On local expansions

by

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Abstract. Local expansions on continua are studied. It is shown, among other things, that every generalized local expansion on a dendrite with a convex metric is an isometry.

§ 1. Introduction. The well known theorem of Banach (see [2], § 2, Theorem 6, p. 160) called *Banach Contraction Mapping Principle* or *Banach's fixed point theorem* (cf. e.g. [25], Proposition 40, p. 453) says that if f is a continuous mapping of a complete metric space X with a metric ϱ into itself such that for every $x, x' \in X$

(1) $$\varrho(f(x), f(x')) \leqslant M\varrho(x, x'), \quad \text{where} \quad 0 \leqslant M < 1,$$

then f has a fixed point. Mappings satisfying (1) are called *contractions*. They are of a great importance in the functional analysis and in its applications, and have a large bibliography which treates not only on contraction mappings in the above sense, but also on various generalizations of the notion. However, the research in this field did not pay till now so great attention to investigate and study other general classes of mappings, in particular extensions, i.e., such continuous mappings f of X into itself that for every $x, x' \in X$ we have

$$\varrho(f(x), f(x')) \geqslant M\varrho(x, x'), \quad \text{where} \quad M > 1.$$

The results presented here are a contribution to this theme.

The paper contains some investigations of continuous mappings of metric spaces, especially so called *generalized local expansions*, i.e., such mappings which do not diminish distances between points locally, or — more precisely — continuous mappings of a metric space X to Y having a property that for each point x of X there exist a neighborhood of x and a constant $M \geqslant 1$ that the distance between any two points of the neighborhood (in X) is at least M times less than the distance between their images under f (in Y). These mappings were created as generalizations of local expansions (for which the constant M is strictly greater than one) investigated e.g. in [42], where a fixed point theorem is proved for such mappings of continua, and in [15], where a necessary and sufficient condition is found, under which a linear graph admits a local expansion onto itself (cf. also [14]).

In § 2 we collect the necessary information concerning local isometries (mappings which locally preserve distances). It is shown that any real-valued local isometry defined on a connected subset of the real line is an isometry. § 3 concerns some general properties of local expansions, and — mostly — of generalized local expansions. § 4 is devoted to generalized local expansions defined on unions of arcs. In particular, arc-preserving mappings are studied. It is proved that every generalized local expansion of $[0, 1]$ onto itself has to be either the identity or the symmetry with respect to $\frac{1}{2}$. The results of this paragraph are applied in the next one which contains investigations of generalized local expansions of some convex spaces. The main result of § 5 says that a generalized local expansion of a dendrite with a convex metric onto itself is an isometry. Finally, generalized local expansions are studied on some spaces with a radially convex metric.

The following standard notation will be used in the paper. The space of all numbers with the natural (i.e. euclidean) metric will be called the *real line* and will be denoted by R. We put (p, q) and $[p, q]$ for the open and the closed interval of reals from p to q respectively. The unit circle will be denoted by S^1, i.e., $S^1 = \{z \in R^2\colon |z| = 1\}$, where z means a complex number. Usually S^1 will be considered as a subspace of R^2, i.e., it will be equipped with the euclidean metric (provided the opposite is not said). If a and b are points in a topological space, then an arc from a to b will be denoted by ab. We shall say that a subset A of a metric space X (with a metric ϱ) is linear if there exists an isometry $\varphi\colon A \to R$ of A into R, i.e., a mapping φ satisfying $\varrho(x, y) = |\varphi(x) - \varphi(y)|$ for each $x, y \in A$. An arc contained in X is said to be a *metric segment* if it is linear. We shall use the symbol $\overline{ab}$ to denote a metric segment from a to b. In particular, if X is the euclidean n-space, R^n, then $\overline{ab}$ will mean the straight line segment joining a and b.

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§ 2. Local isometries. Let X and Y be metric spaces with metrics ϱ_X and ϱ_Y respectively. A mapping $f\colon X \to Y$ of X onto Y is said to be an *isometry* if the equality $\varrho_Y(f(y), f(z)) = \varrho_X(y, z)$ holds for every two points y and z of X.

Manifestly each isometry is a continuous one-to-one mapping.

DEFINITION 2.1. A mapping $f\colon X \to Y$ of X onto Y is said to be a *local isometry* if for every point x of X there is an open neighborhood U of x such that for every two points y and z of U we have

$$\varrho_Y(f(y), f(z)) = \varrho_X(y, z).$$

Obviously each isometry is a local isometry (taking the whole space X for U) but not invertedly, as it can be shown by the following example, which is due to S. T. Czuba.

EXAMPLE 2.1. Let — in the euclidean plane — the continuum X be the union of three straight line segments forming the letter H, i.e., let $X = \overline{ab} \cup \overline{pq} \cup \overline{cd}$, where the points p and q are centres of the straight line segments $\overline{ab}$ and $\overline{cd}$ respectively and the segment $\overline{pq}$ is perpendicular to $\overline{ab}$ and $\overline{cd}$. Further, let a mapping $f\colon X \to X$ of X onto itself be defined as the central symmetry with respect to q on the segment $\overline{cd}$ and as the identity out of it. It is easy to see that f is a local isometry. It is not an isometry since

$$\varrho(f(a), f(c)) = \varrho(a, d) \neq \varrho(a, c)$$

(here ϱ means the euclidean metric in the plane).

The following proposition is obvious.

PROPOSITION 2.1. *Every local isometry is a continuous mapping.*

A local isometry need not be a one-to-one mapping. To see this, consider

EXAMPLE 2.2. Let R be the real line and let the unit circle $S^1 = \{z\colon |z| = 1\}$ have the length of the shortest arc from z_1 to z_2 contained in S^1 as the distance between z_1 and z_2. The mapping $f\colon R \to S^1$ defined by $f(x) = \exp(2\pi i x)$ is a local isometry but not one-to-one.

It is easy to verify the following two propositions (see [29], p. 17 for the definition of the product mapping):

PROPOSITION 2.2. *If $f\colon X \to Y$ is a local isometry and if $A \subset X$, then $f|A\colon A \to f(A) \subset Y$ is a local isometry.*

PROPOSITION 2.3. *The mapping*

$$f_1 \times f_2 \times \ldots \times f_n\colon X_1 \times X_2 \times \ldots \times X_n \to Y_1 \times Y_2 \times \ldots \times Y_n$$

is a local isometry if and only if $f_i\colon X_i \to Y_i$ is a local isometry for every $i = 1, 2, \ldots, n$.

To show the main result of the present paragraph (Theorem 2.1 below) we need two lemmas.

LEMMA 2.1. *Let A be a connected subset of the real line R and let $f\colon A \to f(A) \subset R$ be a local isometry. Then f has not a local extremum at interior points of A.*

Indeed, suppose on the contrary that there is an interior point x_0 of A at which the mapping f has a local extremum. Without loss of generality we can assume that the extremum is a maximum. Thus there is a positive number δ such that $(x_0 - \delta, x_0 + \delta) \subset A$ and that the following implications hold:

(2) if $x \in (x_0 - \delta, x_0 + \delta)$, then $f(x) \leqslant f(x_0)$,

(3) if $y, z \in (x_0 - \delta, x_0 + \delta)$, then $|f(y) - f(z)| = |y - z|$.

Putting $K = \max\{f(x_0 - \frac{1}{2}\delta), f(x_0 + \frac{1}{2}\delta)\}$ we have $K < f(x_0)$ by (2) and (3). By Darboux property for the (continuous) mapping f (see e.g. [25], Proposition 31, p. 171) applied to the intervals $[x_0 - \frac{1}{2}\delta, x_0]$ and $[x_0, x_0 + \frac{1}{2}\delta]$ we can find reals x_1 and x_2 such that $x_0 - \frac{1}{2}\delta < x_1 < x_0 < x_2 < x_0 + \frac{1}{2}\delta$ and $f(x_1) = f(x_2) = \frac{1}{2}(K + f(x_0))$, contrary to (3).

LEMMA 2.2. *Let A be a connected subset of the real line R and let $f\colon A \to f(A) \subset R$ be a local isometry. Then f is one-to-one.*

In fact, it is a consequence of Lemma 2.1 and of Weierstrass' theorem (see e.g. [25], Proposition 29, p. 170).

THEOREM 2.1. *Let A be a connected subset of the real line R and let $f\colon A \to f(A) \subset R$ be a local isometry. Then f is an isometry.*

Proof. Let a and b be distinct points of A. If $f(a) = f(b)$, then (since f, being a local isometry, is not a constant mapping) we see that f has a local extremum at an interior point of the closed interval $[a, b] \subset A$ by Weierstrass' theorem ([25], Proposition 29, p. 170) contrary to Lemma 2.1. Thus $f(a) \neq f(b)$ and we can assume — without loss of generality — that $f(a) < f(b)$. Then the mapping f, being continuous and one-to-one (see Proposition 2.1 and Lemma 2.2), is increasing in $[a, b]$ (see [25], Proposition 35, p. 181). Further, since f is a local isometry, for every point $x \in [a, b]$ there is a number $\delta_x > 0$ such that if $y, z \in (x-\frac{1}{2}\delta_x, x+\frac{1}{2}\delta_x) \cap A$, then

$$|f(y)-f(z)| = |y-z|. \tag{4}$$

The family of open intervals $\{(x-\frac{1}{2}\delta_x, x+\frac{1}{2}\delta_x)\colon x \in [a, b]\}$ is a covering of $[a, b]$, and by compactness of $[a, b]$ there is a finite sequence of points $x_1, x_2, \ldots, x_k$ in $[a, b]$ such that

$$[a, b] \subset \bigcup_{i=1}^{k} (x_i-\tfrac{1}{2}\delta_{x_i}, x_i+\tfrac{1}{2}\delta_{x_i}). \tag{5}$$

Put $\delta = \min\{\frac{1}{2}\delta_{x_i}\colon i = 1, 2, \ldots, k\} > 0$. We show now that for any two points y and z of $[a, b]$ with $|y-z| < \delta$ equality (4) holds. Indeed, it follows from (5) that there is a point x_{i_0} such that $|y-x_{i_0}| < \frac{1}{2}\delta_{x_{i_0}}$. Thus $|z-x_{i_0}| \leqslant |z-y|+|y-x_{i_0}| < \delta+\frac{1}{2}\delta_{x_{i_0}} \leqslant \delta_{x_{i_0}}$ and therefore for the points y and z we have $|y-z| < \delta_{x_{i_0}}$, whence (4) follows.

Now take a natural j such that $(b-a)/j < \delta$, and define $a_i = a+i(b-a)/j$ for $i = 0, 1, \ldots, j$. Thus we have $a = a_0 < a_1 < a_2 < \ldots < a_{j-1} < a_j = b$, whence, the mapping f being increasing, $f(a) = f(a_0) < f(a_1) < f(a_2) < \ldots < f(a_{j-1}) < f(a_j) = f(b)$. Moreover, $f(a_{i+1})-f(a_i) = a_{i+1}-a_i$ for every $i = 0, 1, \ldots, j-1$, which implies immediately that $f(b)-f(a) = \sum_{i=0}^{j-1} [f(a_{i+1})-f(a_i)] = \sum_{i=0}^{j-1} (a_{i+1}-a_i) = b-a$, and the proof is complete.

Let us observe that Theorem 2.1 cannot be generalized to a local isometry defined on an arbitrary connected set A or even on the connected union of finitely many straight line segments. In other words, the condition $A \subset R$ is essential. Indeed, it can be seen again by Example 2.1. Similarly, the condition $f(A) \subset R$ is essential too, as Example 2.2 shows. It is quite easy to verify that the connectedness of A is also an essential hypothesis in Theorem 2.1.

§ 3. Local expansions. Let X and Y be metric spaces with metrics ϱ_X and ϱ_Y respectively.

DEFINITION 3.1. A continuous mapping $f\colon X \to Y$ of X onto Y is said to be an *expansion* (a *generalized expansion*) if there exists a constant $M > 1$ ($M \geqslant 1$) such that for every two points y and z of X the inequality

$$\varrho_Y(f(y), f(z)) \geqslant M\varrho_X(y, z) \tag{6}$$

holds.

DEFINITION 3.2. A continuous mapping $f\colon X \to Y$ of X onto Y is said to be a *local expansion* (a *generalized local expansion*) if for every point x of X there exist an open neighborhood U of x and a constant $M > 1$ ($M \geqslant 1$) such that for every two points y and z of U inequality (6) holds.

The following is an immediate consequence of the above definitions.

PROPOSITION 3.1. (a) *Each expansion is a generalized expansion.* (b) *Each expansion is a local expansion.* (c) *Each generalized expansion is a generalized local expansion.* (d) *Each local expansion is a generalized local expansion.* (e) *Each isometry is a generalized expansion.* (f) *Each local isometry is a generalized local expansion.*

The function tangent tan: $(-\frac{1}{2}\pi, \frac{1}{2}\pi) \to R$ is an example of a generalized expansion which is neither an expansion nor an isometry. The same function serves as an example of a generalized local expansion which is not a local expansion. For every natural $n > 1$ the mapping $f\colon S^1 \to S^1$ defined by $f(z) = z^n$ for $z \in S^1$ is a (generalized) local expansion which is not a (generalized) expansion.

PROPOSITION 3.2. *Each generalized expansion is a one-to-one mapping.*

Indeed, if $y \neq z$, then $\varrho_X(y, z) > 0$, which implies $\varrho_Y(f(y), f(z)) \geqslant \varrho_X(y, z) > 0$, and thus $f(y) \neq f(z)$.

It follows from Proposition 3.2 that if $f\colon X \to Y$ is a generalized expansion, then there exists an inverse mapping $f^{-1}\colon Y \to X$. Obviously we have

PROPOSITION 3.3. *If $f\colon X \to Y$ is a generalized expansion, then the inverse mapping $f^{-1}\colon Y \to X$ does not increase distances of points, and thus it is continuous.*

PROPOSITION 3.4. *Let $f\colon X \to Y$ be a generalized local expansion, let $x \in X$ and let U be an open neighborhood of x as in Definition 3.2. Then the partial mapping $f|U\colon U \to f(U) \subset Y$ is one-to-one.*

In fact, for fixed distinct points y and z of U we have

$$\varrho_Y((f|U)(y), (f|U)(z)) = \varrho_Y(f(y), f(z)) \geqslant M\varrho_X(y, z) \geqslant \varrho_X(y, z) > 0,$$

which implies $f(y) \neq f(z)$.

COROLLARY 3.1. *Each generalized local expansion is a locally one-to-one mapping.*

Similarly to Propositions 2.2 and 2.3 for local isometries, we have — for local expansions — the following two propositions, the proofs of which are quite easy and thus are left to the reader.

PROPOSITION 3.5. *If $f\colon X \to Y$ is a (generalized) local expansion and if $A \subset X$, then $f|A\colon A \to f(A) \subset Y$ is a (generalized) local expansion.*

PROPOSITION 3.6. *The mapping*

$$f_1 \times f_2 \times \ldots \times f_n\colon X_1 \times X_2 \times \ldots \times X_n \to Y_1 \times Y_2 \times \ldots \times Y_n$$

is a (generalized) local expansion if and only if $f_i\colon X_i \to Y_i$ *is a (generalized) local expansion for every* $i = 1, 2, \ldots, n$.

Let us recall that a mapping $f\colon X \to Y$ is said to be *open* if the image of an open set in X is open in Y. A mapping $f\colon X \to Y$ of X onto Y is said to be a *local homeomorphism* provided that for every point $x \in X$ there exists an open neighborhood U of x such that $f(U)$ is an open subset of Y and that $f|U\colon U \to f(U) \subset Y$ is a homeomorphism. Obviously each local homeomorphism is an open mapping.

PROPOSITION 3.7. *If a generalized local expansion is open, then it is a local homeomorphism.*

Indeed, let $f\colon X \to Y$ be an open generalized local expansion, let $x \in X$ and let U be an open neighborhood of x as in Definition 3.2. Then $f(U)$ is an open subset of Y. Further, the partial mapping $f|U$ is continuous by continuity of f, and it is one-to-one by Proposition 3.4. Thus there exists an inverse mapping

$$(f|U)^{-1}\colon f(U) \to U \subset X.$$

We have to show that $(f|U)^{-1}$ is continuous. To this end fix a point $p \in f(U)$ and a number $\varepsilon > 0$. Let $q \in f(U)$ be such that $\varrho_Y(p, q) < \varepsilon$. Then there are points y and z in U with $f(y) = p$ and $f(z) = q$, i.e., $y = (f|U)^{-1}(p)$ and $z = (f|U)^{-1}(q)$. The mapping f being a generalized local expansion and $y, z \in U$, there exists a constant $M \geqslant 1$ such that inequality (6) holds, whence

$$\varrho_X(y, z) \leqslant M\varrho_X(y, z) \leqslant \varrho_Y(f(y), f(z)) = \varrho_Y(p, q) < \varepsilon,$$

and the proof is complete.

The hypothesis of openness of the mapping is essential in the above proposition; it can be seen from the example of a local expansion $f\colon X \to X$ defined on the union X of three circles, which is neither an open mapping nor a local homeomorphism (see [42], Example, p. 3).

A statement similar to Proposition 3.7 was used in [42] to prove that every open local expansion of a continuum onto itself has a fixed point. However, this result cannot be extended to generalized local expansions: we must have the sharp inequality $M > 1$ to reach a fixed point of the mapping. Namely the central symmetry $f\colon S^1 \to S^1$ defined on the unit circle $S^1 = \{z \in R^2\colon |z| = 1\}$ by $f(z) = -z$ is a generalized local expansion (it is an isometry even) without fixed points. Also openness of the mapping is essential in this fixed point theorem, as it was shown in [42], p. 3 by the example mentioned above of a local expansion on the union of three circles. But a question arises if this hypothesis can be weakened in some way. Let us recall that a mapping $f\colon X \to Y$ of a continuum X onto Y is said to be *confluent* (see [11], p. 213) if for every continuum $Q \subset Y$ and for every component C of $f^{-1}(Q)$ we have $f(C) = Q$. It is known that each open mapping of a continuum is confluent (see [11], VI, p. 214). Observe that the local expansion on the union of three circles ([42], Example, p. 3) discussed above is not only non-open, but also non-confluent. So we have

PROBLEM 3.1. Does there exist a confluent local expansion of a continuum onto itself which is fixed point free?

Let us recall a result of Eilenberg who proved that each local homeomorphism of a continuum X onto a dendrite (i.e. a locally connected continuum containing no simple closed curve) is a homeomorphism (see [18], Theorem III, p. 42; compare also [45], Corollary, p. 199). This result has firstly been extended to λ-dendroids (i.e. hereditarily decomposable and hereditarily unicoherent continua) and secondly to tree-like continua by Maćkowiak (see [33], Theorem 9, p. 857; [34], Proposition 7 and Theorem 8, p. 287; and [35], Theorem, p. 64). Thus, as an immediate consequence of Proposition 3.7 we get the following particular version of Maćkowiak's result quoted above:

PROPOSITION 3.8. *If* $f\colon X \to Y$ *is an open generalized local expansion of a continuum* X *onto a tree-like continuum* Y, *then* f *is a homeomorphism.*

We put this proposition only to ask whether openness of the mapping is an essential hypothesis here. In other words we have the following

PROBLEM 3.2. Can openness of the generalized local expansion f of a continuum X onto a tree-like continuum Y be omit to still get the conclusion that X is tree-like and f is a homeomorphism?

Note a partial answer to this question: the answer is positive if X is arcwise connected (see Corollary 4.2 below).

§ 4. Local expansions on the unions of arcs. Consider a class of continua called arc-continua and defined as follows.

DEFINITION 4.1. A continuum X is called an *arc-continuum* provided for each point x of X there exists a non-degenerate arc $A \subset X$ such that $x \in A$.

Obviously all arcwise connected continua are arc-continua, while hereditarily indecomposable ones are not. The simplest indecomposable continuum of Knaster (see [30], § 48, V, Example 1, p. 204) is an arc-continuum, but Janiszewski's irreducible continuum without arcs (see [24], p. 128; compare [30], § 48, V, p. 207, the footnote) is an example of a hereditarily decomposable one which is not.

Observe that being an arc-continuum is not a hereditary property, as an example of the cone over the pseudo-arc shows. Observe further that continuous mappings do not preserve being an arc-continuum. For example, if P denotes the pseudo-arc and I denotes the unit segment of reals, then $P \times I$ is an arc-continuum, while P is a continuous image of $P \times I$ under the natural projection and P is not an arc-continuum. However, if the mappings under consideration are generalized local expansions, then they do. Namely we have

PROPOSITION 4.1. *If* $f\colon X \to Y$ *is a generalized local expansion of an arc-continuum* X *onto* Y, *then* Y *is an arc-continuum.*

In fact, let p be a point of Y, and let $x \in f^{-1}(p) \subset X$. The mapping f being a generalized local expansion, there exists an open neighborhood U as in Definition 3.2. Take an arc A such that $x \in A \subset X$. Thus there exists a non-degenerate arc B

such that $x \in B \subset A \cap U \subset U$. The partial mapping $f|U$ being continuous and one-to-one by Proposition 3.4, we see that $f|B: B \to f(B)$ is a homeomorphism. Thus $f(B)$ is a non-degenerate arc containing $p = f(x)$ and contained in Y, and therefore Y is an arc-continuum.

By an n-od with vertex p we mean a continuum homeomorphic to the union of n straight line segments such that they all and every two of them have only the point p in common. Given a continuum X, a point $p \in X$ is said to be of *order greater than or equal to* n (writing $\mathrm{Ord}_p X \geqslant n$) if there exists in X an n-od with vertex p (cf. [10], p. 230). A point $p \in X$ is called an *end point of* X if p lies on a non-degenerate arc contained in X and if p is an end point of every arc A such that $p \in A \subset X$ (see [10], p. 230). We denote the set of all end points of a continuum X by $E(X)$.

PROPOSITION 4.2. *Let $f: X \to Y$ be a generalized local expansion of a continuum X onto Y, and let K be an arc-continuum contained in X. If $p \in K$ and $\mathrm{Ord}_p K \geqslant n$, then $\mathrm{Ord}_{f(p)} f(K) \geqslant n$.*

Indeed, let L be the n-od with vertex p which is contained in K by assumption, and let U be an open neighborhood of p taken from Definition 3.2. Thus there exists an n-od $N \subset L \cap U \subset K$ having p as its vertex. We conclude from Proposition 3.4 that $f|N: N \to f(N)$ is a homeomorphism. Then $f(p)$ is the vertex of the n-od $f(N) \subset f(K)$, and the proof is complete.

As an immediate consequence of Proposition 4.2 we get

PROPOSITION 4.3. *Let $f: X \to Y$ be a generalized local expansion of a continuum X onto Y, and let K be an arc-continuum contained in X. If $p \in K$ and $f(p) \in E(f(K))$, then $p \in E(K)$.*

PROPOSITION 4.4. *Let $f: X \to Y$ be a generalized local expansion of a continuum X onto Y, and let Y contain no simple closed curve. For each arc $ab \subset X$ its image $f(ab)$ under f is the arc $f(a)f(b)$ having $f(a)$ and $f(b)$ as its end points.*

Proof. The mapping f being continuous, the set $f(ab)$ is a locally connected continuum (see e.g. [30], § 50, II, Theorem 2, p. 256) that contains no simple closed curve. Thus it is a dendrite. Recall that each dendrite has at least two end points ([45], Chapter III, Theorem (6.1), p. 54, and Chapter V, Theorem (1.1) (ii), p. 88). If $f(ab)$ would have more than two end points, then there would be a point $x \in ab \setminus \{a, b\}$ having $f(x)$ as an end point of $f(ab)$, contrary to Proposition 4.3. Therefore $f(ab)$ has exactly two its end points, say $f(p)$ and $f(q)$, where p and q are some points of the arc ab. Then — since a dendrite is a continuum irreducible about the set of its end points (in fact, it follows easily from Theorem (1.1) (ii) in Chapter V of [45], p. 88) — it is a continuum irreducible between these two end points, and, being locally connected, it is an arc. Thus it is the only arc in Y joining $f(p)$ and $f(q)$. Now it follows from Proposition 4.3 that p and q are end points of ab, and therefore we have $f(ab) = f(a)f(b)$, which finishes the proof.

As an immediate consequence of Proposition 4.4 we get

COROLLARY 4.1. *Let $f: X \to Y$ be a generalized local expansion of a continuum X onto Y and let Y contain no simple closed curve. Then for each arc $ab \subset X$ the partial mapping $f|ab: ab \to f(ab)$ is a homeomorphism.*

Note that we cannot omit the assumption of Proposition 4.4 that Y contain no simple closed curve; e.g. if $f: S^1 \to S^1$ is a local expansion defined by $f(z) = z^2$ for each $z \in S^1$, then the arc $\{z = \exp(i\varphi): 0 \leqslant \varphi \leqslant \pi\}$ is mapped onto the whole S^1.

Mappings for which the conclusion of Proposition 4.4 holds have been extensively studied by many authors under the name of arc-preserving mappings. More precisely, a continuous mapping f of a topological space X onto a topological space Y is said to be *arc-preserving* provided that the image under f of any arc in X is either an arc or a single point in Y (see [44], p. 305). A true arc-preserving mapping is defined so as to eliminate the possibility of an arc being carried into a point, that is, the image of every non-degenerate arc in X is a non-degenerate arc in Y. A tree-preserving and true tree-preserving mappings have been defined in a similar manner (see [43], p. 576). It is known that arc-preserving mappings of locally connected continua are tree-preserving (dendrite-preserving) (see [43], Theorem 7, p. 588, and [22], p. 70). Mappings of this kind were studied also in [20] and [21], mainly for locally connected continua. From some other points of view they were investigated in [16] and [17]. Arc-preserving mappings are related in some way to monotone ones for some types of continua, see [13], § 6, especially Proposition 3, p. 307.

The following problem is a slight modification of one asked in [12].

PROBLEM 4.1. Characterize all the continua Y such that every continuous mapping of a continuum X onto Y is arc-preserving.

Proposition 4.4 is a contribution in this direction. Namely it can be reformulated as

PROPOSITION 4.5. *Every generalized local expansion of a continuum onto one which contains no simple closed curve is true arc-preserving.*

As a consequence of the above proposition we get

COROLLARY 4.2. *If $f: X \to Y$ is a generalized local expansion of an arcwise connected continuum X onto a continuum Y which contains no simple closed curve, then f is a homeomorphism.*

Observe that arcwise connectedness of X is a necessary assumption in the above corollary. Indeed, it can be seen by

EXAMPLE 4.1. Put in the rectangular cartesian coordinates x, y in the plane,

$$A = \{(x, y): y = \sin(\pi/x) \text{ and } 0 < x \leqslant 1\} \cup \{(0, y): -1 \leqslant y \leqslant 4\}$$

and $s = (\frac{1}{2}, 2)$. Let g denote the central symmetry with respect to the point s, i.e., we define $g((x, y)) = (1-x, 4-y)$ for every point (x, y) in the plane. Put $X = A \cup g(A)$. Thus g maps X onto itself and point-inverses of g are two-point-sets composed of the points just opposite with respect to s. Let $Y = X/G$ denote the quotient space of X under an equivalence relation G defined by pGq if and only if $q = g(p)$. Then Y is homeomorphic to so called *Warsaw circle* (i.e. $\sin(1/x)$-circle) and the quotient mapping (the natural projection) $f: X \to Y$ can be considered as a generalized local expansion (we multiply the metric in Y by a sufficiently large constant if necessary). We see that X is not arcwise connected, Y contains no simple closed curve, and f is not a homeomorphism. Note that X is an arc-continuum, and thus the example shows that we cannot replace arcwise connectedness of X in Corollary 4.2 by the property of being an arc-continuum.

Let us recall that a dendroid means an arcwise connected continuum which is hereditarily unicoherent (and, consequently, contains no simple closed curve). Thus Corollary 4.2 implies

COROLLARY 4.3. *Every generalized local expansion of a dendroid onto itself is a homeomorphism.*

Note that it follows also from Corollary 4.2 that any generalized local expansion of the Warsaw circle (i.e. $\sin(1/x)$-circle) onto itself is a homeomorphism. Thus it is natural to ask the following

PROBLEM 4.2. Characterize all the continua X having the property that every generalized local expansion of X onto itself is a homeomorphism.

Recall that $[p, q]$ denotes the closed interval of reals from p to q.

PROPOSITION 4.6. *If* $f\colon [a, b] \to f([a, b]) \subset R$ *is a generalized local expansion and if*

$$|f(b)-f(a)| \leqslant b-a\,, \tag{7}$$

then $f([a, b]) = [f(a), f(b)]$ *and* f *is an isometry.*

Proof. The image $f([a, b])$ is a compact and connected subspace of the real line R, so the first part of the conclusion is an immediate consequence of Corollary 4.2.

The mapping f being a generalized local expansion, for every point $p \in [a, b]$ there exist an open neighborhood U_p of p and a constant $M_p \geqslant 1$ such that for every two points x and y of U_p we have

$$|f(x)-f(y)| \geqslant M_p \cdot |x-y|\,. \tag{8}$$

Since the family $\{U_p\colon p \in [a, b]\}$ is an open covering of $[a, b]$, there exists a finite sequence of points $p_1, p_2, \ldots, p_n$ in $[a, b]$ such that $\{U_{p_i}\colon i = 1, 2, \ldots, n\}$ is a finite covering of $[a, b]$ and there is the Lebesgue coefficient of this covering, i.e., such a positive number ε that if some two points of $[a, b]$ differ less than ε, then they belong to the same element of the covering $\{U_{p_i}\colon i = 1, 2, \ldots, n\}$ (see [30], § 41, VI, Corollaries 4c and 4d, p. 23 and 24). Thus for every two points x, y of $[a, b]$ we see by (8) that

$$\text{if}\quad |x-y| < \varepsilon\,, \quad\text{then}\quad |f(x)-f(y)| \geqslant |x-y|\,. \tag{9}$$

Now we shall show that the mapping f is a local isometry. Suppose the contrary. It means that there exists a point $x \in [a, b]$ such that for each open neighborhood U of x there are two points y and z in U with $|f(y)-f(z)| \neq |y-z|$. Since f is a generalized local expansion, hence every point of $[a, b]$ has an open neighborhood with the property that for any two points of this neighborhood the mapping f doesnot

diminish distances between them. Therefore we conclude that the point x has an open neighborhood $U_x \subset [a, b]$ containing two points x' and x'' such that

$$|f(x')-f(x'')| > |x'-x''|\,. \tag{10}$$

Let m be a natural and let us take a set of $m+1$ points x_j (for $j = 0, 1, \ldots, m$) in $[a, b]$ such that

$$a = x_0 < x_1 < x_2 < \ldots < x_m = b\,, \tag{11}$$

$$\{x', x''\} \subset \{x_0, x_1, x_2, \ldots, x_m\}\,, \tag{12}$$

$$x_{j+1}-x_j \leqslant \frac{1}{m}(b-a) < \varepsilon \quad \text{for every } j = 0, 1, \ldots, m-1\,, \tag{13}$$

where ε is — as previously — the Lebesgue coefficient for the covering

$$\{U_{p_i}\colon i = 1, 2, \ldots, n\}\,.$$

Thus every two consecutive points x_j and x_{j+1} belong to the same element of this covering, whence by (9) and (13) we have

$$|f(x_{j+1})-f(x_j)| \geqslant x_{j+1}-x_j \quad \text{for every } j = 0, 1, \ldots, m-1\,. \tag{14}$$

For every $i = 1, 2, \ldots, n$ the partial mapping $f|U_{p_i}$ is a generalized expansion, whence it follows that f is monotone. Thus we have

$$|f(b)-f(a)| = \sum_{j=0}^{m-1} |f(x_{j+1})-f(x_j)| \geqslant \sum_{j=0}^{m-1} (x_{j+1}-x_j) = b-a\,, \tag{15}$$

where the inequality is a consequence of (14). Further, (10) and (12) imply that the inequality in (15) is a proper one, i.e., that it cannot be replaced by the equality. So we have $|f(b)-f(a)| > b-a$ contrary to (7). Therefore we have shown that f is a local isometry. Now the conclusion follows from Theorem 2.1, and the proof is complete.

Proposition 4.6 implies

COROLLARY 4.4. *If* $f\colon [0, 1] \to [0, 1]$ *is a generalized local expansion of the unit interval of reals onto itself then* f *is either the identity or the central symmetry with respect to* $\frac{1}{2}$, *i.e. a mapping defined by* $f(x) = 1-x$ *for each* $x \in [0, 1]$.

Corollary 4.2 and Proposition 3.6 imply that if a product-mapping $f\colon I^n \to I^n$ (i.e. such that $f = f_1 \times f_2 \times \ldots \times f_n$) of the euclidean unit cube I^n onto itself is a generalized local expansion, then it is an isometry. Thus the following problem seems to be interesting.

PROBLEM 4.3. Is every generalized local expansion of I^n onto itself an isometry?

It is remarkable that the suspected affirmative answer to this question cannot be generalized to Hilbert cube. Namely Prof. David P. Bellamy in a conversation with the first author has shown a local expansion of Hilbert cube onto itself. As we know, the result has not been published.

PROPOSITION 4.7. *Let $f\colon X\to Y$ be a generalized local expansion of a continuum X with a metric ϱ_X onto a continuum Y with a metric ϱ_Y. If an arc ab is a metric segment in X such that its image under f is a metric segment in Y, i.e.,*

$$f(\overline{ab}) = \overline{f(a)f(b)}\,, \tag{16}$$

and if

$$\varrho_Y\big(f(a), f(b)\big)\leqslant \varrho_X(a, b)\,, \tag{17}$$

then $f|ab\colon ab\to f(ab)$ is an isometry.

Proof. Since $\overline{ab}$ and $f(\overline{ab})$ are metric segments in X and Y respectively, there are isometries

$$i_1\colon \overline{ab}\to R \quad \text{and} \quad i_2\colon f(\overline{ab})\to R$$

of these segments into the real line R. Let us define a mapping

$$f^*\colon\ i_1(\overline{ab})\to f^*(i_1(\overline{ab}))\subset R$$

putting $f^* = i_2(f|\overline{ab})i_1^{-1}$. We have then the following commutative diagram:

$$\begin{array}{ccc} X\supset \overline{ab} & \xrightarrow{f|\overline{ab}} & f(\overline{ab})\subset Y \\ {\scriptstyle i_1}\downarrow & & \downarrow{\scriptstyle i_2} \\ R\supset i_1(\overline{ab}) & \xrightarrow[f^*]{} & f^*(i_1(\overline{ab}))\subset R \end{array}$$

The mapping f^* is a generalized local expansion as the composite of two isometries i_1^{-1} and i_2, and of the generalized local expansion $f|\overline{ab}$ (see Proposition 3.5). Observe that the length of the closed interval $i_1(\overline{ab})$ in the real line R is equal to $\varrho_X(a, b)$. Similarly, it follows from (16) that the length of the closed interval $i_2(f(\overline{ab}))$ in R is equal to $\varrho_Y(f(a), f(b))$. But $i_2(f(\overline{ab})) = f^*(i_1(\overline{ab}))$ by the definition of f^*. Thus we see that inequality (17) implies inequality (7) and we conclude from Proposition 4.6 that f^* is an isometry. Further, it follows from the definition of the mapping f^* that $f|\overline{ab} = i_2^{-1}f^*i_1$ (see the diagram above). Therefore $f|\overline{ab}$ is an isometry as a composite of three isometries, and the proof is complete.

§ 5. Local expansions on convex spaces. Let a metric space X with a metric ϱ be given. Let x, y, z be points of X. The point z is said to *lie between the points x and y* provided that $\varrho(x, y) = \varrho(x, z)+\varrho(z, y)$ (see [36], p. 77; cf. [8], p. 317).

A metric space X is said to be *convex* (in the well known sense of Menger [36], p. 81 and 82) provided that for each two distinct points x and y of X there exists a point $z\in X$ different from x and y which lies between x and y. It was proved by Menger ([36], p. 89, cf. [37], p. 498; see also Aronszajn [1]; cf. [8], p. 41) that in every complete convex metric space X each two points of X can be joined by a metric segment. Moreover, it is known (cf. e.g. [41], 2.3, p. 116) that a complete metric space X is convex if and only if for every two points x and y in X and for every t, where $0\leqslant t\leqslant 1$, there exists at least one point $z\in X$ such that

$$\varrho(x, z) = (1-t)\varrho(x, y) \quad \text{and} \quad \varrho(z, y) = t\cdot\varrho(x, y)\,.$$

This condition is related to a definition of a convex space due to Wilson ([46], p. 112; cf. [31], p. 324): a metric space X with a metric ϱ is called *convex* in this sense if for every two points $a, b\in X$ and for every real number t with $0\leqslant t\leqslant\varrho(a, b)$ there exists exactly one point c in X such that $\varrho(a, c) = t$ and $\varrho(c, b) = \varrho(a, b)-t$.

If a metric space equipped with a metric ϱ is convex, then the metric ϱ is called a *convex metric*.

In 1928 Menger proved that every metric continuum with a convex metric is locally connected ([36], p. 98) and asked if the inverse implication holds in the following sense. Let us call a topological or metric space convexifiable if it is homeomorphic with a topological or metric space convexifiable if it is homeomorphic with a convex metric space. Menger asked ([36], p. 98 and 99) if every locally connected metric continuum is convexifiable. Some partial answers to this question have been proved in various papers untill Bing solved the problem in the affirmative in 1949 [6]. Let us recall some of the most important ones. Already in the above mentioned paper of Menger it is proved (see [36], p. 96) that a continuum X is convexifiable if there is a metric ϱ on X such that for every point p of X and for every positive number ε there exists an open set containing p whose each point q can be joined with p by a rectifiable arc of length (with respect to ϱ) less than ε. Two years later Kuratowski and Whyburn proved in [31], p. 324 that convexifiability is an extensive property, i.e., if every cyclic element (see e.g. [45], Chapter IV, p. 66; cf. [30], § 52, II, p. 312) of a locally connected continuum X is convexifiable, then the whole X is convexifiable too. This theorem implies that acyclic curves can be convexified, whence — in particular — it follows that every dendrite has a convex metric.

In 1938 Beer in [3] constructed, with a considerable difficulty, a convex metric on every one-dimensional locally connected metric continuum. A year later Harold shown [23] that some three types of locally connected metric continua are convexifiable. In particular it is proved there that a plane continuum having finitely many components of its complementary has a convex metric. Some other partial results were also obtained by Bing [5]. Finally Bing ([6], Theorem 8, p. 1109) and — in the same time — Moise ([38], Theorem 4, p. 1119) have proved that every locally connected metric continuum is convexifiable. However, Moise's paper [38] contained an error (see [39]) rectified later by a result of Bing in [4]. Methods used by Bing to get the result are related to partitioning of a set — see an expository article [7], where the result is repeated as Theorem 6, p. 546.

After this short summary of selected results concerning convexification of a space, we come back to the topic of the paper, i.e. to local expansions. We begin with

PROPOSITION 5.1. *Let $f\colon X\to Y$ be a generalized local expansion of a continuum X onto a dendrite Y equipped with a convex metric. If an arc ab is a metric segment in X, then its image under f is a metric segment in Y, i.e. condition* (16) *holds.*

Indeed, it follows from Proposition 4.4 that $f(ab)$ is the arc $f(a)f(b)\subset Y$. Since the metric on the dendrite Y is convex (and since Y is a complete space) hence every arc in Y is a metric segment, and thus (16) follows.

The assumption of convexity of the metric in Y is essential in Proposition 5.1, as it can be seen from

EXAMPLE 5.1. Put — in the cartesian coordinates in the euclidean plane — $X = \{(x, -x)\colon -1 \leqslant x \leqslant 1\}$ and $Y = \{(0, 2x)\colon 0 \leqslant x \leqslant 1\} \cup \{(2x, 0)\colon 0 \leqslant x \leqslant 1\}$, and define $f\colon X \to Y$ by

$$f((x, -x)) = \begin{cases} (0, -2x) & \text{if } -1 \leqslant x \leqslant 0, \\ (2x, 0) & \text{if } \ \ 0 < x \leqslant 1. \end{cases}$$

In other words, f is a projection in direction of the vector $[1, 1]$ of the segment X onto Y, and Y is the union of two metric segments lying in the two axes of the coordinate system (the metric is euclidean). The reader can easily verify that f is a generalized local expansion.

COROLLARY 5.1. *Let $f\colon X \to Y$ be a generalized local expansion of a continuum X with a metric ϱ_X onto a dendrite Y equipped with a convex metric ϱ_Y. If an arc ab is a metric segment in X, and if condition* (17) *holds, then $f|ab\colon ab \to f(ab)$ is an isometry.*

In fact, the conclusion follows from Proposition 4.7 by Proposition 5.1.

THEOREM 5.1. *A generalized local expansion of a dendrite with a convex metric onto itself is an isometry.*

Proof. Let ϱ be a convex metric on a dendrite X, and let $f\colon X \to X$ be a generalized local expansion. We shall prove that

$$\varrho(f(a), f(b)) \geqslant \varrho(a, b) \quad \text{for every } a, b \in X. \tag{18}$$

In fact, if $\varrho(f(a), f(b)) > \varrho(a, b)$, we are done. In the opposite case, i.e., if $\varrho(f(a), f(b)) \leqslant \varrho(a, b)$ observe that the arc ab is just the metric segment $\overline{ab}$ by the convexity of the metric ϱ. So Corollary 5.1 can be applied and we conclude that $f|ab$ is an isometry, whence $\varrho(f(a), f(b)) = \varrho(a, b)$. Therefore (18) is established.

We conclude that f is one-to-one, and thus the inverse mapping $f^{-1}\colon X \to X$ can be considered. Obviously it is continuous and the inequality

$$\varrho(f^{-1}(p), f^{-1}(q)) \leqslant \varrho(p, q)$$

holds for every two points p and q of X by (18). Thus f^{-1} does not increase the distances of points. It is known that if a continuous mapping of a completely bounded space does not increase the distances of points, then it does not diminish them, i.e. it is an isometry (see [19], Theorem IV, p. 121; cf. [40], Theorems 1 and 2, p. 29 and 31). Therefore f^{-1} is an isometry, whence f is an isometry, and the proof is complete.

Note that the convexity of the metric in Theorem 5.1 cannot be omitted, as Example 2.1 shows. However, it is a consequence of Corollary 4.2 that every local expansion of a dendrite (with an arbitrary metric) onto itself is a homeomorphism.

COROLLARY 5.2. *There is no local expansion of a dendrite with a convex metric onto itself.*

PROBLEM 5.1. Is it true that, given a dendrite X (with an arbitrary metric), there is no local expansion of X onto itself?

Now let us turn our attention to a generalization of the notion of a convex metric, namely to a metric which is radially convex with respect to a point. This concept has been introduced by Koch and McAuley (see [27], p. 343, and [28], p. 3) and used extensively by many authors, especially to characterize smooth continua ([9], p. 229; [13], p. 298 and Theorem 10, p. 310; [32], p. 181). Let us recall some related notion. By a partial order on a set we mean a reflexive, transitive and anti-symmetric binary relation. Let X be a metric space equipped with a partial order Γ. A metric ϱ on X is called *radially convex with respect to* Γ if $(x, y) \in \Gamma$, $(y, z) \in \Gamma$ and $y \neq z$ imply $\varrho(x, y) < \varrho(x, z)$ (see [9], p. 229). It is proved in [9], Theorem 1, p. 229, that if Γ is a closed partial order on the compact metric space X, then there exists an equivalent metric on X which is radially convex with respect to Γ.

A dendroid X is said to be *smooth* (see [13], p. 298) if there is a point $p \in X$ (called an *initial point of* X) such that for every point $a \in X$ and for every sequence of points $a_n \in X$ which is convergent to a, the sequence of arcs pa_n is convergent to the arc pa. It is known ([26], p. 679) that if a dendroid X is smooth, then the relation Γ defined by $(x, y) \in \Gamma$ provided that $x \in py$ is a closed partial order on X, whence it follows that there exists an equivalent metric ϱ on X that is radially convex with respect to Γ. We call this metric ϱ *radially convex with respect to the point* p, or shortly a *radially convex metric*. In other words, a metric ϱ on a dendroid X is said to be *radially convex with respect to a point* $p \in X$ provided that, for each points x and y of X conditions $x \in py$ and $x \neq y$ imply $\varrho(p, x) < \varrho(p, y)$ (see [13], p. 310).

Let us note that we cannot replace, in Proposition 5.1, the convex metric on the dendrite Y by a radially convex one: namely in Example 5.1 the euclidean metric on Y is radially convex with respect to the origin $(0, 0)$. Putting in the same example $a = (-1, 1)$ and $b = (1, -1)$ we see that (17) holds, and thus convexity of the metric on Y cannot be replaced in Corollary 5.1 by radial convexity. Further, observe that the euclidean metric for the continuum X of Example 2.1 is radially convex with respect to the point p, the mapping $f\colon X \to X$ considered there is a generalized local expansion, and this shows that Theorem 5.1 is not longer true if one replaces a convex metric of the dendrite by a radially convex one.

Consider a smooth dendroid X with an initial point p, and let the metric on X be radially convex with respect to p. Take a generalized local expansion f of X onto itself. Then, according to Corollary 4.3, the mapping f is a homeomorphism, and therefore $f(p)$ is an initial point of X.

PROBLEM 5.2. Is it true that, for every point x of a smooth dendroid X with an initial point p and with a metric which is radially convex with respect to p, if f is a generalized local expansion of X onto itself, then the partial mapping $f|px\colon px \to f(px)$ is an isometry?

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