References

[1] J. Akiyama, T. Hamada and I. Yoshimura, *On characterizations of middle graphs*, TRU Mathematics 11 (1975), pp. 35–39.

[2] M. Borowiecki, *Connected bijection method in hypergraph theory and some results concerning the structure of graphs and hypergraphs*, Wydawnictwo Uczelniane W. S. Inż., Zielona Góra 1979.

[3] T. Hamada and I. Yoshimura, *Traversability and connectivity of the middle graph of a graph*, Discrete Math. 14 (1976), pp. 247–256.

[4] R. J. Wilson, *Introduction to Graph Theory*, Olivier and Boyd, Edinburgh 1972.

COLLEGE OF ENGINEERING
DEPARTMENT OF MATHEMATICS
Zielona Góra, Poland

Accepté par la Rédaction le 1. 9. 1980

Powers of spaces of non-stationary ultrafilters

by

J. E. Vaughan (Greensboro, N. C.)

Abstract. Let X denote the space of all non-stationary ultrafilters on a regular uncountable cardinal $\varkappa$ (or more generally, the space associated with a normal ideal on $\varkappa$). These spaces were recently introduced by Eric van Douwen, who showed that X is strongly $\varkappa$-compact but not $\varkappa$-bounded. We show in this paper that X^ω is not strongly $\varkappa$-compact, $X^{2^\varkappa}$ is not totally initially $\varkappa$-compact and X^μ (assuming GCH) is initially $\varkappa$-compact for all cardinals μ. These results answer two basic questions concerning these compactness-like properties.

1. Introduction. The theory of products of countably compact and related spaces is extensive, but the generalization of this theory to higher cardinals is not as well developed. There are some very basic questions which have been answered in the countable case but not in the uncountable case. Two of these questions are concerned with the notions of strong $\varkappa$-compactness and TI-$\varkappa$-compactness.

A space X is said to be *strongly $\varkappa$-compact* provided that for every filter base $\mathscr{F}$ on X of cardinality $\leqslant\varkappa$, there exists a compact set $K\subset X$ such that $F\cap K\neq\emptyset$ for all F in $\mathscr{F}$. A T_3-space X is TI-*$\varkappa$-compact* provided that for every filter base $\mathscr{F}$ on X of cardinality $\leqslant\varkappa$, there exist a compact set $K\subset X$ and a filter base $\mathscr{G}$ of cardinality $\leqslant\varkappa$ such that $\mathscr{G}$ is finer than $\mathscr{F}$ (i.e., every member of $\mathscr{F}$ contains a member of $\mathscr{G}$) and $\mathscr{G}$ converges to K in the sense that every open set containing K also contains a member of $\mathscr{G}$ (see § 2 for the definition of TI-$\varkappa$-compactness in general spaces and for all other definitions).

Clearly, every strongly $\varkappa$-compact space is TI-$\varkappa$-compact (take $\mathscr{G}$ to be $\{F\cap K\colon F\in\mathscr{F}\}$), and the converse is true if $\varkappa=\omega$ (in the class of T_3-spaces). The simple proof of the equivalence of these two properties for the case $\varkappa=\omega$ does not extend to higher cardinals; so we have the basic question:

1.1. For $\varkappa>\omega$, is every TI-$\varkappa$-compact space strongly $\varkappa$-compact?

An important property of the class of TI-$\varkappa$-compact spaces is that it is stable under $\varkappa$-fold products (i.e., every product of $\leqslant\varkappa$ TI-$\varkappa$-compact spaces is TI-$\varkappa$-compact). The proof of this does not extend to strong $\varkappa$-compactness; so we have a second basic question:

1.2. For $\varkappa>\omega$, is every product of no more than $\varkappa$ strongly $\varkappa$-compact spaces, strongly $\varkappa$-compact?

We first considered these questions about ten years ago, and noticed then that if we could find a strongly $\varkappa$-compact space X such that $X^{\varkappa}$ was not strongly $\varkappa$-compact (thus answering 1.2 in the negative) then the product space $X^{\varkappa}$ also answered 1.1 in the negative since TI-$\varkappa$-compactness is $\varkappa$-fold productive. We had little hope, however, that a space X which answered 1.2 could be easily described or that its productive properties could be easily derived. We were quite pleased, therefore, to find that some spaces recently introduced by Eric van Douwen do all of this and more.

The purpose of this paper is to give a detailed description of the compactness-like properties of powers of these spaces, and to use them to answer questions 1.1 and 1.2 for regular uncountable cardinals $\varkappa$ in the negative. For 1.2 more is possible: We show that a *countable* product of strongly $\varkappa$-compact spaces (for $\varkappa$ regular and uncountable) need not be strongly $\varkappa$-compact.

2. Statements of results. The spaces considered in this paper are subspaces of the Stone-Čech compactification of a regular uncountable cardinal $\varkappa$ (where $\varkappa$ has the discrete topology). In [5] van Douwen defined for each normal ideal $\mathscr{I}$ on $\varkappa$ the subspace $X = X(\mathscr{I})$ of $\beta(\varkappa)$ to be the set of all ultrafilters on $\varkappa$ which contain a member of the ideal $\mathscr{I}$. He proved

2.1. *X is λ-bounded for all $\lambda<\varkappa$.*

2.2. *X is not $\varkappa$-bounded.*

2.3. *X is strongly $\varkappa$-compact.*

In this paper we study the compactness-like properties of powers of X and show

2.4. *X^n is strongly $\varkappa$-compact for all $n<\omega$* (this property is obviously finitely productive [18, p. 281]).

2.5. *X^{ω} is not strongly $\varkappa$-compact* (see § 4).

2.6. *$X^{\varkappa}$ is* TI-*$\varkappa$-compact* (this property is $\varkappa$-fold productive [22, Cor. 3.3]).

2.7. *$X^{2^{\varkappa}}$ is not* TI-*$\varkappa$-compact* (see § 5).

2.8. (GCH) *X^{μ} is initially $\varkappa$-compact for every cardinal μ* (see § 6).

From these results we can answer questions 1.1 and 1.2.

2.9. *Let $\varkappa$ be a regular uncountable cardinal. A countable product of strongly $\varkappa$-compact spaces need not be strongly $\varkappa$-compact. Strong ω-compactness is, however, countably productive* [6, 4.2.3] *or* [22, Cor. 3.2].

2.10. *Let $\varkappa$ be a regular uncountable cardinal. There exists a* TI-*$\varkappa$-compact space which is not strongly $\varkappa$-compact (namely X^{ω} in 2.5). In the class of T_3-spaces, strong ω-compactness and* TI-*ω-compactness are equivalent* [20, Lemma 2.1].

3. Definitions and known results. For an infinite cardinal $\varkappa$, we consider the Stone–Čech compactification of $\varkappa$ (where $\varkappa$ has the discrete topology) as a space of ultrafilters on $\varkappa$ (see [3]). An ideal on a regular, uncountable $\varkappa$ is normal provided

it is non-trivial, $<\varkappa$-complete, and closed under diagonal union (see [5]). For a normal ideal $\mathscr{I}$ *the space associated with $\mathscr{I}$ is*

$$X(\mathscr{I}) = \bigcup \{\mathrm{cl}_{\beta(\varkappa)}(I)\colon I \in \mathscr{I}\}$$

considered as a subspace of $\beta(\varkappa)$. These spaces were introduced by van Douwen in [5]. We need very little about normal ideals in this paper (other than 2.1–2.3). We mention that each singleton is in I; so $\varkappa \subset X(\mathscr{I})$, and by 2.2 $X(\mathscr{I}) \neq \beta(\varkappa)$. We also need that $\mathscr{I}$ is *countably complete*, i.e., every union of countably many elements of $\mathscr{I}$ is an element of $\mathscr{I}$ (this follows from $<\varkappa$-completeness).

The set of all functions from $\varkappa$ into X is denoted by ${}^{\varkappa}X$, and such a function is sometimes called a *$\varkappa$-sequence*. Let $[\varkappa]^{\varkappa}$ denote the set of all subsets of $\varkappa$ having cardinality $\varkappa$, and define $[\varkappa]^{<\varkappa}$ to be the set of all subsets of $\varkappa$ having cardinality $<\varkappa$.

A filter base $\mathscr{F}$ on a set P is a non-empty collection of non-empty subsets of P such that the intersection of any two members of $\mathscr{F}$ contains a member of $\mathscr{F}$. A filter base $\mathscr{F}$ *traces* on a set T provided $F \cap T \neq \emptyset$ for all F in $\mathscr{F}$. A filter base $\mathscr{G}$ is *finer* than a filter base $\mathscr{F}$ provided that for each F in $\mathscr{F}$ there exists a G in $\mathscr{G}$ such that $G \subset F$. A filter base $\mathscr{F}$ on a cardinal $\varkappa$ is called *uniform* provided that for each F in $\mathscr{F}$, $|F| = \varkappa$. Put

$$\mathrm{Unif}(\varkappa) = \{u \in \beta(\varkappa)\colon u \text{ is a uniform ultrafilter on } \varkappa\}.$$

The Generalized Continuum Hypothesis (GCH) is used several times in this paper.

We next give some definitions from topology. A filter base $\mathscr{F}$ on a topological space X is called *total* provided that every filter base $\mathscr{H}$ finer than F has an adherent point (i.e., $\bigcap \{\overline{H}\colon H \in \mathscr{H}\} \neq \emptyset$). Clearly, the notion of total filter base generalizes that of a convergent filter base (see [16], [21], and [22]).

For a family $\{X_{\alpha}\colon \alpha<\mu\}$ of topological spaces, where μ is a cardinal number, let $\prod \{X_{\alpha}\colon \alpha<\mu\}$ denote the Cartesian product of the family endowed with the (Tychonoff) product topology. In case each X_{α} is homeomorphic to a single space X, we write X^{μ} instead of $\prod \{X_{\alpha}\colon \alpha<\mu\}$, and say that X^{μ} is a *power of* X.

A topological space P is called

$\varkappa$-bounded [9] if for every $H \subset P$, and $|H| \leqslant \varkappa$ there exists a compact subset $T \subset P$ such that $H \subset T$,

strongly $\varkappa$-compact [18] if for every filter base $\mathscr{F}$ on P, if $|\mathscr{F}| \leqslant \varkappa$, then there exists a compact set $T \subset P$ such that $\mathscr{F}$ traces on T,

totally initially $\varkappa$-compact (for short: TI-$\varkappa$-compact) [22] if for every filter base $\mathscr{F}$ on P with $|\mathscr{F}| \leqslant \varkappa$ there exists a finer, total filter base $\mathscr{G}$ with $|\mathscr{G}| \leqslant \varkappa$, and

initially $\varkappa$-compact [1] if every open cover of P having cardinality $\leqslant \varkappa$ has a finite subcover.

It is easy to see that for every infinite cardinal $\varkappa$

$\varkappa$-bounded $\to$ strongly $\varkappa$-compact $\to$ TI-$\varkappa$-compact $\to$ initially $\varkappa$-compact.

None of these implications can be reversed for every value of $\varkappa$. For $\varkappa$ regular and uncountable, the space X of van Douwen is strongly $\varkappa$-compact but not $\varkappa$-bounded

(2.2, 2.3), and the space X^ω is TI-$\varkappa$-compact but not strongly $\varkappa$-compact (2.5, 2.6). For $\varkappa$ infinite, Example 3.9 in [20] gives an initially $\varkappa$-compact space which is not TI-$\varkappa$-compact. For the special case $\varkappa = \omega$, we refer the reader to [18, Remark 2.2], [20, Lemma 2.1], and [22, § 4], and mention again that for $\varkappa = \omega$ the two middle properties above are equivalent in T_3-spaces.

It is interesting to note the extent to which each of these four properties is preserved under products (there is substantial variation). We start with

$\varkappa$-boundedness: Since every product of compact sets is compact, it is clear that every product of $\varkappa$-bounded spaces is $\varkappa$-bounded (for $\varkappa \geqslant \omega$) [9],

strong $\varkappa$-compactness: For regular $\varkappa > \omega$, this property is finitely productive (as is easy to see; [18]), but is not countably productive (2.5),

TI-$\varkappa$-compactness: This property is $\varkappa$-fold productive for all $\varkappa \geqslant \omega$ [22]. This nice fact (as well as others) points up the interest in this property. TI-$\varkappa$-compactness is not $2^\varkappa$-fold productive in a strong sense (see Theorem 5.9),

initial $\varkappa$-compactness: For regular $\varkappa$, this property is not finitely productive. Van Douwen has shown (assuming GCH) that there exist spaces X and Y which are initially $\varkappa$-compact, but $X \times Y$ is not initially $\varkappa$-compact [4]. On the other hand, if $\varkappa$ is a singular cardinal, then (assuming GCH) every product of initially $\varkappa$-compact spaces is initially $\varkappa$-compact [18] and [20].

4. Proof of 2.5: X^ω is not strongly $\varkappa$-compact.

4.1. Lemma. *If X is a space such that X^ω is strongly $\varkappa$-compact (for any $\varkappa \geqslant \omega$) then every subset of X having cardinality $\leqslant \varkappa$, is contained in a σ-compact subset of X.*

Proof. Let $H \in [X]^\varkappa$, then $|H| = \lambda \leqslant \varkappa$. For every $A \in [H]^{<\omega}$ put

$$F(A) = \{f \in {}^\omega H \colon \mathrm{Range}(f) \supset A\}.$$

Then $\mathscr{F} = \{F(A) \colon A \in [H]^{<\omega}\}$ is a filter base on X^ω (since $F(A) \cap F(A') = F(A \cup A')$) having cardinality $\leqslant \varkappa$. Since X^ω is strongly $\varkappa$-compact, there exists a compact set $T \subset X^\omega$ such that $\mathscr{F}$ traces on T. We show that $H \subset \bigcup \{\pi_n(T) \colon n < \omega\}$ where π_n is the natural projection. Let $x \in H$, and let $A = \{x\}$. Let $f \in F(A) \cap T$, which is non-empty because $\mathscr{F}$ traces on T. Then there exists $n < \omega$ such that $f(n) = x$. Since $f(n)$ is an element of $\pi_n(T)$, this shows $x \in \pi_n(T)$. Thus, H is a subset of a σ-compact set.

4.2. Lemma. *If every σ-compact subset of X is contained in a compact subset of X, and if X^ω is strongly $\varkappa$-compact for any infinite cardinal $\varkappa$, then X is $\varkappa$-bounded.*

Proof. By 4.1, every subset of X of cardinality $\leqslant \varkappa$ is contained in a σ-compact subset of X which by hypothesis is contained in a compact subset of X. Thus X is $\varkappa$-bounded.

The next lemma is implicit in [5, 3.1].

4.3. Lemma. *Let $\mathscr{I}$ be an ideal on $\varkappa$ ($\varkappa \geqslant \omega$). For every compact set $T \subset X(\mathscr{I})$, there exists $I \in \mathscr{I}$ such that $T \subset \mathrm{cl}_{\beta(\varkappa)}(I)$.*

Proof. This follows from the fact that $\{\mathrm{cl}_{\beta(\varkappa)}(I) \colon I \in \mathscr{I}\}$ is an open cover of $X(\mathscr{I})$, and an ideal is stable under finite unions.

4.4 Lemma. *If $\mathscr{I}$ is a countably complete ideal on $\varkappa$ ($\varkappa \geqslant \omega_1$), then every σ-compact subset of $X(\mathscr{I})$ is contained in a compact subset of $X(\mathscr{I})$.*

Proof. This is an immediate consequence of 4.3 and countable completeness.

Proof of 2.5: X^ω is not strongly $\varkappa$-compact ($\varkappa > \omega$). This follows from Lemmas 4.4 and 4.2.

5. Proof of 2.7: $X^{2^\varkappa}$ is not totally initially $\varkappa$-compact. According to S. H. Hechler [10], the cardinal number K_C is defined to be the smallest cardinal number which is the cardinality of a family $\mathscr{F} \subset {}^\omega\omega$ such that for all $H \in [\omega]^\omega$ there exists $f \in \mathscr{F}$ such that $f(H) = \omega$. He proved

5.1. Theorem (Hechler [10]). *If a product of at least K_C spaces is strongly ω-compact, then at least one of the coordinate spaces is ω-bounded.*

A straightforward generalization of this result goes as follows. Define for every cardinal $\varkappa$, the cardinal $K_C(\varkappa)$ to be the smallest cardinal number which is the cardinality of a family $\mathscr{F} \subset {}^\varkappa\varkappa$ such that for all $H \in [\varkappa]^\varkappa$ there exists $f \in \mathscr{F}$ such that $f(H) = \varkappa$. In analogy with 5.1, we have

5.2. Theorem. *If a product of at least $K_C(\varkappa)$ spaces is strongly $\varkappa$-compact, then at least one coordinate space is $\varkappa$-bounded.*

Proof. For regular cardinals $\varkappa$, the proof is very similar to Hechler's proof of 4.1. For singular $\varkappa$, we use in addition Lemma 3.6 in [20].

In order to prove 2.7, we will first prove an analogue of Theorem 5.2 for TI-$\varkappa$-compactness. Since TI-ω-compactness is weaker than strong ω-compactness in T_2 (not T_3) spaces (e.g. the space S of Example 5.13), this analogue will give a new result even in the countable case. This analogue is not quite as straightforward as was Theorem 5.2 because we must work with filter bases instead of $\varkappa$-sequences, and we have to take into account what happens in the T_2 (not T_3) case. In particular, $\varkappa$-boundedness is not the correct property to use here. For this reason we define the notion of a "$\varkappa$-total" space. We also need an analogue of the cardinal $K_C(\varkappa)$.

5.3. Definition. For an infinite cardinal $\varkappa$, let $h(\varkappa)$ denote the smallest cardinal which is the cardinality of a family $\mathscr{F} \subset {}^\varkappa\varkappa$ with the property that for every uniform filter base $\mathscr{H}$ on $\varkappa$ with $|\mathscr{H}| \leqslant \varkappa$ there exists $f \in \mathscr{F}$ such that $f(H)$ contains a final segment of $\varkappa$ for all $H \in \mathscr{H}$.

To see that $h(\varkappa)$ is well-defined, we show that $\mathscr{F} = {}^\varkappa\varkappa$ has the property mentioned in 5.3: Let $\mathscr{H}$ be a uniform filter base on $\varkappa$ with $|\mathscr{H}| \leqslant \varkappa$. By the disjoint refinement lemma [3, 7.5] there exists a family $\{B_H \colon H \in \mathscr{H}\}$ of mutually disjoint subsets of $\varkappa$ such that $B_H \subset H$ and $|B_H| = \varkappa$ for all $H \in \mathscr{H}$. By mapping each B_H onto $\varkappa$ we can construct a function $f \in {}^\varkappa\varkappa$ such that $f(H) = \varkappa$ for all $H \in \mathscr{H}$. Hence, $h(\varkappa)$ is well-defined and $h(\varkappa) \leqslant 2^\varkappa$.

5.4. Remark. It is easy to see that $\varkappa < h(\varkappa)$. Recall [22, Cor. 3.3] that every product of no more than $\varkappa$ TI-$\varkappa$-compact spaces is TI-$\varkappa$-compact. Now $\boldsymbol{X}$ is

TI-$\varkappa$-compact (by 2.2), but by Corollary 5.10 below, $X^{h(\varkappa)}$ is not TI-$\varkappa$-compact. Thus $h(\varkappa)\nleqslant\varkappa$.

5.5. Definition. A set $T\subset X$ is said to be *total in X* provided that the singleton filter base $\{T\}$ is total in X.

5.6. Lemma. *The following are equivalent for a set $T\subset X$.*

1. *T is total in X.*
2. *Every filter base (ultrafilter) which traces on T has an adherent point (converges to a point) in X.*
3. *Every open cover of X contains a finite subfamily which covers T.*

The property considered in this lemma has been discovered independently by at least five authors (see [7], [11], [12], [13], [15]).

If $\bar{T}$ is compact, then T is a total set, and in T_3-spaces the converse is true. Examples abound of T_2 (not T_3) spaces which contain total sets whose closures are not compact: [7, Example 1], [12], [15, Example 2.1], [22, Example 2], and Example 5.13 below.

5.7. Definition. For $\varkappa\geqslant\omega$, a space X is called *$\varkappa$-total* provided that every subset of X of cardinality $\leqslant\varkappa$ is total.

5.8. Remark. The notion of ultracompactness was introduced by A. Bernstein [2] and extended by V. Saks to $\varkappa$-ultracompactness [17]. Contrary to what the name suggests, ultracompactness is strictly weaker than compactness (the prefix refers to ultrafilters). Now it is easy to check that a space is $\varkappa$-total ($\varkappa\geqslant\omega$) if and only if it is $\varkappa$-ultracompact, and therefore we may use the term $\varkappa$-total instead of $\varkappa$-ultracompactness, and the term ω-total instead of ultracompactness. We believe this shorter terminology is clearer.

With these two definitions (5.3 and 5.7) we can prove the following result following the basic outline of Hechler's proof of Theorem 5.1.

5.9. Theorem. *If $X=\prod\{X_\alpha\colon \alpha<h(\varkappa)\}$ is a product space which is* TI-$\varkappa$-*compact, then there exists $\alpha<h(\varkappa)$ such that X_α is $\varkappa$-total.*

Proof. The proof is by contradiction. Assume for each $\alpha<\varkappa$ there exists a set $S_\alpha\subset X_\alpha$ which is not total and $|S_\alpha|\leqslant\varkappa$. List $S_\alpha=\{x_\beta^\alpha\colon \beta<\varkappa\}$ in such a way that each point in S_α appears $\varkappa$ times in the list. Let $\{f_\alpha\colon \alpha<h(\varkappa)\}$ be a subset of ${}^\varkappa\varkappa$ which satisfies the condition in the definition of $h(\varkappa)$. Define a $\varkappa$-sequence $\langle y_\gamma\rangle$ in the product space X as follows: Let π_α denote the usual projection map $\pi_\alpha\colon X\to X_\alpha$. Define y_γ so that

$$\pi_\alpha(y_\gamma)=x^\alpha_{f_\alpha(\gamma)}.$$

Let $\mathscr{S}$ be a filter base on $\varkappa$ such that (1) $|\mathscr{S}|\leqslant\varkappa$ and (2) for every $V\in[\varkappa]^{<\varkappa}$ there exists $S\in\mathscr{S}$ such that $S\cap V=\emptyset$ (for $\varkappa$ singular, use [20, Lemma 3.6]). For each S in $\mathscr{S}$ define $T(S)=\{y_\alpha\colon \alpha\in S\}$, and put $\mathscr{T}=\{T_S: S\in\mathscr{S}\}$. Then $\mathscr{T}$ is a filter base on X of cardinality $\leqslant\varkappa$; so by hypothesis, there exists a filter base $\mathscr{G}$ of cardinality $\leqslant\varkappa$ which is total and which is finer than $\mathscr{T}$. For each G in $\mathscr{G}$ define $H(G)=\{\alpha<\varkappa\colon y_\alpha\in G\}$ and put $\mathscr{H}=\{H(G)\colon G\in\mathscr{G}\}$.

Then $\mathscr{H}$ is a filter base on $\varkappa$ of cardinality $\leqslant\varkappa$ and furthermore (by property 2 of $\mathscr{S}$) $\mathscr{H}$ is uniform. Thus there exists $\alpha<\varkappa$ such that $f_\alpha(H_G)$ contains a final segment, say $(\tau_G,\varkappa)$, for all G in $\mathscr{G}$. By the definition of $\langle y_\gamma\rangle$ this implies that for each $G\in\mathscr{G}$, $\pi_\alpha(G)\supset\{x^\alpha_\beta\colon \tau_G<\beta<\varkappa\}$. Hence (by the redundant listing of S_α) $\pi_\alpha(G)\supset S_\alpha$ for all G in $\mathscr{G}$. Since the continuous image of a total filter base is total [21, Lemma 3], $\pi_\alpha(\mathscr{G})$ is total, and therefore the finer filter base $\{S_\alpha\}$ is also total. Thus S_α is a total set, and this contradiction completes the proof.

5.10. Corollary. *If the product of at least $h(\varkappa)$ T_3-spaces is* TI-$\varkappa$-*compact, then at least one of the coordinate spaces is $\varkappa$-bounded.*

That $X^{h(\varkappa)}$ (a fortiori $X^{2^\varkappa}$) is not TI-$\varkappa$-compact follows from this corollary and 2.2.

We now turn to the countable case of Theorem 5.9.

5.11. Lemma. $K_C=h(\omega)$.

Proof. That $h(\omega)\leqslant K_C$ follows at once from the well-known fact that for every countable filter base $\mathscr{H}$ of infinite subsets of ω there exists $A\in[\omega]^\omega$ such that $(A-H)$ is finite for all $H\in\mathscr{H}$. We show that $K_C\leqslant h(\omega)$. Let $\mathscr{F}\subset{}^\omega\omega$ satisfy the definition of $h(\omega)$. For every $f\in\mathscr{F}$ and $F\in[\omega]^{<\omega}$ define $g(f,F)\in{}^\omega\omega$ as follows: let $F=\{x_i\colon i<n\}$ where $n=|F|$ and set

$$g(f,F)=\begin{cases} f(i) & \text{if } i\notin F,\\ j & \text{if } i\in F \text{ and } i=x_j.\end{cases}$$

Thus $g(f,F)=f$ on $\omega\setminus F$ and $g(f,F)(F)=n$ if $n=|F|$. Let

$$\mathscr{G}=\{g(f,F)\colon f\in\mathscr{F} \text{ and } F\in[\omega]^{<\omega}\}.$$

Thus $|\mathscr{G}|=|\mathscr{F}|$. We have only to show that $\mathscr{G}$ satisfies the property in the definition of K_C. Let $H\in[\omega]^\omega$, and let A be an infinite subset of H such that $H-A$ is infinite. There exists $f\in\mathscr{F}$ such that $f(A)$ contains a final segment of ω; say

$$f(A)\supset\{i<\omega\colon i\geqslant n\}.$$

Let F be a subset of $(H-A)$ having cardinality n. Then $g(f,F)(H)=\omega$.

From Theorem 5.9 we get the following analogue of Hechler's theorem (5.1).

5.12. Corollary. *If a product of at least K_C spaces is* TI-ω-*compact then at least one of the coordinate spaces is ω-total.*

We mentioned that the concept of "ω-total" is not new. It was first considered by A. Bernstein under the name "ultracompact" [2] (a space is called *ultracompact* provided every sequence has a D-limit for every D in $\beta(\omega)\setminus\omega$). It has also been considered by Hechler under the name "e-$\aleph_0$-bounded." Victor Saks [17, 5.6] gave (modulo the existence of a wP-point in $\beta(\omega)\setminus\omega$) an example of a T_2 (not T_3) space which is ω-total but not ω-bounded. Saks's example is now complete since K. Kunen has shown that there exist wP-points in $\beta(\omega)\setminus\omega$ [14].

5.13. Example. An ω-total space P which is not strongly ω-compact. Let S be the ω-total not ω-bounded space of Saks. The space S is not the desired example because it is strongly ω-compact. It is not difficult, however, to modify the construction of S so as to get the desired example. We give here another method which is based on the preceding results. Let $P = S^{2^\omega}$. Every product of $\varkappa$-total spaces is $\varkappa$-total (see [13], [15], or note that this follows easily from the theorem of Pettis which states that any product of total filter bases is a total filter base). Thus, since S is ω-total, so is P. By Hechler's theorem (5.1) P is not strongly ω-compact.

We have mentioned that there are a number of examples of total sets whose closures are not compact. We point out that any $\varkappa$-total space which is not $\varkappa$-bounded (e.g. S) must have a total set of cardinality $\varkappa$ which does not have compact closure.

6. Proof of 2.8: (GCH) **Every power of X is initially $\varkappa$-compact.** We say that a space X is $<\varkappa$-bounded provided that X is λ-bounded for every cardinal $\lambda<\varkappa$. By 2.1 and 2.2, the space X is $<\varkappa$-bounded but not $\varkappa$-bounded.

The following result, which is a direct analogue of Theorem 2.6 in [8] shows what we must do to prove 2.8.

6.1. THEOREM. *Let X be $<\varkappa$-bounded, where $\varkappa \geqslant \omega$. The following are equivalent.*

1. *Every power of X is initially $\varkappa$-compact.*
2. *$X^{2^{2^\varkappa}}$ is initially $\varkappa$-compact.*
3. *$X^{|X|^\varkappa}$ is initially $\varkappa$-compact.*
4. *There exists $D \in \mathrm{Unif}(\varkappa)$ such that X is D-$\varkappa$-compact.*

Our plan of attack is to show that there exists $D \in \mathrm{Unif}(\varkappa)$ such that X is D-$\varkappa$-compact. First we recall the relevant definitions. Let $D \in \mathrm{Unif}(\varkappa)$, let X be a space, $x \in X$, and $f \in {}^\varkappa X$. The point x is called a *D-$\varkappa$-limit of f in X* provided that for every neighborhood U of x, $\{\alpha<\varkappa\colon f(\alpha) \in U\} \in D$. A space X is *D-$\varkappa$-compact* provided that every $\varkappa$-sequence in X has a D-$\varkappa$-limit [17].

The hypotheses of the next lemma were dictated by the space X.

6.2. LEMMA. *If a space X is $<\varkappa$-bounded, and $X^{2^\varkappa}$ is initially $\varkappa$-compact and $X = Y \cup Z$ where $|Y| \leqslant 2^\varkappa$ and Z is $\varkappa$-bounded, there exists $D \in \mathrm{Unif}(\varkappa)$ such that X is D-$\varkappa$-compact.*

Proof. We give a proof of this which follows the idea of $3 \to 4$ in Theorem 6.1 (or Theorem 2.6 of [8]). This method, which is a little shorter than our original proof, was suggested by Victor Saks. If $|Y|<\varkappa$, then X is $\varkappa$-bounded and there is nothing to prove. We assume that $\varkappa \leqslant |Y| \leqslant 2^\varkappa$. Let $\{f_\alpha\colon \alpha<2^\varkappa\}$ list all $\varkappa$-sequences in Y, and define a $\varkappa$-sequence $\langle y_\alpha \rangle$ in $X^{2^\varkappa}$ by the rule $\pi_\alpha(y_\alpha) = f_\alpha(\gamma)$. Since there are $f \in {}^\varkappa Y$ which are one-one we have that the map $\gamma \mapsto y_\gamma$ is also one-one. Since $X^{2^\varkappa}$ is initially $\varkappa$-compact there exists $y \in X^{2^\varkappa}$ such that for every neighborhood U of y, $|\{\gamma<\varkappa\colon y_\gamma \in U\}| = \varkappa$ (i.e., y is a complete accumulation point of $\{y_\gamma\colon \gamma<\varkappa\}$). Thus, we may choose $D \in \mathrm{Unif}(\varkappa)$ such that y is a D-$\varkappa$-limit of the $\varkappa$-sequence $\langle y_\gamma \rangle$. Since D-$\varkappa$-limits are preserved by continuous maps, we have $\pi_\alpha(y)$ is a D-$\varkappa$-limit of f_α in X for all $\alpha<2^\varkappa$. Thus, for this D, every $\varkappa$-sequence in Y has a D-$\varkappa$-limit point in X. Now we can see that X is D-$\varkappa$-compact, because every $\varkappa$-sequence in X must map a member of D into Y or Z. In either case, the $\varkappa$-sequence

will have a D-$\varkappa$-limit in X. It follows from Theorem 6.1, that every power of X is initially $\varkappa$-compact.

Proof of 2.8. The space X is $<\varkappa$-bounded, and van Douwen has shown that $X = Y \cup Z$ where $Y = [\beta(\varkappa) \setminus \mathrm{Unif}(\varkappa)]$ and $Z = X \setminus Y$, and Z is $\varkappa$-bounded. Under (GCH) $|Y| = 2^\varkappa$. Since X is strongly $\varkappa$-compact, hence TI-$\varkappa$-compact, $X^{\varkappa^+} = X^{2^\varkappa}$ is initially $\varkappa$-compact by Corollary 3.3 in [22]. Thus X satisfies the hypothesis of Lemma 6.2, and therefore every power of X is initially $\varkappa$-compact.

We conclude with some open problems.

PROBLEM 1. For $\varkappa>\omega$, is every product of strongly $\varkappa$-compact spaces initially $\varkappa$-compact? This problem was first raised by R. M. Stephenson, Jr. [19, p. 317].

PROBLEM 2. For $\varkappa$ a singular cardinal, does there exist a strongly $\varkappa$-compact space which is not $\varkappa$-bounded?

PROBLEM 3. Answer 1.1 and 1.2 for singular cardinals $\varkappa$.

PROBLEM 4. Can (GCH) be deleted from 2.8?

References

[1] P. S. Alexandroff and P. Urysohn, *Mémoire sur les espaces topologiques compacts*, Verh. Koninkl. Akad. Wetensch., Amsterdam 14 (1929), pp. 1–96.

[2] A. R. Bernstein, *A new kind of compactness for tcpological spaces*, Fund. Math. 66 (1970), pp. 185–193.

[3] W. W. Comfort and S. Negrepontis, *The Theory of Ultrafilters*, Die Grundlehren der Math. Wissenschaften Band 211, Springer–Verlag, New York 1974.

[4] E. van Douwen, *The product of two initially $\varkappa$-compact spaces*, preprint.

[5] — *Compactness-like properties and nonnormality of the space of nonstationary ultrafilters*, preprint.

[6] Z. Frolík, *The topological product of two pseudo-compact spaces*, Czechoslovak Math. J. 10 (85) (1960), pp. 339–349.

[7] S. Gagola and M. Gemignani, *Absolutely bounded sets*, Math. Japon. 13 (1969), pp. 129–132.

[8] J. Ginsburg and V. Saks, *Some applications of ultrafilters in topology*, Pacific J. Math. 57 (1975), pp. 403–418.

[9] S. L. Gulden, W. F. Fleischman and J. H. Weston, *Linearly ordered topological spaces*, Proc. Amer. Math. Soc. 24 (1970), pp. 197–203.

[10] S. H. Hechler, *On some weakly compact spaces and their products*, Gen. Top. and Appl. 5 (1975), pp. 83–93.

[11] — *On a notion of weak compactness in non-regular spaces*, Studies in Topology, edited by N. M. Stavrakas and K. R. Allen, Academic Press, 1975, pp. 215–238.

[12] N. Hindman, *Basically bounded sets and a generalized Heine-Borel theorem*, Amer. Math. Monthly 80 (1973), pp. 549–552.

[13] S. M. Karnik, *On boundedness*, Math. Japon. (1974), pp. 165–172.

[14] K. Kunen, *Weak P-points in N**, preprint.

[15] P. Lambrinos, *A topological notion of boundedness*, Manuscripta Math. 10 (1973), pp. 289–296.

[16] B. J. Pettis, *Cluster sets nets*, Proc. Amer. Math. Soc. 22 (1969), pp. 386–391.

[17] V. Saks, *Ultrafilter invariants in topological spaces*, Trans. Amer. Math. Soc. 241 (1978), pp. 79–97.

[18] V. Saks and R. M. Stephenson, Jr., *Products of m-compact spaces*, Proc. Amer. Math. Soc. 28 (1971), pp. 279–288.
[19] R. M. Stephenson, Jr., *Products of nearly compact spaces*, Proc. Univ. Okla. Top. Conf., Dept. Math. Univ. Oklahoma, Norman 1972, pp. 310–320.
[20] — and J. E. Vaughan, *Products of initially m-compact spaces*. Trans. Amer. Math. Soc. 196 (1974), pp. 177–189.
[21] J. E. Vaughan, *Total nets and filters*, Proc. Memphis State Univ. Topology Conference, Lecture Notes in Pure and Appl. Math. 24, New York 1976, pp. 259–265.
[22] — *Products of topological spaces*, Gen. Top. and Appl. 8 (1978), pp. 207–217.

UNIVERSITY OF NORTH CAROLINA AT GREENSBORO
Greensboro, N.C. 27412

Accepté par la Rédaction le 1. 9. 1980

On locally expansive selfcoverings of compact metrizable spaces

by

Aleksander Całka (Wrocław)

Abstract. This paper is concerned with characterizing, in terms of certain properties of their compositions, the open local expansions defined on compact locally connected Hausdorff spaces.

1. Introduction. Let (M, ϱ) be a metric space and $f\colon M \to M$ a continuous selfmapping of M. We will call f a *local expansion on* (M, ϱ) [3] (cf. also [2]) if

(A) for each $z \in M$ there is a neighborhood U of z and a number $\lambda_U > 1$ such that

$$\varrho(f(x), f(y)) \geqslant \lambda_U \varrho(x, y) \quad \text{for} \quad x, y \in U. \tag{1}$$

If there exists a number $\lambda > 1$ such that condition (A) holds with $\lambda_U \geqslant \lambda$, we say that f is a *local λ-expansion on* (M, ϱ).

Now let M be a metrizable topological space and $f\colon M \to M$ a continuous selfmapping of M. We will say that f is a *topological local expansion* (resp. *topological local λ-expansion*) *on* M if M admits a metric ϱ compatible with the given topology and such that f is a local expansion (resp. local λ-expansion) on (M, ϱ).

(Note that if M is compact then f is a local expansion on (M, ϱ) iff for some $\lambda > 1$ it is a local λ-expansion on (M, ϱ)).

A sequence A_n, $n = 0, 1, \ldots$, of subsets of a topological space M is said to be *fine* if for each open covering $\mathscr{C}$ of M there exists an integer n such that for $m \geqslant n$, each connected component of A_m is a subset of some member of $\mathscr{C}$.

It is easily shown (cf. [2] or [3]) that if M is compact, locally connected and metrizable and f is an open topological local expansion of M onto itself, then f is a local homeomorphism (and therefore a selfcovering of M) and

(B) for each point z of M there exists a neighborhood U of z such that the sequence $f^{-n}(U)$, $n = 0, 1, \ldots$, is fine.

Since this condition does not involve the metric and has a topological character, it is natural to ask the following question. Let M be a compact, locally connected Hausdorff space and $f\colon M \to M$ a local homeomorphism of M onto itself satisfying the condition (B). Is it possible to find a metric ϱ generating the given topology of M such that the mapping f is a local expansion on (M, ϱ)?