A representation theorem for compact-valued multifunctions

by

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Abstract. It is proved that every weakly measurable compact-valued multifunction defined on an arbitrary set and ranging in a metric space of weight $\leqslant \mathfrak{c}$ admits a measurable selector; moreover a single valued representation is found. The proof is based on a theorem on decomposition of point-finite completely additive families.

This paper is another attempt to extend the general selector theorem of Kuratowski and Ryll-Nardzewski [9] under some additional assumptions to the non separable case. Our results generalize in a certain sense the theorem of Ioffe [6] and answer a question from [7]. The results are based on a theorem on the decomposition of point-finite completely additive-Borel families which yields also that the class of the members of such a family is bounded. This result was obtained independently also by Hansell in [4].

1. Notations and terminology. We use notations and terminology of [1] and [8]. For a set X and a space Y we understand by $F\colon X \to \mathscr{P}(Y)$ a set-valued mapping or multifunction mapping points of X to non-empty subsets of Y. For a given family $\mathscr{M}$ of subsets of X a multifunction F is said to be *weakly $\mathscr{M}$-measurable* if the set $F^{-}(U) = \{x \in X\colon F(x) \cap U = \emptyset\}$ belongs to $\mathscr{M}$ for each open set U in Y. A mapping $f\colon X \to Y$ is a *selector* for F provided $f(x) \in F(x)$ for each $x \in X$.

Let $\mathscr{A} = \{A_t\}_{t \in T}$ be an indexed family of subsets of a set X. Then we say that $\mathscr{A}$ is *point-finite* if the set $\{t \in T\colon x \in A_t\}$ is finite for all $x \in X$. For a family $\mathscr{M}$ of subsets of X $\mathscr{A}$ is called *completely additive-$\mathscr{M}$* provided $\bigcup_{t \in T'} A_t \in \mathscr{M}$ for any $T' \subset T$. An indexed family $\mathscr{A} = \{A_t\}_{t \in T}$ of subsets of a space X is said to be *σ-discretely-decomposable* (see [5]) if there exist sets A_t^n ($t \in T$, $n = 1, 2, \ldots$) such that $\{A_t^n\}_{t \in T}$ is discrete for fixed n and $A_t = \bigcup_{n=1}^{\infty} A_t^n$ for every $t \in T$.

Let $\mathscr{M}_0$ be a field of subsets of a set X and $\mathscr{M}$ the σ-field generated by $\mathscr{M}_0$. For each ordinal $\alpha < \omega_1$ we have defined in a natural way families

$$\mathscr{F}_0, \mathscr{F}_1, \ldots, \mathscr{F}_\alpha, \ldots, \mathscr{F}_0 = \mathscr{M}_0$$

where the sets of the family $\mathcal{F}_\alpha$ are countable intersections or unions of sets belonging to $\mathcal{F}_\varrho$ with $\varrho<\alpha$ according to whether α is even or odd (see [8] § 30, p. 345). By the class of $M\in\mathcal{M}$ with respect to $\mathcal{M}_0$ we understand $\inf(\alpha\colon M\in\mathcal{F}_\alpha)$. For a metric space we denote by Σ_α the Borel sets of additive class α. R_+ are the non-negative reals and $\mathfrak{b}$ (continuum) is the power of the reals. $D(\mathfrak{m})^N$ means the countable product of a discrete space of cardinality $\mathfrak{m}$. We consider it in its usual product metric (see [1], p. 326).

2. Point-finite completely additive families.

LEMMA. *Let $\mathcal{M}$ be a σ-field of subsets of an arbitrary set X and $\mathcal{A}=\{A_t\}_{t\in T}$ a point-finite completely additive-$\mathcal{M}$ family with $\bigcup\mathcal{A}=X$ and $|T|\leqslant\mathfrak{c}$. Suppose that the index set T is a subset of R_+. Then the function f defined by $f(x)=\min\{t\colon x\in A_t\}$ is $\mathcal{M}$ measurable with respect to the discrete topology on R_+. Moreover $\{A_t\setminus f^{-1}(t)\}_{t\in T}$ is again a point-finite completely additive family.*

Proof. We shall prove the equality

$$f^{-1}(T')=\bigcap_{m=0}^{\infty}\bigcup_{n=0}^{\infty}[\bigcup\{A_t\colon t\in T' \text{ and } n2^{-m}\leqslant t<(n+1)2^{-m}\}\setminus$$
$$\setminus\bigcup\{A_t\colon t\in T \text{ and } t<n2^{-m}\}]\in\mathcal{M}, \quad \text{where } T'\subset T.$$

To this purpose let $f(x)=t'\in T'$. For each m we can find a number $n(m)$ such that $n(m)2^{-m}\leqslant t'<(n(m)+1)2^{-m}$. Of course $x\notin A_t$ for $t<n(m)2^{-m}$. So x belongs to the right-hand side.

If conversely $f(x)=t_0\notin T'$ then there exists an m such that for all n with $n2^{-m}\leqslant\min\{t\neq t_0\colon x\in A_t\}<(n+1)2^{-m}$ we have $t_0<n2^{-m}$ and therefore x does not belong to the right-hand side.

Put now $C_t=A_t\setminus f^{-1}(t)$. The family $\{C_t\}_{t\in T}$ is of course point-finite and it is completely additive because we can prove for any $T'\subset T$

$$\bigcup_{t\in T'}C_t=\bigcup_{k=0}^{\infty}\bigcap_{m=k}^{\infty}\bigcup_{n=0}^{\infty}[\bigcup\{A_t\colon t\in T' \text{ and } n2^{-m}\leqslant t<(n+1)2^{-m}\}\cap$$
$$\cap\bigcup\{A_t\colon t\in T \text{ and } t<n2^{-m}\}]\in\mathcal{M}.$$

Indeed, if $x\in C_t$ for some $t\in T'$ we simply take $k>1/|f(x)-t|$. On the other hand, if x belongs to the right-hand side, we find n and m such that $x\in A_t$ for some $t\in T'$ with $t\geqslant n2^{-m}$ and $f(x)<t$. We conclude that $x\in C_t$.

LEMMA 2. *Let X, $\mathcal{M}$, $\mathcal{A}$, T be as in Lemma 1. Then $\mathcal{A}$ admits a σ-disjoint completely additive-$\mathcal{M}$ decomposition. More specifically, there exist disjoint completely additive families $\{A_t^n\}_{t\in T}$, $n=1,2,\ldots$, such that*

(1) $A_t=\bigcup_{n=1}^{\infty}A_t^n$ *and*

(2) $\bigcup_{t\in T}A_t^n=\{x\in X\colon x$ *is contained in at least n members of* $\mathcal{A}\}$.

Proof. Applying Lemma 1 to the family $\mathcal{A}$ we put $A_t^1=f^{-1}(t)$ and obtain the point-finite completely additive family $\{A_t\setminus A_t^1\}_{t\in T}$.

In the induction step we assume that all A_t^i for $i<n$ have been defined and that $\{A_t\setminus\bigcup_{i=1}^{n-1}A_t^i\}_{t\in T}$ is a point-finite completely additive family. Applying Lemma 1 to the latter family we define $A_t^n=f^{-1}(t)$. Again by Lemma 1 $\{A_t\setminus\bigcup_{i=1}^{n}A_t^n\}_{t\in T}$ is point-finite completely additive and the induction is finished.

In order to prove conditions (1) and (2) let us assume that $A_{t1},\ldots,A_{tn}$, $t1<t2<\ldots<tn$, are all elements of $\mathcal{A}$ containing a given point x. Then $x\in A_{t1}^1$, $x\in A_{t2}^2$, $x\in A_{t3}^3$, ..., $x\in A_{tn}^n$ and hence $x\in\bigcup_{t\in T}A_t^m$ whenever $m\leqslant n$. Further we conclude $x\notin A_{t1}^m, x\notin A_{t2}^m,\ldots,x\notin A_{tn}^m$ hence $x\notin\bigcup_{t\in T}A_t^m$ whenever $m>n$ and therefore condition (2) holds. On the other hand, if $x\in A_t$ then $t=ti$ for some $i\leqslant n$ and $x\in A_{ti}^i$. This proves condition (1).

THEOREM 1 (¹). *Let $\mathcal{M}_0$ be a field of subsets of an arbitrary set X and $\mathcal{M}$ the σ-field generated by $\mathcal{M}_0$. Then the class with respect to $\mathcal{M}_0$ of the members of a point-finite completely additive-$\mathcal{M}$ family is bounded; i.e. there exists an $\alpha<\omega_1$ that all members of the family are of class α.*

This is an easy combination of a general result of Preiss [10] and our Lemma 2. Let us however give a short proof of this fact.

Proof. If the class were not bounded this would be so for a subfamily $\mathcal{A}=\{A_t\}_{t\in T}$ of cardinality $\aleph_1$, $T\subset R_+$. By Lemma 2 there is a decomposition of $\mathcal{A}$ into disjoint completely additive families $\{A_t^n\}_{t\in T}$ and we have $\bigcup_{t\in T'}A_t=\bigcup_{n=1}^{\infty}\bigcup_{t\in T'}A_t^n$ for any $T'\subset T$. So it is sufficient to show that the families $\{A_t^n\}_{t\in T}$ are of bounded class.

Let $A_{ab}=\bigcup\{A_t^n\colon t\in[a,b]\subset R\}$. Of course

$$\beta=\sup\{\text{class of } A_{ab}\colon a,b \text{ rational}\}<\omega_1.$$

Now for an arbitrary $t\in T$ the class of $A_t^n=\bigcap_{\substack{a<t<b\\ a,b\\ \text{rational}}}A_{ab}$ is less or equal to $\beta+1$.

COROLLARY 1. *Let X be a metric space and $\mathcal{A}=\{A_t\}_{t\in T}$ a point-finite completely additive-Borel family which is σ-discretely-decomposable. Then $\mathcal{A}$ is completely additive-Σ_α for some $\alpha<\omega_1$.*

Proof (compare Lemma 4 in [7]). Take α as in Theorem 1. Let $A_t=\bigcup_{n=1}^{\infty}A_t^n$ where $\{A_t^n\}_{t\in T}$ is discrete for each fixed n. We may assume that $A_t^n=A_t^n\setminus A_t$. Take

(¹) After this paper had been written the author obtained a preprint [4] from R. W. Hansell where Theorem 1 is also proved in a different way.

$T' \subset T$ and put $G^n = \bigcup_{t \in T'} A_t^n$. By a classical result of Montgomery (see [8] § 30) G^n is a Borel set of class α. We get $\bigcup_{t \in T'} A_t = \bigcup_{t \in T'} \bigcup_{n=1}^{\infty} A_t^n = \bigcup_{n=1}^{\infty} G^n$, a set of additive class α.

3. A selection theorem for absolutely analytic spaces. A metric space is called *absolutely analitic* provided that it is an analytic subset whenever embedded in a complete metric space. In [5] Hansell proved the deep theorem that a disjoint completely additive analytic in an absolutely analytic space is σ-discretely-decomposable. This was extended to the case of point-finite families in [7], Theorem 1. The theorem below answers Question 2 in [7].

THEOREM 2. *Let X be an absolutely analytic metric space and Y an arbitrary metric space. Every compact-valued weakly-Borel-measurable multifunction $F\colon X \to \mathscr{P}(Y)$ admits a Borel-measurable selector. Moreover the selector is of class α for some $\alpha < \omega_1$.*

The theorem follows immediately from Theorem 2 in [7] and the lemma below.

LEMMA 3. *Let $F\colon X \to \mathscr{P}(Y)$ be as in Theorem 2. Then F is weakly Σ_α-measurable for some $\alpha < \omega_1$.*

Proof. Let $\{U_t^n\}_{t \in T}$, $n = 1, 2, \ldots$, be a base for the topology of the space Y, the families $\{U_t^n\}_{t \in T}$ being discrete. Now the family $\mathscr{A}_n = \{F^-(U_t^n)\}_{t \in T}$ is point-finite, for the compact set $F(x)$ intersects only finitely many of the sets U_t^n. It is also completely additive-Borel because $\bigcup_{t \in T'} F^-(U_t^n) = F^-\big(\bigcup_{t \in T'} U_t^n\big)$ for any $T' \subset T$. By Theorem 1 in [7] and Corollary 1 there is an $\alpha_n < \omega_1$ that $\mathscr{A}_n$ is additive-$\Sigma_{\alpha n}$. To end the proof it is sufficient to put $\alpha = \sup_n \alpha_n$. For any open set U in Y the set $F^-(U)$ is the countable sum of sets of class α and therefore of additive class α.

COROLLARY 2. *Each point-finite completely additive-Borel covering $\mathscr{A} = \{A_t\}_{t \in T}$ of an absolutely analytic space X has a completely additive-Borel disjoint refinement.*

Proof. In Theorem 2 put $Y = T$ with the discrete topology and

$$F(x) = \{t\colon x \in A_t\}.$$

If f is a measurable selector then $\{F^{-1}(t)\}_{t \in T}$ is the requested refinement.

QUESTION. Does Corollary 2 hold in general or at least for a metric space? (Compare Lemma 2 and see also [4]). Notice that under additional set-theoretical assumptions a more general fact for metric spaces was proved by Fleissner in [2].

4. A representation theorem. The representation of the multifunction in the following theorem is analogous to the theorem of Ioffe [6] where it is proved for closed-valued maps into a Polish space and $D(\mathfrak{m})^N$ instead of $D(\aleph_0)^N$.

THEOREM 3. *Let X be an arbitrary set with a σ-field $\mathscr{M}$ and Y a metric space of weight $\mathfrak{m} \leqslant \mathfrak{c}$. Then every compact-valued weakly-measurable multifunction $F\colon X \to \mathscr{P}(Y)$ admits a measurable selector. Moreover for the Baire space $Z = D(\mathfrak{m})^N$ there is a mapping $\Phi\colon Z \times X \to \mathscr{P}(Y)$ with $\Phi(\cdot, x)$ continuous, $\Phi(z, \cdot)$ measurable and $\Phi(Z, x) = F(x)$ for each $x \in X$.*

Proof. We shall identify $D(\mathfrak{m})$ with $T \subset R_+$ and we assume that the metric of Y is bounded by 1.

By induction we define for each n and $(r_1, r_2, \ldots, r_n) \in T^n$ a multifunction $F_{r_1 \ldots r_n}$ satisfying the following conditions:

(1) $\emptyset \neq F_{r_1 \ldots r_n}(x) \subset F(x)$,

(2) $F_{r_1 \ldots r_n}$ is weakly $\mathscr{M}$-measurable,

(3) $\operatorname{diam}(F_{r_1 \ldots r_n}(x)) \leqslant 2^{-n}$,

(4) $F_{r_1 \ldots r_n}(x) = \bigcup_{r \in T} F_{r_1 \ldots r_n r}(x)$.

Put $F_{r_1}(x) = F(x)$ for all $r_1 \in T$ and suppose that all $F_{r_1 \ldots r_{n-1}}$ have been defined. Take a locally finite open covering $\mathscr{U} = \{U_r\}_{r \in T}$ of Y by sets of diameter $\leqslant 2^{-n}$ where some U_r may be empty.

For $r \in T$ let $D_r = F^-_{r_1 \ldots r_{n-1}}(U_r)$. Then the family $\{D_r\}_{r \in T}$ is completely additive-$\mathscr{M}$ and since F is compact-valued this family is also point-finite. We obtain this from (1), and it does not interfere that the sets $F_{r_1 \ldots r_{n-1}}(x)$ may not be closed.

We apply Lemma 1 to the family above. The sets $D'_r = f^{-1}(r)$ form a disjoint completely additive refinement of the family $\{D_r\}_{r \in T}$. Let for $r_n \in T$

$$(5)\quad F_{r_1 \ldots r_{n-1} r_n}(x) = \begin{cases} F_{r_1 \ldots r_{n-1}}(x) \cap U_{r_n} & \text{for } x \in D_{r_n}, \\ F_{r_1 \ldots r_{n-1}}(x) \cap U_r & \text{for } x \in D'_r \setminus D_{r_n}. \end{cases}$$

Obviously the conditions (1), (3) and (4) are fulfilled. The only property to show is the measurability of $F_{r_1 \ldots r_{n-1} r_n}$. For simplicity we shall write in this part F_n and F_{n-1} instead of $F_{r_1 \ldots r_{n-1} r_n}$ and $F_{r_1 \ldots r_{n-1}}$ respectively.

Let U be an open subset of Y. The family $\{E_r\}_{r \in T}$, where

$$E_r = \{x \in X\colon F_{n-1}(x) \cap U_r \cap U \neq \emptyset\}$$

is again point-finite completely additive-$\mathscr{M}$ and, using again Lemma 1 with respect to the family $\{E_r\}_{r \in T}$, we obtain its disjoint completely additive refinement $\{E'_r = f^{-1}(r)\}_{r \in T}$. Now

$$F_n^-(U) \setminus D_{r_n} = F_{n-1}^-(U) \cap \bigcap_{k=0}^{\infty} \bigcup_{n=0}^{\infty} \big[\bigcup\{D'_r \setminus D_{r_n}\colon r \in T \text{ and } m2^{-k} \leqslant r < (m+1)2^{-k}\} \setminus \bigcup\{E'_r\colon r \in T \text{ and } r \geqslant (m+1)2^{-k}\}\big] \in \mathscr{M}.$$

This equality requires some proof. Let $x \in F_n^-(U) \setminus D_{r_n}$. Of course $x \in D'_r$ for exactly one r and so we get $F_{n-1}(x) \cap U_r \cap U \neq \emptyset$. This means that $x \notin E'_s$ for $s > r$ and of course $x \in F_{n-1}^-(U)$. Hence x belongs to the right-hand side.

Now let $x \notin F_n^-(U)$, $x \notin D_{r_n}$ and $x \in D'_r$. It follows that $F_{n-1}(x) \cap U_r \cap U = \emptyset$. From the definition of D'_r we know that $r' = \min\{s \in T\colon F_{n-1}(x) \cap U_s \cap U \neq \emptyset\} > r$. ($r'$ is defined whenever $x \in F_{n-1}^-(U)$.) For sufficiently big k we get $(m+1)2^{-k} \leqslant r'$ whenever $m2^{-k} \leqslant r < (m+1)2^{-k}$. But this means that x does not belong to the right-hand side.

Realize at last that $F_n^-(U) \cap D_{r_n} = F_{n-1}^-(U \cap U_{r_n})$ hence $F_n^-(U) \in \mathscr{M}$. This ends the inductive construction of the system of multifunctions.

We are now able to define $\Phi(z,x) = \bigcap_{n=1}^{\infty} \overline{F_{r_1\ldots r_n}(x)}$, for $z = (r_1, r_2, \ldots)$. The function Φ is well-defined by conditions (1) and (3) and the compactness of $F(x)$. The condition $F(x) = \Phi(Z, x)$ follows from (4).

To prove the measurability of $\Phi_z = \Phi(z, \cdot)$ it is enough to observe that for a closed subset K of Y we have by (2) the relation

$$\Phi_z^{-1}(K) = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \{x\colon F_{r_1\ldots r_n}(x) \cap \{y \in Y\colon \operatorname{dist}(y, K) < 2^{-m}\} \neq \emptyset\} \in \mathscr{M}.$$

It remains to show the continuity of $\Phi(\cdot, x)$. But whenever in the product metric

$$a(z, z') = \sum_{n=1}^{\infty} 2^{-n}\, d_0(r_n, r_n') \leqslant 2^{-k-2} \qquad \text{for } z = (r_1, r_2, \ldots),\ z' = (r_1', r_2', \ldots),$$

d_0 being the discrete 0-1 metric, then $r_i = r_i'$ for $i \leqslant k$ and therefore $F_{r_1\ldots r_k} = F_{r_1'\ldots r_k'}$. Whence by (4) and the definition of Φ we get $\operatorname{dist}(\Phi(z,x), \Phi(z',x)) \leqslant 2^{-k}$. This proves that $\Phi(\cdot, x)$ is continous for any $x \in X$.

Remark 1. If $\mathscr{M}$ is generated by a field $\mathscr{M}_0$ and F is weakly Σ_α-measurable for some $\alpha < \omega$, then, thoroughly examinating the proof of Theorem 3 and using Theorem 1, we see that the class of the selector $\mathscr{M}$ is bounded.

Remark 2. Examinating the proof of [7], Theorem 2, one obtains in Theorem 2 a representation of the multifunction similar to that in Theorem 3.

Remark 3. We may of course in Theorem 3 remove the weight restriction on Y by assuming that $\mathscr{M}$ consists of at most $\mathfrak{c}$ elements (which holds, for example, if X is separable metric), but this is rather artificial. So the question is, how to get rid of the cardinality restriction. Another way of generalizing this theorem is to consider non-metrizable Y. This seems to be even more complicated. The so far best result in this direction is that of Graf in [3]. Notice that the assertion of Theorem 3 with Y compact 0-dimensional would yield the existence of a Borel lifting for the unit intervall with Lebesgue-measure (which was obtained so far under the assumption of CH; cf. [3]).

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