-1})$ and $\psi_A\colon \mathcal{H}(A)\to\mathcal{H}(A_{-1})$ for each $A\in M^*\setminus\{\emptyset\}$, satisfying the following conditions (4.1)–(4.11) for each $A\in M^*\setminus\{\emptyset\}$:

(4.1) $\mathcal{G}(\emptyset)=\{\emptyset\}$ and $\mathcal{R}(\emptyset)=\{X\times Y\}=\{H(X\times Y)\}=\mathcal{H}(\emptyset)$.

(4.2) $\mathcal{G}(A)$ is a collection by open rectangles.

(4.3) $\mathcal{R}(A)$ is a collection by closed × open rectangles.

(4.4) $\mathcal{H}(A)=\{H(R)\colon R\in\mathcal{R}(A)\}$ and it is a collection by open rectangles such that $R\subset H(R)$ for each $R\in\mathcal{R}(A)$.

(4.5) If $\mathcal{H}(A)$ is point-finite at p, where $A\in M^*$, then there exists some $B\in M^*$ such that $B_{-1}=A$ and both $\mathcal{G}(B)$ and $\mathcal{H}(B)$ are point-finite at p.

(4.6) $\mathcal{G}(A)\prec\mathcal{O}$.

(4.7) $H(\varphi_A(R))=\psi_A(H(R))$ for each $R\in\mathcal{R}(A)$.

(4.8) $H(R)\subset\psi_A(H(R))$ for each $R\in\mathcal{R}(A)$.

(4.9) If $p\in R\in\mathcal{R}(A_{-1})$ and $p\notin\bigcup\mathcal{G}(A)$, then there exists some $Q\in\mathcal{R}(A)$ such that $p\in Q$ and $\varphi_A(Q)=R$.

(4.10) For each $(R_1,\dots,R_n)\in\prod_{i=1}^{n}\mathcal{R}(A_{i-n})$, where $A\in M(2\times n)$, such that $\varphi_{A_{i-n}}(R_i)=R_{i-1}$ for $2\leqslant i\leqslant n$, the finite sequence $\langle R_1',\dots,R_n'\rangle$ is an admissible choice of Player II for s in $G(\boldsymbol{DC},X)$.

(4.11) For each $(R_0,\dots,R_n)\in\prod_{i=0}^{n}\mathcal{R}(A_{i-n})$, where $A\in M(2\times n)$, such that $\varphi_{A_{i-n}}(R_i)=R_{i-1}$ for $1\leqslant i\leqslant n$,

$$F(R_0',\dots,R_n')\cap H(R_n)'=\emptyset.$$

These constructions are similar to ones in the proof of Theorem 2.1 and can be obtained by their modifications. The detail is left to the reader.

For each $A\in M(2\times n)$ and $n\geqslant 0$, we set

$$\mathcal{U}(A)=\bigcup\{\mathcal{G}(A_{i-n})\colon 0\leqslant i\leqslant n\}\cup\mathcal{O}|(\bigcup\mathcal{H}(A)).$$

Then we show that $\{\mathcal{U}(A)\colon A\in M^*\}$ is a θ-sequence of open refinements of $\mathcal{O}$. Pick any $A\in M^*$ and $p\in X\times Y$. Let $A\in M(2\times n)$. Assume $p\notin\bigcup\mathcal{G}(A_{i-n})$ for $0\leqslant i\leqslant n$. From (4.1) and (4.9), we can choose some $R\in\mathcal{R}(A)$ containing p. By $R\subset H(R)$, we have $p\in\bigcup\mathcal{H}(A)$. Hence each $\mathcal{U}(A)$ is a cover of $X\times Y$. Moreover, by (4.2) and (4.6), each $\mathcal{U}(A)$ is an open refinement of $\mathcal{O}$. Again, pick any $p\in X\times Y$. By (4.1) and (4.5), we can inductively choose some

$$T=\begin{pmatrix}k_1 & k_2 & \dots\\ m_1 & m_2 & \dots\end{pmatrix}$$

such that both $\mathcal{G}(T_n)$ and $\mathcal{H}(T_n)$ are point-finite at p for each $n\in N$, where

$$T_n=\begin{pmatrix}k_1 & \dots & k_n\\ m_1 & \dots & m_n\end{pmatrix}\in M(2\times n).$$

Going through the processes similar to ones of Claims 1–3 in the proof of Theorem 2.1, we obtain $p\notin\bigcap\{\bigcup\mathcal{H}(T_n)\colon n\geqslant 0\}$. So we can choose some $n_0\in N$ such that $p\notin\bigcup\mathcal{H}(T_{n_0})$. Let $A=T_{n_0}$. Since $A\in M(2\times n_0)$ and $\mathcal{G}(A_{i-n_0})$ is point-finite at p for $0\leqslant i\leqslant n_0$, $\mathcal{U}(A)$ is point-finite at p. The proof is complete.

Next, we consider what kinds of submetacompact spaces are strongly submetacompact.

PROPOSITION 4.1. *Spaces which are the countable union of closed metacompact subspaces are strongly submetacompact.*

Proof. Let X be a space which has a countable closed cover $\{X_n\colon n\in N\}$ such that each X_n is metacompact. Since $\bigcup\{X_i\colon i\leqslant n\}$ is metacompact for each $n\in N$, we can assume $X_1\subset X_2\subset\dots$ Let $\mathcal{O}$ be any open cover of X. For each $n\in N$, there exists a collection $\mathcal{V}_n$ of open sets in X such that $\mathcal{V}_n\prec\mathcal{O}$, $\bigcup\mathcal{V}_n\supset X_n$ and it is point-finite in X_n. Let us put $\mathcal{U}_n=\mathcal{V}_n\cup\mathcal{O}|(X\setminus X_n)$ for each $n\in N$. Then the sequence $\{\mathcal{U}_n\colon n\in N\}$ of open covers is a strong θ-sequence of open refinements of $\mathcal{O}$.

PROPOSITION 4.2. *Subparacompact spaces are strongly submetacompact.*

Proof. Let X be a subparacompact space. Let $\mathcal{O}$ be any open cover of X. There exists a closed refinement $\mathcal{F}=\bigcup\{\mathcal{F}_i\colon i\in N\}$ of $\mathcal{O}$ such that $\mathcal{F}_i=\{F_\lambda\colon\lambda\in\Lambda_i\}$ is discrete in X for each $i\in N$. For each $\lambda\in\Lambda_i$ and $i\in N$, we choose an $O_\lambda\in\mathcal{O}$ containing F_λ. Since X is discretely subexpandable (cf. [4, Theorem 3.2]), for each $i\in N$ there exists a sequence $\{\mathcal{V}_{i,n}=\{V_{\lambda,n}\colon\lambda\in\Lambda_i\}\colon n\in N\}$ of collections of open sets in X, satisfying

(i) $F_\lambda\subset V_{\lambda,n}\subset V_{\lambda,n-1}\subset V_{\lambda,0}=O_\lambda$ for each $\lambda\in\Lambda_i$ and $n\in N$, and

(ii) for each $x\in X$ we can choose some $n\in N$ such that x is contained in at most one member of $\mathcal{V}_{i,n}$.

Let $E_0=\emptyset$ and $E_i=\bigcup\{\bigcup\mathcal{F}_j\colon j\leqslant i\}$ for each $i\in N$. Then $\{E_i\colon i\in N\}$ is a monotone closed cover of X. For each $i,n\in N$, we put $\mathcal{W}_{i,n}=\{V_{\lambda,n}\setminus E_{i-1}\colon\lambda\in\Lambda_i\}$. Moreover, for each $i,n\in N$ we set

$$\mathcal{U}_{i,n}=\bigcup\{\mathcal{W}_{j,n}\colon j\leqslant i\}\cup\mathcal{O}|(X\setminus E_i).$$

Now, we show that $\{\mathcal{U}_{i,n}\colon i,n\in N\}$ is a strong θ-sequence of open refinements of $\mathcal{O}$. It is easy to check that each $\mathcal{U}_{i,n}$ is a cover of X. So each $\mathcal{U}_{i,n}$ is an open refinement of $\mathcal{O}$. Let $x\in X$. We take some $i_0\in N$ such that $x\in E_{i_0}\setminus E_{i_0-1}$. For each $j\leqslant i_0$ we choose some $n_j\in N$ such that x is contained in at most one member of $\mathcal{V}_{j,n_j}$. Here we put $n_0=\max\{n_j\colon j\leqslant i_0\}$. Let us pick any $i,n\in N$ such that $i\geqslant i_0$ and $n\geqslant n_0$. It follows from the choice of n_0 and (i) that x is contained in at most i_0 members of $\bigcup\{\mathcal{W}_{j,n}\colon j\leqslant i_0\}$. For each $j>i_0$, $\bigcup\mathcal{W}_{j,n}$ and E_{i_0} are disjoint. So x is contained in no member of $\bigcup\{\mathcal{W}_{j,n}\colon j>i_0\}$. Clearly, we have $x\notin X\setminus E_i$. Hence x is contained in at most i_0 members of $\mathcal{U}_{i,n}$. Therefore, $\mathcal{U}_{i,n}$ is point-finite at x for each $i\geqslant i_0$ and $n\geqslant n_0$. This implies that $\{\mathcal{U}_{i,n}\colon i,n\in N\}$ is a strong θ-sequence of open refinements of $\mathcal{O}$. The proof is complete.

§ 5. Outer-almost $\aleph_0$-expandability. In this section, let $\boldsymbol{K}$ and $\boldsymbol{L}$ be non-void classes of spaces which are hereditary with respect to closed sets. According to [9], we denote by $\boldsymbol{\sigma K}$ the class of all spaces $X=\bigcup\{X_n\colon n\in N\}$ such that

$$\{X_n\colon n\in N\}\subset 2^X\cap\boldsymbol{K},$$

and by FK the class of all spaces $X = \bigcup\{X_i\colon i \leqslant n\}$ such that $\{X_i\colon i \leqslant n\} \subset 2^X \cap K$ and $n \in N$. Moreover, we denote by DK the class of all spaces which have a discrete cover consisting of members of K.

It was asked in [9, Question 4.10] whether $I(\sigma K, X)$ implies $I(K, X)$. Professor Telgársky and the referee of this paper have kindly informed the author that this question is negative under the continuum hypothesis. Indeed, the author has been pointed out the following.

EXAMPLE 5.1. Assuming CH, there exists a Tychonoff space X such that $I(\sigma 1, X)$ but not $I(1, X)$ where 1 denotes the class of all one-point spaces.

Let Y be a Lusin set such that $G(1, X)$ is undetermined. Under CH, the existence of Y is guaranteed in [1, Corollary 4]. Let D be a countable dense subset of Y. Put $X = Y_D$, which is defined as follows; neighborhood base of each $x \in D$ is the same as in Y and $\{x\}$ is open in X for each $x \in X \setminus D$. It is easily verified that X is a desired space.

Now, take a class L such that, for a space X, $K \cap 2^X \subset L \cap 2^X \subset \sigma K \cap 2^X$. Here, we consider when $I(L, X)$ implies $I(K, X)$.

DEFINITION. A class L is said to be *outer-almost* $\aleph_0$*-expandable* in a space X with respect to K if each $F \in 2^X \cap L$ is a union of $\{F_n\colon n \in N\} \subset 2^X \cap K$ such that there exists a collection $\{W_n\colon n \in N\}$ of open sets in X, satisfying the following;

(i) it is point-finite at each point of $X \setminus F$,

(ii) $F_n \subset W_n$ for each $n \in N$.

Note that the condition (i) can be replaced by

(i)′ $W_{n+1} \subset W_n$ for each $n \in N$ and $\bigcap\{W_n\colon n \in N\} \setminus F = \emptyset$.

THEOREM 5.1. *If a class L is outer-almost $\aleph_0$-expandable in a space X with respect to K, then $I(L, X)$ implies $I(K, X)$.*

Proof. Let t be a winning strategy of Player I in $G(L, X)$. We use the notations of N^*, $e \oplus n$, e_{-j}, $|e|$, Σe and $l(e)$, which have been explained before Theorem 2.2.

Now, assume that we have already constructed an admissible sequence $\langle E_1, H_1, \ldots, E_m, H_m\rangle$ in $G(FK, X)$ such that $E_{i+1} = s(H_0, H_1, \ldots, H_m)$ for $0 \leqslant i \leqslant m-1$, where $H_0 = X$, and such that there exist two families

$$\{\mathscr{F}(e)\colon e \in N^* \text{ with } \Sigma e \leqslant m-1\}$$

and $\{\mathscr{W}(e)\colon e \in N^* \text{ with } \Sigma e \leqslant m-1\}$ of collections of subsets in X and a collection $\{R(e)\colon e \in N^* \text{ with } \Sigma e \leqslant m-1\}$ of closed sets in X, satisfying the following conditions (5.1)–(5.9) for each $e \in N^*$ with $\Sigma e \leqslant m-1$:

(5.1) $\mathscr{F}(e) = \{F(e \oplus n)\colon n \in N\} \subset 2^X \cap K$.

(5.2) $\mathscr{W}(e) = \{W(e \oplus n)\colon n \in N\}$ consists of open sets in X.

(5.3) $W(e \oplus n+1) \subset W(e \oplus n)$ for each $n \in N$.

(5.4) $\bigcap\{W(e \oplus n)\colon n \in N\} \setminus \bigcup \mathscr{F}(e) = \emptyset$.

(5.5) $F(e \oplus n) \subset W(e \oplus n)$ for each $n \in N$.

(5.6) $E_k = \bigcup\{F(e)\colon \Sigma e = k\}$ for each $k \leqslant m$.

(5.7) $R(\emptyset) = X$ and $R(e) = R(e_{-1}) \cap H_m \setminus W(e)$.

(5.8) $\langle R(e_{1-|e|}), \ldots, R(e_{-1}), R(e)\rangle$ is an admissible choice of Player II for t in $G(L, X)$.

(5.9) $t(R(\emptyset), R(e_{1-|e|}), \ldots, R(e_{-1}), R(e)) = \bigcup \mathscr{F}(e)$.

We pick any $e \in N^*$ with $\Sigma e = m$. We put $R(e) = R(e_{-1}) \cap H_m \setminus W(e)$. For each $n \leqslant l(e)$, we have $F(e_{-1} \oplus n) \subset E_{\Sigma e_{-1}+n} \subset X \setminus H_m$. For each $n > l(e)$, we have $F(e_{-1} \oplus n) \subset W(e)$. These follows from the inductive assumptions. Since

$$t(R(\emptyset), R(e_{1-|e|}), \ldots, R(e_{-1}))$$

is represented by $\bigcup \mathscr{F}(e_{-1}) = \bigcup\{F(e_{-1} \oplus n)\colon n \in N\}$, it is disjoint from $R(e)$. By $R(e_{-1}) \supset R(e)$, $\langle R(e_{1-|e|}), \ldots, R(e_{-1}), R(e)\rangle$ is an admissible choice of Player II for t in $G(L, X)$. So there exist two collections $\mathscr{F}(e) = \{F(e \oplus n)\colon n \in N\}$ and $\mathscr{W}(e) = \{W(e \oplus n)\colon n \in N\}$ of subsets of X, satisfying the conditions (5.1)–(5.5) and (5.9). Let us put

$$E_{m+1} = \bigcup\{F(e)\colon \Sigma e = m+1\} = s(H_0, \ldots, H_m).$$

Clearly, $E_{m+1} \in FK$. Thus the conditions (5.1)–(5.9) are satisfied. Finally, Player II choose any closed set H_{m+1} in X such that $H_{m+1} \subset H_m$ and $E_{m+1} \cap H_{m+1} = \emptyset$. The desired constructions are finished.

In order to show that s is a winning strategy of Player I in $G(FK, X)$, it suffices to show $\bigcap\{H_m\colon m \in N\} = \emptyset$. Assume $x_0 \in \bigcap\{H_m\colon m \in N\}$. Then we have $x_0 \notin \bigcup\{\bigcup \mathscr{F}(e)\colon e \in N^*\}$. Indeed, assume $x_0 \in \bigcup \mathscr{F}(e_0)$ for some $e_0 \in N^*$. So, x_0 is in some $F(e_0 \oplus n_0)$. By (5.6), $F(e_0 \oplus n_0)$ is contained in $E_{\Sigma e_0 + n_0}$, which is disjoint from $H_{\Sigma e_0 + n_0}$. This contradicts to $x_0 \in H_{\Sigma e_0 + n_0}$. Hence, by (5.4), one can inductively choose some infinite sequence $(n_1, n_2, \ldots) \in N^N$ such that $x_0 \notin W(n_1, \ldots, n_k)$ for each $k \in N$. Let $e(k) = (n_1, \ldots, n_k)$ for each $k \in N$. Then it is easily seen $x_0 \in \bigcap\{R(e(k))\colon k \in N\}$. However, by (5.8), we can obtain $\bigcap\{R(e(k))\colon k \in N\} = \emptyset$. This is a contradiction. Hence we have $I(FK, X)$. By [9, Theorem 4.1], $I(K, X)$ is true. The proof is complete.

For a regular Lindelöf space X, DK is outer-almost $\aleph_0$-expandable in X with respect to K. So we obtain, by Theorem 5.1, the following Telgársky's result, which follows from [11, Theorem 3.4].

COROLLARY 5.1. *For a regular Lindelöf space X, $I(DK, X)$ implies $I(K, X)$.*

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