A representation theorem for compact-valued multifunctions

by

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Abstract. It is proved that every weakly measurable compact-valued multifunction defined on an arbitrary set and ranging in a metric space of weight ≤ α admits a measurable selector; moreover a single valued representation is found. The proof is based on a theorem on decomposition of point-finite completely additive families.

This paper is another attempt to extend the general selector theorem of Kuratowski and Ryll-Nardzewski [9] under some additional assumptions to the non separable case. Our results generalize in a certain sense the theorem of Ioffe [6] and answer a question from [7]. The results are based on a theorem on the decomposition of point-finite completely additive Borel families which yields also that the class of the members of such a family is bounded. This result was obtained independently also by Hansel in [4].

1. Notations and terminology. We use notations and terminology of [1] and [8]. For a set X and a space Y we understand by \( F : X \to \mathcal{P}(Y) \) a set-valued mapping or multifunction mapping points of X to non-empty subsets of Y. For a given family \( \mathcal{M} \) of subsets of X a multifunction F is said to be \( \mathcal{M} \)-measurable if the set \( F^{-1}(U) = \{ x \in X : F(x) \cap U = \emptyset \} \) belongs to \( \mathcal{M} \) for each open set U in Y.

A mapping \( f : X \to Y \) is a selector for F provided \( f(x) \in F(x) \) for each \( x \in X \).

Let \( \mathcal{A} = (A_t)_{t \in T} \) be an indexed family of subsets of a set X. Then we say that \( \mathcal{A} \) is point-finite if the set \( \{ t \in T : x \in A_t \} \) is finite for all \( x \in X \). For a family \( \mathcal{M} \) of subsets of X \( \mathcal{M} \) is called completely additive \( \mathcal{M} \) provided \( \bigcup A_t \in \mathcal{M} \) for any \( T \subset T \). An indexed family \( \mathcal{A} = (A_t)_{t \in T} \) of subsets of a space X is said to be \( \sigma \)-discretely-decomposable (see [5]) if there exist sets \( \mathcal{A}_n = (A_{n,t})_{t \in T} \), \( n = 1, 2, \ldots \) such that \( \mathcal{A}_n \) is discrete for fixed \( n \) and \( A_t = \bigcup \mathcal{A}_n \) for every \( t \in T \).

Let \( \mathcal{M}_0 \) be a field of subsets of a set X and \( \mathcal{M} \) the \( \sigma \)-field generated by \( \mathcal{M}_0 \). For each ordinal \( \alpha \leq \omega_1 \) we have defined in a natural way families

\[ F_0, F_1, \ldots, F_{\alpha}, \ldots, F_\alpha = \mathcal{M}_0. \]
where the sets of the family $\mathcal{F}_\alpha$ are countable intersections or unions of sets belonging to $\mathcal{G}_\alpha$ with $\alpha < \omega$ according to whether $\alpha$ is even or odd (see [8], §30, p. 345). By the class of $M \in \mathcal{M}$ with respect to $\mathcal{A}_0$ we understand $\inf \{ \alpha \in \mathbb{R}_+ : M \in \mathcal{F}_\alpha \}$. For a metric space we denote by $\mathcal{E}$ the Borel sets of additive class $\alpha$. $\mathcal{E}$ are the non-negative reals and $\mathcal{B}$ (continuum) is the power of the reals. $D(\alpha)^n$ means the countable product of a discrete space of cardinality $m$. We consider it in its usual product metric (see [1], p. 326).

2. Point-finite completely additive families.

**Lemma**. Let $\mathcal{A}$ be a $\sigma$-field of subsets of an arbitrary set $X$ and $\mathcal{A}_0 = \{ A_i \}_{i \in I}$ a point-finite completely additive-$\mathcal{A}_0$ family with $\mu(\mathcal{A}_0) = X$. Suppose that the index set $I$ is a subset of $\mathbb{R}_+$. Then the function $f$ defined by $f(x) = \min \{ r : x \in A_r \}$ is $\mathcal{A}$ measurable with respect to the discrete topology on $\mathbb{R}_+$. Moreover, $\{ A_i \}^{-1}(f^{-1}(r)) \in \mathcal{A}$ is again a point-finite completely additive family.

**Proof**. We shall prove the equality

$$f^{-1}(T) = \bigcap_{n=0}^{\infty} \bigcup_{m=0}^{\infty} \left( \bigcup_{r \in B} \{ A_r \} : t \in T \text{ and } n2^{-m} \leq r \leq (n+1)2^{-m} \right) \cap \left( \bigcup_{r \in B} \{ A_r \} : t \in T \text{ and } n2^{-m} < r \leq (n+1)2^{-m} \right),$$

where $T \subseteq B$.

To this purpose let $f(x) = r \in B$. For each $n$ we can find a number $m(n)$ such that $n2^{-m} \leq r < (n+1)2^{-m}$. Of course $x \in A_r$, for $r \in B$ and $x \in A_r$ for $r \in B$. So $x$ belongs to the right-hand side.

If conversely $f(x) = r \notin T$ then there exists an $m$ such that for all $n$ with $n2^{-m} \leq m < (n+1)2^{-m}$ we have $x \notin A_r$ and therefore $x$ does not belong to the right-hand side.

Put now $C_i = \{ A_r \}^{-1}(r)$. The family $\{ C_i \}_{i \in I}$ is of course point-finite and it is completely additive because we can prove for any $T \subseteq B$

$$\bigcup_{i \in I} C_i = \bigcup_{n=0}^{\infty} \left( \bigcup_{m=0}^{\infty} \left( \bigcup_{r \in B} \{ A_r \} : t \in T \text{ and } n2^{-m} \leq r \leq (n+1)2^{-m} \right) \cap \left( \bigcup_{r \in B} \{ A_r \} : t \in T \text{ and } n2^{-m} < r \leq (n+1)2^{-m} \right) \right).$$

Indeed, if $x \in C_i$, for some $r \in B$ we simply take $x \in \{ A_r \}^{-1}(r)$. On the other hand, if $x \in C_i$, for some $r \in B$ with $r \in 2^{-m}$ and $f(x) < r$, we conclude that $x \in C_i$.

**Lemma**. Let $X, \mathcal{A}, \mathcal{A}_0, T$ be as in Lemma 1. Then $\mathcal{A}_0$ admits a $\sigma$-disjoint completely additive-$\mathcal{A}$ decomposition. More specifically, there exist disjoint completely additive families $\{ A_i \}_{i \in I}$ such that

1. $A_i = \bigcup_{r \in B} \{ A_r \}$ and
2. $\bigcup_{r \in B} \{ A_r \} \subseteq \{ x \in X : x \text{ is contained in at least } n \text{ members of } \mathcal{A}_0 \}$.

**Proof**. Applying Lemma 1 to the family $\mathcal{A}_0$ we put $A_i = f^{-1}(i)$ and obtain the point-finite completely additive family $\{ A_i \}_{i \in I}$. In the induction step we assume that all $A_i$ for $i < n$ have been defined and that $\{ A_i \}_{i \in I}$ is a point-finite completely additive family. Applying Lemma 1 to the latter family we define $A_n = f^{-1}(n)$. Again by Lemma 1 $\bigcap_{i \in I} \{ A_i \}_{i \in I}$ is point-finite completely additive and the induction is finished.

In order to prove conditions (1) and (2) let us assume that $A_{11}, \ldots, A_{1n}$, $t_1 \leq t_2 \leq \ldots \leq t_m$ are all elements of $\mathcal{A}_0$ containing a given point $x$. Then $x \in A_{111}, x \in A_{12}, \ldots, x \in A_{1n}$, and hence $x \in \bigcup_{i \in I} \{ A_i \}$ whenever $m < n$. Further we conclude $x \in \bigcup_{i \in I} \{ A_i \}$, $x \notin \bigcup_{i \in I} \{ A_i \}$, $x \notin \bigcup_{i \in I} \{ A_i \}$, $x \notin \bigcup_{i \in I} \{ A_i \}$, whenever $m > n$. Hence, condition (2) holds. On the other hand, if $x \in A_i$, then $t = t_i$ for some $i < n$ and $x \in A_i$. This proves condition (1).

**Theorem** (1). Let $\mathcal{M}_0$ be a field of subsets of an arbitrary set $X$ and $\mathcal{A}_0$ the $\sigma$-field generated by $\mathcal{M}_0$. Then the class with respect to $\mathcal{M}_0$ of the members of a point-finite completely additive-$\mathcal{M}_0$ family is bounded; i.e. there exists an $\alpha < \omega_1$ such that all members of the family are of class $\alpha$.

This is an easy combination of a general result of Preiss [10] and our Lemma 2. Let us however give a short proof of this fact.

**Proof**. If the class were not bounded this would be so for a subfamily $\mathcal{A}_0 = \{ A_i \}_{i \in I}$ of cardinality $n$, $T \subseteq B$. By Lemma 2 there is a decomposition of $\mathcal{A}_0$ into disjoint completely additive families $\{ A_i \}_{i \in I}$ and we have $\bigcup_{i \in I} \{ A_i \} = \bigcup_{i \in I} \{ A_i \}$ for any $T \subseteq B$. So it is sufficient to show that the families $\{ A_i \}_{i \in I}$ are of bounded class.

Let $A_{ab} = \{ A_i : t \in [a, b] \}$ of course

$$\beta = sup \{ \text{class of } A_{ab} : a, b \text{ rational} \} < \omega_1.$$ 

Now for an arbitrary $t \in T$ the class of $A_t = \bigcap_{n=0}^{\infty} \{ A_i \}_{i \in I}$ is less or equal to $\beta + 1$.

**Corollary** 1. Let $X$ be a metric space and $\mathcal{A}_0 = \{ A_i \}_{i \in I}$ a point-finite completely additive-Borel family which is $\sigma$-discretely-decomposable. Then $\mathcal{A}_0$ is completely additive-$\mathcal{F}_\alpha$ for some $\alpha < \omega_1$.

**Proof** (compare Lemma 4 in [7]). Take $x$ as in Theorem 1. Let $A_i = \bigcup_{r \in B} \{ A_r \}$ where $\{ A_r \}_{r \in B}$ is discrete for each fixed $n$. We may assume that $\mathcal{A}_0 = \mathcal{A}_0 \cap \mathcal{B}$. Take

(1) After this paper had been written the author obtained a preprint [4] from R. W. Hansell where Theorem 1 is also proved in a different way.
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Proof. We shall identify \( D(m) \) with \( T = R \), and we assume that the metric of \( Y \) is bounded by 1.

By induction we define for each \( n \) and \( r_1, r_2, \ldots, r_n \in T^n \) a multifunction \( F_{r_1, \ldots, r_n} \) satisfying the following conditions:

1. \( \emptyset \neq F_{r_1, \ldots, r_n}(x) \subset F(x) \).
2. \( F_{r_1, \ldots, r_n} \) is weakly measurable.
3. \( \text{diam}(F_{r_1, \ldots, r_n}(x)) \leq 2^{-n} \).
4. \( F_{r_1, \ldots, r_n}(x) = \bigcup_{r \in T} F_{r_1, \ldots, r_n}(x) \).

Put \( F_r(x) = F(x) \) for all \( r \in T \) and suppose that all \( F_{r_1, \ldots, r_n} \) have been defined.

Take a locally finite open covering \( \mathcal{U} = \{ U_i \}_{i \in T} \) of \( Y \) by sets of diameter \( \leq 2^{-n} \) where some \( U_i \) may be empty.

For \( r \in T \) let \( D_r = F_{r_1, \ldots, r_n}(U_i) \). Then the family \( \{ D_r \}_{r \in T} \) is completely additive-\( \mathcal{M} \) and since \( F \) is compact-valued this family is also point-finite. We obtain this from (1), and it does not interfere that the sets \( F_{r_1, \ldots, r_n}(x) \) may not be closed.

We apply Lemma 1 to the family above. The sets \( D_r = f^{-1}(r) \) form a disjoint completely additive refinement of the family \( \{ D_r \}_{r \in T} \).

For \( x \in T \) let \( F_{r_1, \ldots, r_n}(x) = \bigcup_{r \in T} F_{r_1, \ldots, r_n}(x) \). Obviously the conditions (1), (3) and (4) are fulfilled. The only property to show is the measurability of \( F_{r_1, \ldots, r_n} \). For simplicity we shall write in this part \( F_r \) and \( D_r \) instead of \( F_{r_1, \ldots, r_n} \), and \( D_{r_1, \ldots, r_n} \) respectively.

Let \( U \) be an open subset of \( Y \). The family \( \{ E_r \}_{r \in T} \) where

\[
E_r = \{ x \in X : F_r(x) \cap U \neq \emptyset \}
\]

is again point-finite completely additive-\( \mathcal{M} \) and, using again Lemma 1 with respect to the family \( \{ E_r \}_{r \in T} \), we obtain its disjoint completely additive refinement \( \{ E_r' \}_{r \in T} \). Now

\[
F_r(U) \cap D_m = F_{r_1, \ldots, r_n}(U) \cap \bigcup_{r \in T} \bigcup_{m=0}^{\infty} \bigcup_{n=0}^{\infty} \bigcup_{E_r \cap T \cap (r+m+1/2)^{-1/2}} \bigcup_{E_r' \cap T \cap (r+m+1/2)^{-1}} \in \mathcal{M}.
\]

This equality requires some proof. Let \( x \in F_r(U) \cap D_m \). Of course \( x \in D_r \) for exactly one \( r \) and so we get \( F_{r_1, \ldots, r_n}(x) \cap U \neq \emptyset \). This means that \( x \notin E_r' \) for \( s > r \) and of course \( x \notin F_{r_1, \ldots, r_n}(U) \). Hence \( x \) belongs to the right-hand side.

Now let \( x \notin F_r(U) \). \( x \notin D_m \), and \( x \notin D_r \). It follows that \( F_{r_1, \ldots, r_n}(x) \cap U = \emptyset \). From the definition of \( D_r' \) we know that \( r' = \min \{ x \in D_r : F_{r_1, \ldots, r_n}(x) \cap U = \emptyset \} \).

(\( r' \) is defined whenever \( x \in F_{r_1, \ldots, r_n}(U) \)). For sufficiently big \( k \) we get \( (m+1)^{-1} \leq r' \leq (m+1)^{-1} \). But this means that \( x \) does not belong to the right-hand side.

Realize at last that \( F_r(U) \cap D_m = F_{r_1, \ldots, r_n}(U) \cap D_m \) hence \( F_r(U) \in \mathcal{M} \). This ends the inductive construction of the system of multifunctions.
We are now able to define $\Phi(z, x) = \bigcap_{n=1}^{m} F_{r_n, r_n}(z^n)$ for $z = (r_1, r_2, \ldots)$. The function $\Phi$ is well-defined by conditions (1) and (3) and the compactness of $F(x)$. The condition $F(x) = \Phi(Z, x)$ follows from (4).

To prove the measurability of $\Phi$, it is enough to observe that for a closed subset $K$ of $Y$ we have by (2) the relation

$$\Phi_{r_n}^{-1}(K) = \bigcap_{n=1}^{m} \bigcup_{r_n} \{ x : F_{r_n, r_n}(x) \cap \{ y \in Y : \text{dist}(y, K) < 2^{-n} \} \neq \emptyset \} \in \mathcal{M}.$$

It remains to show the continuity of $\Phi(\cdot, x)$. But whenever in the product metric

$$d(x, x') = \sum_{n=1}^{m} 2^{-n} \cdot d_n(r_n, r'_n) < 2^{m-2}$$

$d_n$ being the discrete $0$-$1$ metric, then $r_i = r'_i$ for $i \leq k$ and therefore $F_{r_1, r_k} = F_{r_1, x}$. Whence by (4) and the definition of $\Phi$ we get $\text{dist}(\Phi(x, x), (x, x')) < 2^{-n}$. This proves that $\Phi(\cdot, x)$ is continuous for any $x \in X$.

**Remark 1.** If $\mathcal{M}$ is generated by a field $\mathcal{A}$ and $F$ is weakly $\Sigma_1$ measurable for some $\sigma$-$\sigma$, then, thoroughly examining the proof of Theorem 3 and using Theorem 1, we see that the class of the selector $\mathcal{M}$ is bounded.

**Remark 2.** Examining the proof of [7], Theorem 2, one obtains in Theorem 2 a representation of the multifunction similar to that in Theorem 3.

**Remark 3.** We may of course in Theorem 3 remove the weight restriction on $Y$ by assuming that $\mathcal{M}$ consists of at most $c$ elements (which holds, for example, if $X$ is separable metric), but this is rather artificial. So the question is, how to get rid of the cardinality restriction. Another way of generalizing this theorem is to consider non-metrizable $Y$. This seems to be even more complicated. The so far best result in this direction is that of Graf in [3]. Notice that the assertion of Theorem 3 with $Y$ compact $0$-dimensional would yield the existence of a Borel lifting for the unit interval with Lebesgue-measure (which was obtained so far under the assumption of $CH$; cf. [3]).

**References**