On products of incompressible AR's

by

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Abstract. It is shown that there exists a 3-dimensional incompressible AR X whose product $X\times X$ is homeomorphic to the 6-dimensional ball B^6; moreover, the family of spaces X's satisfying these properties is uncountable. A similar result holds for n-dimensional AR's with $n>3$. This answers a question posed by Jo Ford and R. W. Heath [FH].

1. Introduction. A space X is called *incompressible* if X admits no homeomorphism onto a proper subset of X, see P. Fletcher and J. Sawyer [FS] for more details. L. S. Husch [H] has shown, in response to a question of Fletcher and Sawyer [FS], that *there exists a* 3-*dimensional incompressible metric space whose product with the circle* S^1 *is compressible*. Recently, Jo Ford and R. W. Heath [FH] has shown that *there exist two* 1-*dimensional incompressible metric continua whose product is compressible*. We shall show that *there exist two incompressible* AR's (by AR we mean compact metric absolute retract) *whose product is compressible*. This answers a question of Jo Ford and R. W. Heath [FH].

We wish to thank R. J. Daverman and C. D. Bass for some helpful discussions.

2. Decompositions and AR's. All decompositions will be upper semi-continuous. The n-ball $B^n = \{(x_1, x_2, \ldots, x_n)\colon x_1^2+x_2^2+\ldots+x_n^2 \leqslant 1\}$ has boundary sphere $S^{n-1} = \{(x_1, x_2, \ldots, x_n)\colon x_1^2+x_2^2+\ldots+x_n^2 = 1\}$. The following is immediate from [S].

(2.1) THEOREM. *There exists a decomposition G of B^3 satisfying*: (a) *the non-degenerate elements of G form a null collection of arcs each of which is contained in* (B^3-S^2), (b) *the decomposition space B^3/G is a* 3-*dimensional* AR, *and* (c) B^3/G *does not contain any proper subset which is a* 3-*dimensional* AR.

The following is our main result.

(2.2) THEOREM. *There exists a* 3-*dimensional incompressible* AR Q *such that* $Q\times Q$ *is compressible*; *moreover*, $Q\times Q$ *is homeomorphic to* B^6.

Proof. Put $Q = B^3/G$, B^3/G as in Theorem (2.1). Let $\pi\colon B^3 \to Q$ denote the projection associated with the decomposition G. Consider the product decomposition $G\times G$ of $B^3\times B^3$ and observe that the boundary $\partial(B^3\times B^3) = (S^2\times B^3)\cup$

$\cup (B^3 \times S^2)$ and the interior $\mathrm{Int}(B^3 \times B^3) = (B^3 - S^2) \times (B^3 - S^2)$ are saturated subsets of $B^3 \times B^3$ with respect to $G \times G$. It is easy to see directly (or cf. [DS]) that the image of $\partial(B^3 \times B^3)$ under the projection $\pi \times \pi$, $\pi \times \pi\colon B^3 \times B^3 \to Q \times Q$, has DDP; and moreover, it follows from a theorem of C. D. Bass [BA] that the image of $\mathrm{Int}(B^3 \times B^3)$ has DDP. This means that each of the decompositions of $\partial(B^3 \times B^3)$ and $\mathrm{Int}(B^3 \times B^3)$ induced by $G \times G$ is shrinkable. This follows from Edwards' Approximation Theorem [E]. We next show that this suffices to prove that $Q \times Q$ is homeomorphic to $B^3 \times B^3$. Let $\alpha\colon B^3 \times B^3 \to Q' \times Q'$ and $\beta\colon Q' \times Q' \to Q \times Q$ be projections associated with the decompositions, which are called α and β, of $B^3 \times B^3$ where the nondegenerate elements of α are those of $G \times G$ which lie on $\partial(B^3 \times B^3)$ and the nondegenerate elements of β are of the form $\alpha(g)$ where g is a nondegenerate element of $G \times G$ which is contained in $\mathrm{Int}(B^3 \times B^3)$. It is easy to see that $Q' \times Q'$ is homeomorphic to $B^3 \times B^3$ (more specifically, α can be approximated by a homeomorphism). Since all the nondegenerate elements of β lie in the interior, the map β can be approximated by a homeomorphism. This proves that $Q \times Q$ is homeomorphic with $B^6 = B^3 \times B^3$. It is clear that Q is incompressible and B^6 is compressible. ■

The following more general result is an easy consequence of our results in [S] and the discussions given above.

(2.3) THEOREM. *For each integer* $n \geqslant 3$, *there exists an uncountable family* $\mathcal{F}_n$ *consisting of* n*-dimensional incompressible* AR*'s such that* (a) $A \times S^1$ *is homeomorphic to* $B^n \times S^1$ *for any* AR A *belonging to* $\mathcal{F}_n$, *and* (b) $A \times B$ *is homeomorphic to* $B^n \times B^n$ *for any two* AR*'s* A *and* B *belonging to* $\mathcal{F}_n$.

Since $B^n \times S^1$ is clearly compressible, our assertion (b) in Theorem (2.3) may be compared with some results of Husch [H]. We may point out that K. Borsuk (cf. [B] or [M]) and R. Molski [M] (or cf. [B]) have also constructed, for each $n \geqslant 2$, an uncountable family of incompressible n-dimensional AR's. The following question appears to be an open problem.

(2.4) QUESTION. Does there exist two incompressible AR's X and Y such that $1 \leqslant \dim(X) \leqslant 3$, $1 \leqslant \dim(Y) \leqslant 2$, and $X \times Y$ is compressible?

Many others, but, somewhat related results concerning products of some AR's may also be found in [DS].

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