On the classical Krull dimension of rings

by

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Abstract. If $X = \operatorname{spec}(R)$ is the set of prime ideals of a ring R, then X with a certain topology has derived dimension if and only if R has classical Krull dimension. Moreover the two dimensions then differ by at most 1.

Introduction. We recall the definition of the classical Krull dimension of a ring R. Let $X = \operatorname{spec}(R)$ be the set of all prime ideals of R. Let $\operatorname{spec}_0(R)$ denote the set of all maximal ideals of R. Then if $\alpha > 0$ is an ordinal, denote by $\operatorname{spec}_\alpha(R)$ the set of prime ideals P of R such that each prime Q properly containing P belongs to $\operatorname{spec}_\beta(R)$ for some $\beta < \alpha$. Then the smallest ordinal α for which $X = \operatorname{spec}_\alpha(R)$ is called the *classical Krull-dimension* cl. K-dim(R) of R (for more details see [1], [4]). We study derived dimension of X with respect to certain topologies which we define on X and show that derived dimension of X exists if and only if cl. K-dim(R) exists and the two dimensions differ by at most 1. We now establish some notation to be preserved throughout the paper. If A is a two-sided ideal of a ring R, we let $V(A)$ denote the subset of $X = \operatorname{spec}(R)$ consisting of those prime ideals that contain A, and let $D(A) = X - V(A)$. Now one can easily see that the sets $D(A)$ satisfy the axioms for open sets in a topological space and we call this the Z-topology on X (for more details see [6]). Now put $B_1 = \{V(A) \cap D(B)\colon A, B \text{ are ideals in } R\}$, $B_2 = \{V(A)\colon A \text{ is an ideal of } R\}$, then clearly each B_i, $i = 1, 2$, can be a base for a topology on X. The topology on X which has B_1 as a base is clearly stronger than the Z-topology and we call it the SZ-topology on X and the one with B_2 as a base is called the *V-topology*. Let us recall that in a topological space X an element $x \in X$ is called a *limit point of a subset A of X* if each open set containing x contains at least one point of A distinct from x. The set of all limit points of A is denoted by A' and is called the *derived set of A* and point $a \in A$ is called *isolated* whenever $a \in A - A'$. The α-derivative of a topological space X is defined by transfinte induction: $X_0 = X$, $X_{\alpha+1} = X'_\alpha$, and $X_\alpha = \bigcap_{\beta<\alpha} X_\beta$ for a limit ordinal α. Clearly each X_α is a closed subset of X and if for an ordinal α we have $X_\alpha = \emptyset$, then X is called *scattered*, see [5]. If X is scattered and α is the smallest ordinal such that $X_\alpha = \emptyset$, then α is called *derived dimension of X* and is denoted by $d(X) = \alpha$.

Rings in this paper are associative with identity.

Krause [4] shows that having classical Krull-dimension is equivalent to having acc on prime ideals. Using König Graph Theorem, Gordon and Robson [1] have shown that acc on prime ideals implies acc on finite intersections of prime ideals. We give a proof to a slight generalization of this result.

We need the following lemma.

LEMMA 1. *Let S be a partially ordered set with the minimum condition and $A_1, A_2, \ldots, A_n$ be nonempty subsets of S and let $T = A_1 \times A_2 \times \ldots \times A_n$ be ordered such that whenever $(a_1, a_2, \ldots, a_n) \geqslant (b_1, b_2, \ldots, b_n)$ then $a_i \geqslant b_i$ for some $1 \leqslant i \leqslant n$. Then T has the minimum condition.*

Proof. Let $(a_1^1, a_2^1, \ldots, a_n^1) \geqslant (a_1^2, a_2^2, \ldots, a_n^2) \geqslant \ldots \geqslant (a_1^k, a_2^k, \ldots, a_n^k) \geqslant \ldots$ be an inifinite chain in T. For each $1 \leqslant m \leqslant n$ let $a_m^{r_m}$ be a minimal element in the set $\{a_m^k\}_{k=1,2,\ldots}$ and put $r = \mathrm{Max}(r_1, r_2, \ldots, r_n)$ then we get

$$(a_1^k, a_2^k, \ldots, a_n^k) = (a_1^{k+1}, a_2^{k+1}, \ldots, a_n^{k+1}), \qquad \forall k \geqslant r.$$

Proposition 1. *If a ring R has acc on prime ideals, then it has acc on ideals I of the form $I = \bigcap_{P \in F} P_i^{k_i}$, where F is a finite set of noncomparable prime ideals and k_i is a positive integer.*

Proof. Let $I_1 \subset I_2 \subset \ldots \subset I_n \subset \ldots$ be an infinite ascending chain of ideals, each of which is of the form $I_n = \bigcap_{P \in F_n} P_i^{k_i}$, where F_n is a finite set of noncomparable prime ideals and each k_i is an integer. If it happens that $F_{r_1} = F_{r_2} = \ldots = F_{r_n} = \ldots$, where $r_1 < r_2 < \ldots < r_n < \ldots$ is an infinite sequence, then the previous lemma shows that the chain $I_{r_1} \subset I_{r_2} \subset \ldots \subset I_{r_n} \subset \ldots$ can not be infinite and we are through. Therefore without loss of generality we can assume $F_{n+1} - F_n \neq \emptyset$, $\forall n$ and complete the proof by obtaining a contradiction. We note that $F_{i-1} \cap F_r \supseteq F_i \cap F_r$ for all r and $i-1 \geqslant r$, for if not then there exists $P_i \in F_i \cap F_r$ such that $P_i \notin F_{i-1}$. Hence there exists $P_{i-1} \in F_{i-1}$ such that $P_{i-1} \subset P_i$ and since $r \leqslant i-1$, there exists $P_r \in F_r$ such that $P_r \subseteq P_{i-1} \subset P_i$, but P_r, P_i are both in F_r and can not be comparable. This shows that without loss of generality we can assume that $F_{i-1} \cap F_r = F_i \cap F_r$, for all r and $i-1 \leqslant r$. Now given any integer $m > 0$ let $P_m \in F_m - F_{m-1}$, then $P_m \notin \bigcup_{i=1}^{m-1} F_i$ for otherwise $P_m \in F_r$, for some $r \leqslant m-1$ and $F_m \cap F_r = F_{m-1} \cap F_r$ implies that $P_m \in F_{m-1}$, which is impossible. Hence there exists $P_{m-1} \in F_{m-1}$ such that $P_{m-1} \subset P_m$ and $P_{m-1} \notin \bigcup_{i=1}^{m-2} F_i$, for otherwise $P_{m-1} \in F_{m-1} \cap F_k$ for some $k \leqslant m-2$ implies that $P_{m-1} \in F_m$, which is impossible. Repeating this process we get $P_1 \subset P_2 \subset \ldots \subset P_m$ a chain of length m and each P_i belong to F_i. Now put $F_1^n = \{P_1 \in F_1 :$ there exists a chain $P_1 \subset P_2 \subset \ldots \subset P_n$, where $P_i \in F_i$, $i = 1, \ldots, n\}$. We have already shown that $F_1^n \neq \emptyset$, $\forall n$. Moreover F_1^n is finite and $F_1^n \supseteq F_1^m$ for $m \geqslant n$, therefore the chain $F_1^2 \supseteq F_1^2 \supseteq \ldots \supseteq F_1^2 \supseteq \ldots$ is stationary and we can choose $Q_1 \in \bigcap_{n=1}^{\infty} F_1^n$. Now for each $n \geqslant 2$, let $F_2^n = \{P_2 \in F_2 :$ there exists a chain $Q_1 \subset P_2 \subset \ldots \subset P_n$, where $P_i \in F_i$, $i = 2, \ldots, n\}$, it is clear that $F_2^n \neq \emptyset$, $\forall n \geqslant 2$ and we can choose $Q_2 \in \bigcap_{n=2}^{\infty} F_2^n$. Hence proceeding inductively we get a chain $Q_1 \subset Q_2 \subset \ldots \subset Q_n \subset \ldots \subset$ which is the desired contradiction.

The following lemma is well-known and easy to prove.

LEMMA 2. *Let $X = \mathrm{spec}(R)$ be with the Z-topology, then the followings are equivalent.*

1. *X is Noetherian (acc on open subsets).*
2. *Every subset of X is quasi-compact.*
3. *R has acc on intersections of prime ideals.*

COROLLARY 1. *Let a ring R have classical Krull-dimension and have only finitely prime ideals minimal over any ideal, then every prime ideal is minimal over some finitely generated subideal.*

Proof. If A is an ideal in R, let $P(A)$ denote the intersection of all prime ideals containing A. It is sufficient to show that $P(A) = P(\langle x_1, x_2, \ldots, x_n \rangle)$, where $\langle x_1, x_2, \ldots, x_n \rangle$ is the ideal generated by some elements $x_1, x_2, \ldots, x_n \in A$. It is clear that $V(A) = \bigcap_{x \in A} V(\langle x \rangle)$ and $X - V(A) = \bigcup_{x \in A} (X - V(\langle x \rangle))$. Now by Proposition 1, we note that R has acc on intersections of prime ideals and therefore Lemma 2 shows that every subset and in particular $X - V(A)$ is quasi-compact. Thus there are some elements $x_1, x_2, \ldots, x_n \in A$ such that $X - V(A) = \bigcup_{i=1}^{n} (X - V(\langle x_i \rangle))$. Hence $V(A) = \bigcap_{i=1}^{n} V(\langle x_i \rangle)$ implies that $P(A) = P(\langle x_1, x_2, \ldots, x_n \rangle)$.

Next we prove a stronger result.

PROPOSITION 2. *Let R be a ring with $\mathrm{cl.}K\text{-}\dim(R) = n$ and have only finitely many prime ideals minimal over any ideal, then every prime ideal is minimal over a subideal generated by $\leqslant n$ elements.*

Proof. Let P be a prime ideal and $P = P_0 \supset P_1 \supset \ldots \supset P_m$ be a chain of prime ideals, then by Lemma 1.3 of [4] we have $\mathrm{cl.}K\text{-}\dim(R/P_m) > \mathrm{cl.}K\text{-}\dim(R/P_{m-1}) > \ldots > \mathrm{cl.}K\text{-}\dim(R/P)$. This shows that $\mathrm{rank}(P) \leqslant \mathrm{cl.}\ K\text{-}\dim(R)$. Now if we assume that the zero ideal is generated by the empty set, then one can proceed by induction on $k = \mathrm{rank}(P) \leqslant n$ and show that P is minimal over a subideal generated by $\leqslant k$ elements. For $k = 0$ it is clear by our assumption. Let us assume it true when $\mathrm{rank}(P) \leqslant k-1$ and let $\mathrm{rank}(P) = k$. Now let $P_1, P_2, \ldots, P_r$ be all minimal prime ideals, then since $k > 0$ we have $P \nsubseteq \bigcup_{i=1}^{r} P_i$. Thus there exists $x_1 \in P$ such that $x_1 \notin P_i$, $\forall_i$. Consider $\bar{R} = R/\langle x_1 \rangle$, $\bar{P} = P/\langle x_1 \rangle$, where $\langle x_1 \rangle$ is the ideal generated by x_1. Now it is clear that $\mathrm{rank}(\bar{P}) \leqslant k-1$ and by the induction hypothesis $\bar{P}$ is minimal over $\langle \bar{x}_2, \bar{x}_3, \ldots, \bar{x}_k \rangle$. Now suppose $x_i = f^{-1}(\bar{x}_i)$, $i = 2, 3, \ldots, k$ where $f\colon R \to \bar{R}$ is the natural epimorphism, then it is clear that P is minimal over $\langle x_1, x_2, \ldots, x_k \rangle$.

The following result must be well-known, but we give a proof for the convenience of the reader.

LEMMA 3. *Let X be a topological space, then the followings are equivalent.*

1. *Every nonempty subset of X contains an isolated point.*
2. *There is an ordinal $\alpha>0$ such that $X_\alpha = \emptyset$.*

Proof. (1) → (2): let $X_\alpha \neq \emptyset$ for all ordinal α. It is clear that $X_{\alpha+1} = X_\alpha - S_\alpha$, where S_α is the set of all isolated points of X_α and since $S_\alpha \neq \emptyset$ we get $X_\alpha \supsetneqq X_{\alpha+1}$ for each ordinal α which is impossible.

(2) → (1): Assume $X_\alpha = \emptyset$ for some ordinal $\alpha>0$, and let S be a nonempty subset of X. Let β be the smallest ordinal among the ordinals $\leqslant\alpha$ for which $S \cap X_\beta = \emptyset$. It is clear that for each α we have $X_\alpha = X - \bigcup_{\gamma<\alpha} S_\gamma$, where S_γ is the set of all isolated points of X_γ. Now $S \cap X_\beta = \emptyset$ implies that $S \subset \bigcup_{\gamma<\beta} S_\gamma$. Let γ be the first ordinal among the ordinals $<\beta$ such that $S \cap S_\gamma \neq \emptyset$. Suppose that $x \in S \cap S_\gamma$, then we claim that x is an isolated point of S. To see this it is sufficient to show that $S \subseteq X_\gamma$. But $X_\gamma = X - \bigcup_{\lambda<\gamma} S_\lambda$ and $S \cap S_\lambda = \emptyset$, $\forall \lambda<\gamma$ implies that $S \subseteq X_\gamma$.

COROLLARY 2. *Let R have classical Krull-dimension equal to α, then $X = \operatorname{spec}(R)$ with either the SZ-topology or the V-topology have derived dimension and $d(X) \leqslant \alpha+1$.*

Proof. Let S be a nonempty subset of X, then since R has acc on prime ideals, there exists a maximal $P \in S$ of S. We note that $V(P) \cap S = \{P\}$. This shows that P is an isolated point of S with respect to the V-topology on X, but clearly SZ-topology is stronger than V-topology, therefore P is also an isolated point of S with respect to the SZ-topology. Hence $d(X)$ exists and to show that $d(X) \leqslant \alpha+1$, it is sufficient to prove $\operatorname{spec}_\alpha(R) \subseteq \bigcup_{\beta\leqslant\alpha} S_\beta$, where S_β is the set of all isolated points of X_β, for $X = \operatorname{spec}_\alpha(R)$ implies that $X_{\alpha+1} = X - \bigcup_{\beta\leqslant\alpha} X_\beta = \emptyset$. We proceed by induction on α. For $\alpha = 0$ we must show that $\operatorname{spec}_0(R) \subseteq S_0$, but clearly each maximal ideal is an isolated point of X. Let us assume that for ordinals $\beta>\alpha$ we have $\operatorname{spec}_\beta(R) \subseteq \bigcup_{\gamma\leqslant\beta} S_\gamma$. Now suppose that $P \in \operatorname{spec}_\alpha(R)$, then $P \subsetneq Q$ implies that $Q \in \operatorname{spec}_\beta(R)$ for some $\beta<\alpha$, then the induction hypothesis shows that $Q \in S_\gamma$ for some $\gamma<\alpha$. Now if $P \notin \bigcup_{\gamma<\alpha} S$ then $P \in X_\alpha = X - \bigcup_{\gamma<\alpha} S_\gamma$ and we claim that $P \in S_\alpha$. To see this we prove that P is a maximal element in X_α. So let $Q \supsetneq P$, then we have already shown that $Q \in \operatorname{spec}_\beta(R)$ for some $\beta<\alpha$, therefore $Q \notin X_\alpha$. Thus $\operatorname{spec}_\alpha(R) \subseteq \bigcup_{\beta\leqslant\alpha} S$.

PROPOSITION 3. *Let $\text{cl.}k\text{-dim}(R) = \alpha$ and suppose R has only finitely many prime ideals minimal over any ideal, then $X = \operatorname{spec}(R)$ with the SZ-topology have derived dimension which is not a limit ordinal and $d(X) \leqslant \alpha+1$.*

Proof. All we have to do is to show that $d(X)$ is not a limit ordinal. Let $d(X) = \alpha$, where α is a limit ordinal. But we have $X_\alpha = \bigcap_{\beta<\alpha} X_\beta$ and $\{X_\beta\}_{\beta<\alpha}$ is a family of closed subsets with the finite intersection property. We claim that $X_\alpha \neq \emptyset$, which is a contradiction. It is sufficient to prove that X with the SZ-topology is quasi-compact. We observe that X with the Z-topology is T_0 and every subset is quasi-compact and also every nonempty irreducible closed subset has a generic point (see [6] Corollary 9.7), therefore X with the Z-topology is spectral in the sense of [2]. Now we note that $S = \{V(A), D(B)\colon A, B \text{ are ideals of } R\}$ is an open sub-basis for X with the SZ-topology, therefore the SZ-topology on X is exactly the patch topology on X (see [2]) and by Theorem 1 of [2] it is compact.

LEMMA 4. *Let $X = \operatorname{spec}(R)$ be with the V-topology and $S \subseteq X$, then an element $P \in S$ is an isolated point of S if and only if it is a maximal element of S.*

Proof. If $P \in S$ is maximal, then $V(P) \cap S = \{P\}$ shows that P is isolated. Now suppose that $P \in S$ is isolated then there exists an open subset G such that $P \in G$, $G \cap S = \{P\}$. But there exists $V(A)$ such that $P \in V(A) \subseteq G$, then $V(A) \cap S = \{P\}$. Now we claim that P is maximal in S, for if $P \subsetneq Q$ and $Q \in S$, then $Q \in V(A)$ which is impossible.

COROLLARY 3. *Let $X = \operatorname{spec}(R)$ be with the V-topology, then $\operatorname{spec}_\alpha(R) = \bigcup_{\beta\leqslant\alpha} S_\beta$, where S_β is the set of isolated points of X_β.*

Proof. We proceed by induction on α. For $\alpha = 0$ it is clear. Let us assume that $\operatorname{spec}_\beta(R) = \bigcup_{\gamma\leqslant\beta} S_\gamma$ for all $\beta<\alpha$. Now let $P \in \bigcup_{\beta\leqslant\alpha} S_\beta$, then $P \in S_\beta$ for some $\beta\leqslant\alpha$. If $P \in S_\alpha$, then P is a maximal element of X_α and so $Q \in X$, $Q \supsetneq P$ implies that $Q \notin X_\alpha = X - \bigcup_{\beta<\alpha} S_\beta$ which implies that $Q \in S_\beta$ for some $\beta<\alpha$. Thus $Q \in \bigcup_{\gamma\leqslant\beta} S_\gamma = \operatorname{spec}_\beta(R)$ implies that $P \in \operatorname{spec}_\alpha(R)$ and if $P \notin S_\alpha$ then $P \in S_\beta$ for some $\beta<\alpha$ implies that $P \in \bigcup_{\gamma\leqslant\beta} S_\gamma = \operatorname{spec}_\beta(R) \subseteq \operatorname{spec}_\alpha(R)$. Therefore we have $\bigcup_{\beta\leqslant\alpha} S_\beta \subseteq \operatorname{spec}_\alpha(R)$. Conversely, let $P \in \operatorname{spec}_\alpha(R)$, then if $P \notin \bigcup_{\beta<\alpha} S_\beta$ we show that $P \in S_\alpha$. To this end let $Q \in X$, $Q \supsetneq P$, then $Q \in \operatorname{spec}_\beta(R) = \bigcup_{\gamma<\beta} S_\gamma$, $\beta<\alpha$ implies that $Q \notin X_\alpha = X - \bigcup_{\gamma<\alpha} S_\gamma$, but $P \in X_\alpha$ shows that P must be a maximal element in X_α, i.e. $P \in S_\alpha$. Thus we have $\operatorname{spec}_\alpha(R) \subseteq \bigcup_{\beta\leqslant\alpha} S_\beta$.

COROLLARY 4. *Let $X = \operatorname{spec}(R)$ be with the V-topology, then $d(X)$ exists if and only if $\text{cl.}K\text{-dim}(R)$ exists and $d(X) = \text{cl.}K\text{-dim}(R)$ if $d(X)$ is a limit ordinal and $d(X) = \text{cl.}K\text{-dim}(R)+1$ if $d(X)$ is not a limit ordinal.*

Proof. $\operatorname{spec}_\alpha(R) = \bigcup_{\beta\leqslant\alpha} S_\beta$ and $X_{\alpha+1} = X - \bigcup_{\beta\leqslant\alpha} S_\beta$ shows that $d(X)$ exists if and only if $\text{cl.}K\text{-dim}(R)$ exists. Now let $d(X) = \alpha$ be a limit ordinal, then $X_\alpha = X - \bigcup_{\beta<\alpha} S_\beta = \emptyset$ implies that $X = \bigcup_{\beta<\alpha} S_\beta = \bigcup_{\beta\leqslant\alpha} S_\beta = \operatorname{spec}_\alpha(R)$. Hence $\text{cl.}K\text{-dim}(R) \leqslant \alpha$. But by Corollary 2, we have $d(X) \leqslant \text{cl.}K\text{-dim}(R)+1$. Thus $\text{cl.}K\text{-dim}(R) = \alpha$. Now let $d(X) = \beta+1$, then we show that $\text{cl.}K\text{-dim}(R) = \beta$. We note that $X_{\beta+1} = \emptyset$ implies that $X = \bigcup_{\gamma\leqslant\beta} S_\beta = \operatorname{spec}_\beta(R)$. Thus $\text{cl.}K\text{-dim}(R) \leqslant \beta$ and $d(X) \leqslant \text{cl.}K\text{-dim}(R)+1$ implies that $\text{cl.}K\text{-dim}(R) = \beta$.

Remark. There are commutative rings with arbitrary classical Krull-dimension, see [1], this and the previous result show that given any nonlimit ordinal α, there exists a topological space with derived dimension $\alpha+1$. We also observe that Lemma 3, Lemma 4 and Corollary 4 immediately yield the well-known fact that asserts acc on prime ideals is equivalent to having classical Krull-dimension.

Added in proof. There exists a commutative Noetherian ring R with an arbitrary classical Krull dimension (see [1]). Hence $X = \operatorname{spec}(R)$ with the V-topology is quasi-compact (see Proposition 3). This immediately shows that given a nonlimit ordinal α, there exists a space X such that $d(X) = \alpha$.

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Strictly convex spheres in V-spaces

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Abstract. A well known theorem of Functional Analysis states that Strict Convexity is equivalent to unique metric lines in a Banach space. In this paper that result is put in a more general setting — the class of V-spaces. The class of V-spaces includes Banach spaces, as well as other metric spaces.

Rotundity or Strict Convexity has been studied extensively in Banach spaces. It is well known that metric lines are unique in a Banach space B if and only if B is strictly convex [1, 4, 5, 7, 10, 14]. The list of conditions in B equivalent to strict convexity (and therefore unique metric lines) is long. Day [7] lists six such conditions, Bumcrot [4] gives four other conditions, Andalafte and Valentine [1] list some of the conditions of Day and Bumcrot as well as four others. In related result Reda [13] proved the equivalency of algebraic and metric lines in Hilbert space and Nitka and Wiatrowska [12] proved that in Minkowski space, both more restricted than Banach space. Freese [9] found a number of conditions equivalent to the monotone property in a complete, convex, externally convex metric space. He also showed that the monotone property was equivalent to unique metric lines in a Banach space. In this paper it will be shown that unique metric lines and strict convexity (redefined in purely metric terms) are equivalent in a larger class of spaces.

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Many of the conditions mentioned above may be defined in purely metric terms and, hence, discussed in that more general setting.

It is not difficult to find examples of complete, convex, externally convex metric spaces in which the concepts of strict convexity and unique metric lines are not equivalent. Therefore the spaces of Freese's result are too general if we wish to show the equivalency of unique metric lines and strict convexity. The spaces considered here are all complete, convex, and externally convex metric spaces, however, and we will call those spaces *line spaces*. In a line space more than one metric line may contain two given points.