A note on singularities in ANR's

by

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Abstract. The behaviour of various singularities in ANR's, and of conditions designed to remove them, is studied. Examples are constructed to show that in general these properties behave badly in Cartesian products, neither passing from the factors to the product nor vice versa.

In this note it is pointed out that recent results, mainly in decomposition space theory, imply negative answers to problems posed by Borsuk about singularities in ANR's.

Let X be an ANR. Then X has the *singularity of Peano* [4] if there is a closed subset F of X which is contractible in X, but not contractible in any subset of X of dimension at most $\dim F+1$.

If there exists an integer n such that $p_n(X)>0$ but there is no closed proper subset F of X satisfying $p_n(F)+1 = p_n(X)$, then X has the *singularity of Brouwer* [4]. (Here $p_n(X)$ denotes $\dim H^n(X; Q)$).

If X cannot be expressed (for each $\varepsilon>0$) as a finite union of AR's of diameter less than ε, then X has the *singularity of Mazurkiewicz* [4].

If given $\varepsilon>0$, there exists $\delta>0$ such that a compact non-empty subset A of diameter at most δ can be contracted in a set of diameter at most ε and dimension at most $\dim A+1$, then X *satisfies condition* Δ [4].

If X has a covering $\{A_1, \ldots, A_n\}$ by AR's such that any finite intersection of the A_i is either empty or an AR, then X *satisfies condition* Γ [4].

If for each $\varepsilon>0$ there is a covering by finitely many AR's of diameter at most ε which satisfies the condition of Γ, then X *satisfies condition* Λ [5].

If X satisfies condition Δ, respectively condition Λ, it does not have the singularity of Peano, respectively the singularity of Mazurkiewicz [4, 5]. Condition Γ nearly rules out the latter singularity (see below).

THEOREM. *There exists a natural transformation T, from the category of* ANR's *and continuous maps to itself, such that $T(X)$ satisfies conditions Δ, Γ and Λ, and is free of the singularities of Peano, Brouwer and Mazurkiewicz. Moreover T^2 is naturally equivalent to T.*

Proof. Define T by $T(X) = X\times Q$, $T(f) = f\times \mathrm{id}_Q$, where Q is the Hilbert cube. Any fixed homeomorphism $Q \cong Q\times Q$ gives the equivalence of T and T^2. Since X is an ANR, $X\times Q$ is a Q-manifold by Edwards' ANR theorem [9]. By

Chapman's triangulation theorem [9] there is a finite complex K such that $K\times Q\cong X\times Q$. Since this homeomorphism is uniformly continuous, it suffices to verify that $K\times Q$ satisfies Δ to show that $T(X)$ does (and so is free of the Mazurkiewicz singularity). To do this one writes $K\times Q$ as $K\times[-1,1]^N\times Q_{N+1}$, where $\operatorname{diam} Q_{N+1}<\frac{1}{2}\varepsilon$, and covers the polyhedron $K\times[-1,1]^N$ by the stars of the vertices in a triangulation of mesh $\frac{1}{2}\varepsilon$.

Since $T(X)$ is a Q-manifold, it satisfies Δ and so is free of the Peano singularity, either by ([11]; 3.4) or else by the triangulation theorem [9] and ([4]; VII. 5) plus the observation that Q satisfies Δ (cf. [6], [10]).

Finally let F be the complement in K of the open star (in the first derived triangulation) of a barycentre of a simplex of highest dimension. Then by the excision and homotopy axioms, we have $H^*(K\times Q, F\times Q)\cong H^*(B^n, S^{n-1})$. Thus from the exact sequence of a pair with rational coefficients, it follows that

$$|\dim H^p(K\times Q;\, Q)-\dim H^p(F\times Q;\, Q)|\leqslant 1\,,$$

and so $T(X)\cong K\times Q$ does not have Brouwer's singularity.

Remark. Of course a similar result follows from West's finite type theorem [14]. Notice that considerable smoothing is occurring — there are only countably many Q-manifolds up to homeomorphism, but uncountably many ANR's.

Now we give the solutions to problems of Borsuk regarding the behaviour of these properties in cartesian products. Let P be a class of ANR's. Then P is said to be:

(1) *multiplicative* if $X\in P$ and $Y\in P$ implies $X\times Y\in P$.

(2) *ideally multiplicative* if $X\in P$ and X nondegenerate implies $X\times Y\in P$ for all ANR's Y.

(3) *factorisable* if $X\times Y\in P$ implies $X\in P$ and $Y\in P$.

(4) *prime factorisable* if $X\times Y\in P$ implies $X\in P$ or $Y\in P$.

(5) *additive* if $X\in P$, $Y\in P$ and $X\cap Y\in P$ imply $X\cup Y\in P$.

In [4; IX. 8 and VI. 5] the question of multiplicative and factorisable behaviour of the singularites was raised.

EXAMPLE A. Armentrout [1] has described a decomposition G of S^3 into a null sequence of arcs and points such that S^3/G has the singularity of Mazurkiewicz. By a result of Bass [3] if H, K are upper semi-continuous finite-dimensional cell-like decompositions of S^n, S^m respectively, then $S^n/H\times S^m/K\cong S^n\times S^m$. (Alternatively a result of Smith [13] would suffice, except in Example B below). Hence $S^3/G\times S^3/G\cong S^3\times S^3$ is a polyhedron having factors with the singularity of Mazurkiewicz. (The likelihood of this was pointed out in [1].)

EXAMPLE B. Cannon and Daverman [8] have described a finite-dimensional upper semi-continuous cell-like decomposition G of S^n ($n\geqslant 4$) such that any contraction of a certain loop in S^n/G contains an open set. Thus S^n/G has the Peano singularity, while by [3] $S^n/G\times S^n/G\cong S^n\times S^n$ is a polyhedron, and hence satisfies Δ.

EXAMPLE C. Take two discs which have a single boundary point in common. In each disc match a 3-ball to an interior arc (cf. [4]; VI, 1) to obtain T say. Then thicken one of the modified discs by embedding it in the face of a cube, calling the resulting space S. Take $X=Y=S$, with the cubes attached to different discs, so that $X\cap Y=T$. Then clearly X, $X\cap Y$ and Y have the Peano singularity. However in $X\cup Y$ the singular parts of the space lie in faces of the cubes, and by first pushing into the interior of the cubes and then using ([4]; VII. 5.6) it is easy to see that the space is free of the singularity of Peano and indeed satisfies Δ. Hence the Peano singularity is not additive.

EXAMPLE D. Singh [12] has described a finite-dimensional cell-like upper semi-continuous decomposition G of S^3 such that $W=S^3/G$ contains no proper ANR of dimension greater than 1. Hence W fails to satisfy Δ or Γ. However by [3] $W\times W\cong S^3\times S^3$.

EXAMPLE E. Mimicking the idea of C, let W be the space of Example D. As in [12], $sW\cong S^4$. Let $X=sW\vee W$ and $Y=W\vee sW$. Then $X\cup Y=sW\vee sW\cong S^4\times S^4$ satisfies Δ and Γ, but X, Y and $X\cap Y$ all clearly fail to do so, since they have the singularity of Mazurkiewicz.

EXAMPLE F. Modify the example X of ([4]; VI. 1) by matching the segment L in Q^2 with the 4-ball Q^4. If B is the boundary of Q^2, then $Q^1\times B$ is contractible in $Q^1\times X$, but is clearly not contractible in any 3-dimensional subset thereof. As Q^1 has Δ, this property is not ideally multiplicative.

EXAMPLE G. We recall the construction of the example X of ([4]; VI. 4). The basic building operation is as follows; from the interior of a disc D delete the interiors of two subdiscs D_1 and D_2; take another disc E, join two points of E by an arc A in $\operatorname{Int} E$ and identify the endpoints; glue $\operatorname{Bd} E$ to $\operatorname{Bd} D_1$, and glue the image of A to $\operatorname{Bd} D_2$. If we have performed this construction along a whole chain of discs ending with a point, and if a contractible set (e.g. an AR) contains $\operatorname{Bd} D_1$, then $\operatorname{Bd} D_1$ can only be contracted in a subset containing $\operatorname{Bd} D_2$, and inductively it follows that the contractible set contains the whole chain. By filling D with a countable number of chains, dense in a suitable sense, one obtains a space X with the singularity of Mazurkiewicz.

We claim that $X\times[0,1]$ also has this singularity. Since $[0,1]$ satisfies Δ, this implies that Δ is not ideally multiplicative and that the singularity of Mazurkiewicz is not factorizable. Now $Z=X\times[0,1]$ is obtained by iterating the following operation. Bore two vertical holes in $B^3=B^2\times[0,1]$; take another copy B' of B^3, and join an arc in the interior of the top face to one vertically below it in the interior of the bottom face by means of a vertical rectangle R; identify the two vertical edges of R, and glue the vertical faces of B' to the boundary of the first bored-out tube, and the image of R after identification to the boundary of the second bored-out tube. It is easy to see by a standard general position argument that if c is a loop running once round the first tube, any singular disc with c as boundary must contain loops running in both directions around the second tube. Thus by arguing just as in [4], it follows that if A is an AR in Z, we have $p(A)=X$ (where p is projection to the first coordinate). Thus Z does have the singularity of Mazurkiewicz.

The information is summarised in the following table, in which 1 means true, 0 means false. A reference to a reason is given, letters referring to the above examples.

	mult.	i. mult.	fact.	p. fact.	add.
Peano	0: *B*	0: *B*, [6] or [10]	0: *F*	?	0: *C*
Mazurkiewicz	0: *D*	0: *D* or Thm	0: *G*	1: see (1)	0: *E*
Δ	1: ([4], p. 169)	0: *F*	0: *B*, [6] or [10]	0: *B*	1: ([4], p. 167)
Γ	1: ([4], p. 177)	?	0: *A* or Thm	0: *A*	0: [2]
Λ	1: [5] or (1)	0: *G*	0: *D* or Thm	0: *D*	?

Remarks. 1. Condition Λ is definitely stronger than condition Γ. Indeed in the example of [2], the space X satisfies Γ. However it is clear by the techniques of that paper that an AR covering part of the interior of the "handle" of X must have an inverse image meeting both of the end discs, and so cannot be small. Thus Λ fails for X.

2. The space constructed at the end of [7] does not satisfy condition Δ but is free of the Peano singularity. It would be interesting to have similar examples for the other conditions.

3. The bad behaviour with respect to dimension of subsets of cartesian products makes it hard to determine whether the Peano singularity is prime factorisable.

4. In so far as they relate to the singularities of Peano and Mazurkiewicz, questions VI. 5.1, VI. 5.3, IX. 8.1, IX. 8.2 and IX. 8.3 of [4] have been answered; VI. 5.2, IX. 13.1, IX. 14.1 and the parts relating to the Brouwer singularity (with the exception of VI. 5.1) remain open. It would also be interesting to know whether the condition Λ is additive.

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Accepté par la Rédaction le 22. 1. 1981