On the classical Krull dimension of rings

by

O. A. S. Karamzadeh (Ahvaz)

Abstract. If $X = \text{spec}(R)$ is the set of prime ideals of a ring $R$, then $X$ with a certain topology has derived dimension if and only if $R$ has classical Krull dimension. Moreover the two dimensions then differ by at most 1.

Introduction. We recall the definition of the classical Krull dimension of a ring $R$. Let $X = \text{spec}(R)$ be the set of all prime ideals of $R$. Let $\text{spec}_0(R)$ denote the set of all maximal ideals of $R$. Then if $\alpha > 0$ is an ordinal, denote by $\text{spec}_\alpha(R)$ the set of prime ideals $P$ of $R$ such that each prime $Q$ properly containing $P$ belongs to $\text{spec}_\beta(R)$ for some $\beta < \alpha$. Then the smallest ordinal $\alpha$ for which $X = \text{spec}_\alpha(R)$ is called the classical Krull dimension cl. $X$-dim$_\alpha(R)$ of $R$ (for more details see [1], [4]). We study derived dimension of $X$ with respect to certain topologies which we define on $X$ and show that derived dimension of $X$ exists if and only if cl. $X$-dim$_\alpha(R)$ exists and the two dimensions differ by at most 1. We now establish some notation to be preserved throughout the paper. If $A$ is a two-sided ideal of a ring $R$, we let $V(A)$ denote the subset of $X = \text{spec}(R)$ consisting of those prime ideals that contain $A$, and let $D(A) = X - V(A)$. Now one can easily see that the sets $D(A)$ satisfy the axioms for open sets in a topological space and we call this the Z-topology on $X$ (for more details see [6]). Now put $B_1 = \{V(A), D(B); A, B$ are ideals in $R\}$, $B_2 = \{V(A); A$ is an ideal of $R\}$, then clearly each $B_i$, $i = 1, 2$, can be a base for a topology on $X$. The topology on $X$ which has $B_1$ as a base is clearly stronger than the Z-topology and we call it the SZ-topology on $X$ and the one with $B_2$ as a base is called the $V$-topology. Let us recall that in a topological space $X$ an element $x \in X$ is called a limit point of a subset $A$ of $X$ if each open set containing $x$ contains at least one point of $A$ distinct from $x$. The set of all limit points of $A$ is denoted by $A'$ and is called the derived set of $A$ and point $a \in A$ is called isolated whenever $a \notin A'$. The $a$-derivative of a topological space $X$ is defined by transfinite induction: $X_0 = X$, $X_{\alpha+1} = X_\alpha'$, and $X_\alpha = \bigcap_{\beta < \alpha} X_\beta$ for a limit ordinal $\alpha$. Clearly each $X_\alpha$ is a closed subset of $X$ and if for an ordinal $\alpha$ we have $X_\alpha = \emptyset$, then $X$ is called scattered, see [5]. If $X$ is scattered and $\alpha$ is the smallest ordinal such that $X_\alpha = \emptyset$, then $\alpha$ is called derived dimension of $X$ and is denoted by $d(X) = \alpha$. 
Rings in this paper are associative with identity.

Krause [4] shows that having classical Krull-dimension is equivalent to having acc on prime ideals. Using König Graph Theorem, Gordon and Robson [1] have shown that acc on prime ideals implies acc on finite intersections of prime ideals. We give a proof to a slight generalization of this result.

We need the following lemma.

**Lemma 1.** Let \( S \) be a partially ordered set with the minimum condition and \( A_1, A_2, \ldots, A_n \) be nonempty subsets of \( S \) and let \( T = A_1 \times A_2 \times \cdots \times A_n \) be ordered such that whenever \((a_1, a_2, \ldots, a_n) \geq (b_1, b_2, \ldots, b_n)\), then \( a_i \geq b_i \) for some \( 1 \leq i \leq n \). Then \( T \) has the minimum condition.

**Proof.** Let \((a_1^0, a_2^0, \ldots, a_n^0) \geq (a_1^1, a_2^1, \ldots, a_n^1) \geq \cdots \geq (a_1^r, a_2^r, \ldots, a_n^r) \geq \cdots \geq (a_1^s, a_2^s, \ldots, a_n^s) \geq \cdots \) be an infinite chain in \( T \). For each \( 1 \leq m \leq n \) let \( a_m^0 \) be a minimal element in the set \( \{a_m^n \mid 1 \leq n \leq s\} \) and put \( r = \max(r_1, r_2, \ldots, r_n) \) then we get

\[
(a_1^0, a_2^0, \ldots, a_n^0) = (a_1^{s+1}, a_2^{s+1}, \ldots, a_n^{s+1}), \quad \forall k \geq r.
\]

**Proposition 1.** If a ring \( R \) has acc on prime ideals, then it has acc on ideals \( I \) of the form \( I = \bigcap_{P \in \mathfrak{p}} P \), where \( P \) is a finite set of noncomparable prime ideals and \( k_0 \) is a positive integer.

**Proof.** Let \( I_1 \subseteq I_2 \subseteq \cdots \subseteq I_k \subseteq \cdots \) be an infinite ascending chain of ideals, each of which is of the form \( I_k = \bigcap_{P \in \mathfrak{p}_k} P \), where \( \mathfrak{p}_k \) is a finite set of noncomparable prime ideals and each \( k \) is an integer. If it happens that \( F_1 = F_2 = \cdots = F_m = \cdots \), where \( r_1 < r_2 < \cdots < r_m < \cdots \) is an infinite sequence, then the previous lemma shows that the chain \( I_1 \subseteq I_2 \subseteq \cdots \subseteq I_m \subseteq \cdots \) can be finite and we are through. Therefore without loss of generality we can assume \( F_{i+1} \notin F_i \), \( \forall k \), and complete the proof by obtaining a contradiction. We note that \( F_{i+1} \subseteq F_i \cap F_i, \forall r \) for all \( i = 1, r \), for if not then there exists \( P \in F_i \cap F_i \) such that \( P \notin F_{i+1} \). Hence there exists \( P_{i+1} \in F_{i+1} \) such that \( P_{i+1} \in F_i \), and since \( r \geq 1 \), there exists \( P \in F_i \) such that \( P \subseteq P_{i+1} \subseteq P_i \), but \( P_i, P_{i+1} \) both in \( F_i \) and can not be comparable. This shows that without loss of generality we can assume \( F_{i+1} \subseteq F_i \cap F_i, \forall r \) for all \( i = 1, r \).

Next we prove a stronger result.

**Proposition 2.** Let \( R \) be a ring with \( \text{cl.K.dim}(R) = n \) and have only finitely many prime ideals minimal over any ideal, then every prime ideal is minimal over some subideal generated by \( \leq n \) elements.

**Proof.** Let \( P \) be a prime ideal and \( P = P_0 \supseteq P_1 \supseteq \cdots \) be a chain of prime ideals, then by Lemma 1.3 of [4] we have \( \text{cl.K.dim}(R/P_0) \geq \text{cl.K.dim}(R/P_0) \supseteq \cdots \geq \text{cl.K.dim}(R/P_n) \) for all \( n \). This shows that \( \text{rank}(P) \leq \text{cl.K.dim}(R) \). Now if we assume that the zero ideal is generated by the empty set, then one can proceed by induction on \( k = \text{rank}(P) \leq n \) and show that \( P \) is minimal over a subideal generated by \( \leq k \) elements. For \( k = 0 \) it is clear by our assumption. Let us assume it true when \( \text{rank}(P) \leq k-1 \) and let \( \text{rank}(P) = k \). Now let \( P_1, P_2, \ldots, P_r \) be all minimal prime ideals, then since \( k > 0 \) we have \( P_1 \notin P_r \). Thus there exists \( x \in P_1 \) such that \( x \notin P_r \), \( \forall P \).

Consider \( R = \mathbb{R}(x) \), \( \mathbb{P} = \mathbb{P}(x) \), where \( (x) \) is the ideal generated by \( x \). Now it is clear that \( \text{rank}(P) \leq k-1 \) and by the induction hypothesis \( P \) is minimal over \( \langle x \rangle \). Now suppose \( x = f^{-1}(S), i = 2, 3, \ldots, k \) where \( f: R \to R \) is the natural epimorphism, then it is clear that \( P \) is minimal over \( \langle x_1, x_2, \ldots, x_k \rangle \).
The following result must be well-known, but we give a proof for the convenience of the reader.

**Lemma 3.** Let \( X \) be a topological space, then the followings are equivalent:
1. Every nonempty subset of \( X \) contains an isolated point.
2. There is an ordinal \( \alpha > 0 \) such that \( X_\alpha = \emptyset \).

**Proof.** (1) \( \rightarrow \) (2): Let \( X_\alpha \neq \emptyset \) for all ordinal \( \alpha \). It is clear that \( X_{\alpha + 1} = X_\alpha - S_\alpha \), where \( S_\alpha \) is the set of all isolated points of \( X_\alpha \) and since \( S_\alpha \neq \emptyset \) we get \( X_\beta \neq X_\beta + 1 \) for each ordinal \( \beta \) which is impossible.

(2) \( \rightarrow \) (1): Assume \( X_\alpha = \emptyset \) for some ordinal \( \alpha > 0 \), and let \( S \) be a nonempty subset of \( X \). Let \( \beta \) be the smallest ordinal among the ordinals \( \leq \alpha \) for which \( S \cap X_\beta = \emptyset \). It is clear that for each \( \pi \) we have \( X_\pi = X - \bigcup S_\pi \), where \( S_\pi \) is the set of all isolated points of \( X_\pi \). Now \( S \cap X_\beta = \emptyset \) implies that \( S \subseteq X_\beta \). Let \( \gamma \) be the first ordinal among the ordinals \( < \beta \) such that \( S \cap S_\gamma \neq \emptyset \). Suppose that \( x \in S \cap S_\gamma \), then we claim that \( x \) is an isolated point of \( S \). To see this it is sufficient to show that \( S \subseteq X_\gamma \). But \( X_\gamma = X - \bigcup S_\gamma \) and \( S \subseteq X_\gamma \), \( \forall \gamma < \beta \) implies that \( S \subseteq X_\gamma \).

**Corollary 2.** Let \( R \) have classical Krull dimension equal to \( a \), then \( X = \text{spec}(R) \) with either the SZ-topology or the V-topology have derived dimension and \( d(X) \leq a + 1 \).

**Proof.** Let \( S \) be a nonempty subset of \( X \), then \( R \) has a socle prime ideals, there is a maximal \( P \in S \). We note that \( V(P) \cap S = \{ P \} \). This shows that \( P \) is an isolated point of \( S \), with respect to the \( P \)-topology on \( X \), clearly \( SZ \)-topology is stronger than \( P \)-topology, therefore \( P \) is also an isolated point of \( S \), with respect to the \( SZ \)-topology. Hence \( d(X) \leq a + 1 \), it is sufficient to prove \( \text{spec}(R) \subseteq \bigcup S_\beta \), where \( S_\beta \) is the set of all isolated points of \( X_\beta \), for \( X = \text{spec}(R) \) implies that \( X_{\alpha + 1} = X - \bigcup S_\alpha = \emptyset \). We proceed by induction on \( \alpha \).

For \( \alpha = 0 \) we must show that \( \text{spec}(R) \subseteq S_\beta \), but clearly each maximal ideal is an isolated point of \( X \). Let us assume that for ordinals \( \beta < \alpha \) we have \( \text{spec}(R) \subseteq S_\beta \).

Now suppose that \( P \in \text{spec}(R) \), then \( P \not\subseteq Q \) implies that \( Q \in \text{spec}(R) \) for some \( \beta < \alpha \), then the induction hypothesis shows that \( Q \subseteq S_\beta \), for some \( \gamma < \beta \). Now if \( P \not\subseteq S_\beta \) then \( P \not\subseteq X_\beta = X - \bigcup S_\beta \), and we claim that \( P \not\subseteq S_\gamma \). To see this we prove that \( P \) is a maximal element in \( X_\beta \). So let \( Q \supseteq P \), then we have already shown that \( Q \in \text{spec}(R) \) for some \( \beta < \alpha \), therefore \( Q \not\subseteq S_\beta \). Thus \( \text{spec}(R) \subseteq S_\beta \).

**Corollary 4.** Let \( X = \text{spec}(R) \) be with the \( V \)-topology, then \( d(X) \) exists if and only if \( cl.K-dim(R) \) exists and \( d(X) = cl.K-dim(R) \) if \( d(X) \) is a limit ordinal and \( d(X) = cl.K-dim(R) + 1 \) if \( d(X) \) is not a limit ordinal.

**Proof.** \( \text{spec}(R) = \bigcup S_\beta \) and \( X_{\alpha + 1} = X - \bigcup S_\alpha \) shows that \( d(X) \) exists if and only if \( cl.K-dim(R) \) exists. Now let \( d(X) = \alpha \) be a limit ordinal, then \( X_\alpha = X - \bigcup S_\alpha \) implies that \( X = \bigcup S_\beta = \bigcup S_\alpha = \text{spec}(R) \). Hence \( cl.K-dim(R) \leq a \). But by Corollary 2, we have \( d(X) \leq cl.K-dim(R) + 1 \). Thus \( cl.K-dim(R) = \alpha \). Now let \( d(X) = \beta + 1 \), then we show that \( cl.K-dim(R) = \beta \).

We note that \( X_{\alpha + 1} = \emptyset \) implies that \( X = \bigcup S_\beta = \text{spec}(R) \). Thus \( cl.K-dim(R) \leq \beta \) and \( d(X) \leq cl.K-dim(R) + 1 \) implies that \( cl.K-dim(R) = \beta \).

**Remark.** There are commutative rings with arbitrary classical Krull-dimension.
see [1], this and the previous result show that given any nonlimit ordinal \( \alpha \), there exists a topological space with derived dimension \( \alpha + 1 \). We also observe that Lemma 3, Lemma 4 and Corollary 4 immediately yield the well-known fact that asserts acc on prime ideals is equivalent to having classical Krull-dimension.

Added in proof. There exists a commutative Noetherian ring \( R \) with an arbitrary classical Krull dimension (see [1]). Hence \( X = \text{spec}(R) \) with the \( V \)-topology is quasi-compact (see Proposition 3). This immediately shows that given a nonlimit ordinal \( \alpha \), there exists a space \( X \) such that \( d(X) = \alpha \).

References


COLLEGE OF MATHEMATICS AND
COMPUTER SCIENCES
JAMMU-HAPUR UNIVERSITY
Aligarh, India

Accepted par la Redaction le 3. 11. 1980

Strictly convex spheres in \( V \)-spaces

by

Raymond Freese (St. Louis, Mo.) and Grattan Murphy (Orono, Maine)

Abstract. A well known theorem of Functional Analysis states that Strict Convexity is equivalent to unique metric lines in a Banach space. In this paper that result is put in a more general setting — the class of \( V \)-spaces. The class of \( V \)-spaces includes Banach spaces, as well as other metric spaces.

Rotundity or Strict Convexity has been studied extensively in Banach spaces. It is well known that metric lines are unique in a Banach space \( B \) if and only if \( B \) is strictly convex \([1, 4, 5, 7, 10, 14]\). The list of conditions in \( B \) equivalent to strict convexity (and therefore unique metric lines) is long. Day \([7]\) lists six such conditions, Bonsall \([4]\) gives four other conditions, Andalafie and Valentine \([1]\) list some of the conditions of Day and Bonsall as well as four others. In related result Reda \([13]\) proved the equivalency of algebraic and metric lines in Hilbert space and Nitska and Wiatrowska \([12]\) proved that in Minkowski space, both more restricted than Banach space. Freese \([9]\) found a number of conditions equivalent to the monotone property in a complete, convex, externally convex metric space. He also showed that the monotone property was equivalent to unique metric lines in a Banach space. In this paper it will be shown that unique metric lines and strict convexity (redefined in purely metric terms) are equivalent in a larger class of spaces.

I

Many of the conditions mentioned above may be defined in purely metric terms and, hence, discussed in that more general setting.

It is not difficult to find examples of complete, convex, externally convex metric spaces in which the concepts of strict convexity and unique metric lines are not equivalent. Therefore the spaces of Freese's result are too general if we wish to show the equivalency of unique metric lines and strict convexity. The spaces considered here are all complete, convex, and externally convex metric spaces, however, and we will call those spaces line spaces. In a line space more than one metric line may contain two given points.