Remark on ANR-divisors

by

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Abstract. In this note we prove the following:

If X is a movable continuum such that pro-$\pi_1(X)$ is stable, and pro-$H_k(X)$ are stable for all k and are trivial for all but finitely many k, then X is an ANR-divisor.

Introduction. In [2] J. Dydak has raised the following question:

Let X be a movable continuum such that pro-$\pi_1(X)$ is stable, and pro-$H_k(X)$ is stable for all k and trivial for all but finitely many k. Is X an ANR-divisor?

The aim of this note is to give a positive answer to the above question.

We assume that the reader is familiar with some elementary facts from shape and ANR-divisors theories (see [3]).

I. Preliminaries. The following theorem is a special case of Theorem 2 in [1]:

(1.1) THEOREM. *Let X be a continuum. Then the following conditions are equivalent*:

a. pro-$H_k(X)$ *is stable for all* k,

b. $H^k(X)$ *is finitely generated for all* k.

Let us prove the following:

(1.2) THEOREM. *Let X be a movable continuum such that* pro-$H_k(X)$ *are stable for all k and are trivial for all but finitely many k. If X is approximatively* 1*-connected then X is a pointed* FANR.

Proof. In [1] J. Dydak proved the following fact (Lemma 3):

Let X be a continuum and let $n>0$. Then $\check{H}^n(X)/\mathrm{Tor}\,\check{H}^n(X)$ is isomorphic to $\mathrm{Hom}_Z(\text{pro-}H_n(X))$ and $\mathrm{Tor}\,\check{H}^n(X)$ is isomorphic to $\mathrm{Ext}_Z(\text{pro-}H_{n-1}(X))$.

In our case the above lemma implies that $\check{H}^n(X)$ are trivial for all but finitely many n. Since X is approximatively 1-connected and movable, the fundamental dimension of X is finite (see [5]). Moreover, by Theorem (1.1) we infer that $\check{H}^n(X)$ is finitely generated for all n. Hence Theorem (1.2) follows from a result of R. Geoghegan and R. C. Lacher (see [4]), which states that the shape of each finite dimensional and approximatively 1-connected continuum X is polyhedral if and only if its integral Čech cohomology is finitely generated.

The next three theorems may be found in [3] p. 119–122.

(1.3) THEOREM. *If* $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$ *and* Y *is an* ANR-*divisor, then* Y *is an* ANR-*divisor.*

(1.4) THEOREM. *If* X *is a pointed* FANR, *then* X *is an* ANR-*divisor.*

(1.5) THEOREM. *Let* X *and* Y *be compacta. If* $X \cup Y$ *and* $X \cap Y$ *are* ANR-*divisors, then* X *and* Y *are* ANR-*divisors.*

II. The main theorem. The aim of this note is to prove the following theorem:

(2.1) THEOREM. *If* X *is a movable continuum such that* pro-$\pi_1(X)$ *is stable, and* pro-$H_k(X)$ *are stable for all* k *and trivial for all but finitely many* k, *then* X *is an* ANR-*divisor.*

Proof. Let $x_0 \in X$ and let $(X, x_0) = \varprojlim\{(X_n, x_0), f_n^m\}$. We denote the natural projection by $f_n\colon X \to X_n$. Since pro-$\pi_1(X, x_0)$ is stable, we may assume that $(f_n^m)_\#\colon \pi_1(X_m, x_0) \to \pi_1(X_n, x_0)$ is an isomorphism for every $m \geqslant n$. Let (S, s_0) be a finite bouquet of 1-spheres such that there is an epimorphism $\varphi\colon \pi_1(S, s_0) \to \check{\pi}_1(X, x_0)$. Then there are maps $g_n\colon (S, s_0) \to (X_n, x_0)$ such that

$$(g_n)_\# = (f_n^m g_m)_\# \quad \text{for all } m \geqslant n$$

and

$$(g_n)_\# = (f_n)_\# \varphi .$$

Let us consider the space

$$(Y_n, y_0) = (M(g_n), x_0)$$

where $M(g_n)$ is a mapping cylinder of g_n.

One can see that the map

$$\hat{h}_n^{n+1}\colon X_{n+1} \cup S \to M(g_n)$$

defined by the formula

$$\hat{h}_n^{n+1}(x) = \begin{cases} f_n^{n+1}(x) & \text{for } x \in X_{n+1}, \\ x & \text{for } x \in S \end{cases}$$

has an extension $h_n^{n+1}\colon M(g_{n+1}) \to M(g_n)$.

Let $(Y, y_0) = \varprojlim\{(Y_n, y_0), h_n^m\}$. It is easy to check that

$$\mathrm{Sh}(X, x_0) = \mathrm{Sh}(Y, y_0) .$$

Theorem (1.3) implies that X is an ANR-divisor if and only if Y is an ANR-divisor. Let us consider the space $(Y \cup CS, y_0)$ where CS denotes the cone over $S \subset Y$. Then by Theorems (1.2) and (1.4) the set $Y \cup CS$ is an ANR-divisor. Since $Y \cap CS$ is an ANR-divisor, our theorem can be derived from Theorem (1.5). The proof is completed.

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References

[1] J. Dydak, *Some algebraic properties of continua*, Bull. Acad. Polon. Sci. 27 (1979), pp. 717–721.
[2] — A. Kadlof and S. Nowak, *Open problems in shape theory*, preprint.
[3] — and J. Segal, *Shape theory*, Lecture Notes in Math. 688, Springer 1978.
[4] R. Geoghegan and R. C. Lacher, *Compacta with the shape of finite complexes*, Fund. Math. 92 (1976), pp. 25–27.
[5] S. Nowak, *On the fundamental dimension of approximatively* 1-*connected compacta*, Fund. Math. 89 (1975), pp. 61–79.

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