see [1], this and the previous result show that given any nonlimit ordinal α, there exists a topological space with derived dimension $\alpha+1$. We also observe that Lemma 3, Lemma 4 and Corollary 4 immediately yield the well-known fact that asserts acc on prime ideals is equivalent to having classical Krull-dimension.

Added in proof. There exists a commutative Noetherian ring R with an arbitrary classical Krull dimension (see [1]). Hence $X = \operatorname{spec}(R)$ with the V-topology is quasi-compact (see Proposition 3). This immediately shows that given a nonlimit ordinal α, there exists a space X such that $d(X) = \alpha$.

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Strictly convex spheres in V-spaces

by

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Abstract. A well known theorem of Functional Analysis states that Strict Convexity is equivalent to unique metric lines in a Banach space. In this paper that result is put in a more general setting — the class of V-spaces. The class of V-spaces includes Banach spaces, as well as other metric spaces.

Rotundity or Strict Convexity has been studied extensively in Banach spaces. It is well known that metric lines are unique in a Banach space B if and only if B is strictly convex [1, 4, 5, 7, 10, 14]. The list of conditions in B equivalent to strict convexity (and therefore unique metric lines) is long. Day [7] lists six such conditions, Bumcrot [4] gives four other conditions, Andalafte and Valentine [1] list some of the conditions of Day and Bumcrot as well as four others. In related result Reda [13] proved the equivalency of algebraic and metric lines in Hilbert space and Nitka and Wiatrowska [12] proved that in Minkowski space, both more restricted than Banach space. Freese [9] found a number of conditions equivalent to the monotone property in a complete, convex, externally convex metric space. He also showed that the monotone property was equivalent to unique metric lines in a Banach space. In this paper it will be shown that unique metric lines and strict convexity (redefined in purely metric terms) are equivalent in a larger class of spaces.

I

Many of the conditions mentioned above may be defined in purely metric terms and, hence, discussed in that more general setting.

It is not difficult to find examples of complete, convex, externally convex metric spaces in which the concepts of strict convexity and unique metric lines are not equivalent. Therefore the spaces of Freese's result are too general if we wish to show the equivalency of unique metric lines and strict convexity. The spaces considered here are all complete, convex, and externally convex metric spaces, however, and we will call those spaces *line spaces*. In a line space more than one metric line may contain two given points.

DEFINITION 1. A line space has *unique metric lines* if the set $l(p, q)$ of points linear with each pair of distinct points p, q is isometric with the real line.

Since line spaces are too general for our purposes we consider other properties to assume valid in our spaces. One of these properties is a generalization of a property of four points which form a parallelogram.

DEFINITION 2. A metric space M is said to have the *vertical angle property* if for each four pairwise distinct points p, q, r, s such that a midpoint of p and s is a midpoint of q and r, then $pq = rs$.

If in the definition we relax the requirement that p, q, r and s be distinct, then we have the *strong vertical angle property*.

If for each p, q, r of M there exist points m and s of M satisfying the conditions of Definition 2, then M has the *weak vertical angle property*.

Spaces with vertical angle properties will be said to be VA, SVA or WVA, as appropriate.

DEFINITION 3. A metric space M is said to be FS if, whenever points $p, q, r, s \in M$ with pqr and pqs, there exists a t such that pqt, qrt, and qst.

V-spaces

The spaces of primary interest here will be spaces having the weak vertical angle property which are FS.

DEFINITION 4. A *V-space* is a line space which is FS and WVA.

THEOREM 1. *A complete metric space M is a V-space iff it is FS and WVA.*

Proof. Since a V-space is already complete, FS and WVA, we need to show that a complete, FS, WVA metric space is a V-space. It is not difficult to show that WVA implies convexity and external convexity.

For the vertical angle properties, $SVA \Rightarrow VA \Rightarrow WVA$ in a line space. The question arises whether a V-space is VA or SVA (i.e. whether FS and WVA imply VA or SVA). It is not, however, as will be shown in the second part of this paper. We also show that every Banach space is a V-space (Theorem 3). There are, however, V-spaces which are not Banach spaces (e.g., the hyperbolic plane). Theorems 2 and 8 are therefore more general than those in [1], [4], [5], [6], [7], and [10].

Slits, forks and bows

If two metric segments with common endpoints have no other points in common, then their union is called a *slit*. If two metric segments have exactly one endpoint in common and intersect in a metric segment their union is called a *fork*. If two metric segments whose union is not a segment have no common endpoints, intersect in a metric segment, and each contains an endpoint of the other, then their union is a *bow*. Lelek and Nitka [11] describe a metric space not having these phenomena as strongly convex, without ramifications and without edges (SC, WR and WE). Blumenthal [2] proved that a complete, convex, externally convex metric space has unique metric lines iff it does not have slits, forks or bows.

LEMMA 1. *If a line space has a bow, then it has a fork.*

Proof. Suppose there exists a bow. Then there are points p, q, r, s with pqr, qrs but not prs. Choose s^* on a metric line containg p, q and r such that prs^* and $rs^* = rs$. Then qrs^* holds by properties of betweenness. But qrs, qrs^* $s \neq s^*$ implies M has a fork.

A normed linear space is said to be strictly convex if, whenever p and q are points such that $\|p\| = \|q\| = 1$, then $\left\|\frac{p+q}{2}\right\| < 1$. Since all spheres of a normed linear space have the same shape (may be translated to the origin), strict convexity is a property of all spheres. In a metric space the same homogeneity is not necessarily assumed, so the equivalent condition in a metric space must be stated for each sphere.

DEFINITION 5. A metric space M has *strictly convex spheres* if for every p, q, r, s of M such that $pq = pr$ and s is metrically between q and r, then $ps < pq$.

LEMMA 2. *If M is a line-space with strictly convex spheres, then there do not exist slits in M.*

Proof. Suppose the contrary, then there exist points p, q with two distinct midpoints, m and m'. Let m^* be a midpoint of m and m'. Since S has strictly convex spheres, $pm^* < pm$ and $qm^* < qm$ so $pq = pm + mq > pm^* + m^*q$, contradicting the triangle inequality.

Since a V-space is a line space, Lemmas 1 and 2 hold for V-spaces.

THEOREM 2. *In a V-space unique metric lines are equivalent to strictly convex spheres.*

Proof. Assume that M is a V-space with strictly convex spheres. We must rule out slits, forks and bows. Lemma 2 rules out slits and, by Lemma 1 bows imply forks, so all that is needed is to eliminate the possibility of forks. We shall show that forks imply slits. Suppose we have a fork. Then there exist points p, q, r, s with pqr and pqs and segments $S(p, r)$, $S(p, s)$ that diverge at q. Since M is FS, there exists a t such that qrt and qst. We must, therefore, have two distinct segments joining q and t one of which contains r and the other s. But this would imply that these segments contain a slit, contradicting Lemma 2.

Now assume that M has unique metric lines. Therefore, let p, q, r, s be points with $pq = pr$ and s is between q and r (q, s, and r are then distinct). We may w.l.o.g. assume that s is a midpoint of q and r. M has strictly convex spheres, unless for some quadruple p, q, r, s as above, $ps \geqslant pq$. Suppose $ps \geqslant pq$. Then, applying, the weak vertical angle property to p, q, r we have points m and p' with m a common midpoint of q, r and p, p'. Since lines are unique, $m = s$ so $ps = sp' = \frac{1}{2}pp'$. From the weak vertical angle property $pq = rp'$ and $pr = qp'$ so all of those distances are equal. Now

$$pp' \leqslant pq+qp' = pq+qp \leqslant ps+sp = pp'$$

implies that q is a midpoint of p and p', which is impossible if M has unique metric lines.

THEOREM 3. *A Banach space is a V-space.*

Proof. We need to show that a Banach space B is FS and WVA. To show B is WVA we assume $p, q, r \in B$. The required points m and s may be taken to be $m = \frac{1}{2}(q+r)$ and $s = q+r-p$. To show B is FS we assume $p, q, r, s \in B$ with pqr and pqs (the betweenness is metric betweenness not necessarily algebraic betweenness). The point t required in Definition 3 may be taken to be $r+s-q$.

II

In this part we investigate some of the properties of VA and SVA line spaces and some further consequences of Theorem 2.

The properties VA and SVA are stronger than WVA in line spaces since each implies WVA. It is surprising, though, that each of the properties VA and SVA imply unique metric lines in a line space. This is proved in Theorem 6. Additional conditions equivalent to strictly convex spheres are given in Theorem 7 for Banach spaces and Theorem 8 for V-spaces.

The first step is to relate forks and slits in V-spaces.

THEOREM 4. *If a VA line space has a fork, then it has a slit.*

Proof. Suppose M is a VA line space with a fork. Then we have points p, q, r and s with pqr, pqs and segments $S(q,r)$ and $S(q,s)$ with $S(q,r) \cap S(q,s) = \{q\}$. Suppose, without loss of generality, that $pq = qr > qs$. Let $pq = a$, $qs = b$. Let u be a point between p and q with $uq = b$. Then q is a midpoint of p and r and u and s and the points p, r, u and s are distinct. The vertical angle property implies $pu = rs = a-b$. Then $qr = a = b+a-b = qs+sr$ implies s is between q and r, so we have a slit.

COROLLARY. *A VA line space is a V-space.*

Proof. Since VA implies WVA in a line space, we need only show that the V-space M is FS. Let $p, q, r, s \in M$ with pqr and pqs. By external convexity, there exists a t, such that the segment $S(p,t)$ contains q and r and $qt > qs$. The proof of Theorem 4 shows that qst. Therefore the point t is the one required by Definition 3.

THEOREM 5. *A VA line space has strictly convex spheres.*

Proof. Suppose M is a VA line space without strictly convex spheres. Then there exist points p, q, r and m (distinct) with $pq = pr$, m a midpoint of q and r and $pm \geqslant pq$. There exists a point p' such that m is a midpoint of p and p'. By the vertical angle property applied to p, q, r, p', we have $pq = rp'$ and $pr = qp'$ so $pq = qp' = rp'$. If $pm > pq$, then $pq+qp' > pm+mp = pm+mp' = pp'$ contradicting the triangle inequality. Hence $pq = pr = pm$. We construct a sequence $\{q_i\}$ as follows: $q_0 = q$, q_1 is on a metric line containing r, m and q such that q is a midpoint of m and q_1, and q_i is on a metric line containing r, m and q_j $(j<i)$ with q_{i-1} a midpoint of q_i and q_{i-2}, for all $i>1$. The vertical angle property, applied to p, m, q_1, p' implies that $pm = q_1p'$ and $p'm = q_1p$ so $pq_1 = q_1p' = pq$. By induction we may show, in the same manner, that $pq_i = pq$. Consider the triangle p, m, q_i. $mq_i = (i+1)\cdot mq$ because m and q_i are on a metric line containing all of the q_j's. The triangle inequality requires $mq_i = (i+1)mq \leqslant mp+pq_i = 2pq$. By choosing i sufficiently large we get a contradiction. Therefore M has strictly convex spheres.

THEOREM 6. *A VA line space has unique metric lines.*

Proof. A VA line space is a V-space with strictly convex spheres by Theorem 5 and the corollary of Theorem 4. By Theorem 2 strictly convex spheres are equivalent to unique metric lines.

Our contribution to the list of conditions equivalent to strict convexity in Banach spaces is given by the following theorem.

THEOREM 7. *In a Banach space, the following are equivalent:*

(a) *Strict convexity (strictly convex spheres).*

(b) *Unique metric lines.*

(c) *The vertical angle property.*

(d) *The strong vertical angle property.*

Proof. We know that (d) ⇒ (c) from part I. By Theorem 6 (c) ⇒ (b). (b) ⇔ (a) is well known. We need only observe that a Banach space with unique metric lines is SVA so (b) ⇒ (d).

Actually, the conditions mentioned in Theorem 7 are all equivalent in a V-space. Many other properties are equivalent to those, also. The following properties are defined in terms of a metric so are possible properties of a V-space.

DEFINITION 6. *Metric lines are Chebyshev sets* in M iff for each x in M and each metric line L $\exists a$ unique point of L nearest x.

DEFINITION 7. M has the *monotone property* provided for each point p and metric line L of M the distance px between p and a point x of L is monotone increasing as x recedes along either half-line of L determined by a foot of p on L.

In a metric space the distances of four points p, q, r, s may satisfy the strict ptolemaic inequality:

$$pq\cdot rs+pr\cdot qs > ps\cdot qr\,.$$

DEFINITION 8. M has the *isosceles weak strict ptolemaic property* if each non-linear quadruple p, q, r, s for which qrs and $pq = ps$ satisfies the strict ptolemaic inequality.

DEFINITION 9. M has the *isosceles feeble strict ptolemaic property* if each non-linear quadruple p, q, r, s for which qrs, $pq = ps$ and $qr = rs$ holds satisfies the strict ptolemaic inequality.

THEOREM 8. *In a V-space the following are equivalent*:

(a) *Strict convexity (strictly convex spheres).*

(b) *Unique metric lines,*

(c) *The vertical angle property.*

(d) *The strong vertical angle property.*

(e) *The monotone property.*

(f) *The isosceles weak strict ptolemaic property.*

(g) *The isosceles feeble strict ptolemaic property.*

Proof. It is easy to extend Theorem 7 to V-spaces. Freese [9] has shown that in a complete, convex, externally convex metric space e, f and g are equivalent. It is not difficult to show that (e) implies (a). We show that (a) implies (e).

Suppose (a) and not (e). If we do not have the monotone property then there exists a point p and a line L not containing p with points q, r, s and f_p on L such that $pq = pr = ps$, and f_p is a foot of p on L. Without loss of generality we may assume the points of L are in the order q, f_p, r, s. Then $pq = ps = pr$ and qrs contradict the assumption of strictly convex spheres.

In [1] it is shown that strict convexity in a Banach space is equivalent to the condition that metric lines are Chebysbev sets. Each of the conditions listed in Theorem 8 will imply that the metric lines are Chebyshev sets in a V-space. It would be interesting to know if this condition is also equivalent to strictly convex spheres in V.

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