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The axiom of determinateness implies ω_2 has precisely two countably complete, uniform, weakly normal ultrafilters

by

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Abstract. A well known consequence of the axiom of determinateness is that over ω_2 there are precisely two ω_2-complete, normal ultrafilters. This can be strengthened to precisely two ω_1-complete, uniform, weakly normal ultrafilters exist over ω_2. The proof is a modification of the original proof of Martin and Paris. One application of this result shows that a theorem of ZFC, due to Ketonen, fails in ZF+AD.

§ 0. Notation and preliminaries. A familiarity with set theory is assumed. For the definitions of ultrafilter, $\varkappa$-complete, normal, fine, and other common set theoretic notions the reader is referred to [2].

If A is a set, then $\bar{\bar{A}}$ denotes the cardinality of A.

If $\varkappa \leqslant \lambda$ are cardinals, $P_\varkappa\lambda = \{a \subseteq \lambda\colon \bar{\bar{a}} < \varkappa\}$.

A filter U over $P_\varkappa\lambda$ is called *weakly normal* if given any $f\colon P_\varkappa\lambda \to \lambda$ such that $\{a \in P_\varkappa\lambda\colon f(a) \in a\} \in U$, then there exists a $\gamma < \lambda$ such that $\{a \in P_\varkappa\lambda\colon f(a) < \gamma\} \in U$. Likewise, a filter F over $\varkappa$ is called *weakly normal* if given any $f\colon \varkappa \to \varkappa$ such that $\{\alpha \in \varkappa\colon f(\alpha) < \alpha\} \in F$, then there exists $\gamma < \varkappa$ such that $\{\alpha \in \varkappa\colon f(\alpha) < \gamma\} \in F$.

A filter F over $\varkappa$ is *uniform* if $a \in F$ implies $\bar{\bar{a}} = \varkappa$.

Let $\varkappa \leqslant \mu$ be regular cardinals. F an ultrafilter over λ is $(\varkappa, \mu)$-*regular* if there exists $\{X_\alpha\colon \alpha \in \mu\} \subseteq F$ such that for every $a \subseteq \varkappa$ with $\bar{\bar{a}} = \varkappa$, then $\bigcap_{\alpha \in a} X_\alpha = \emptyset$.

Let X, Y be sets and F, U ultrafilters over X, Y respectively. U is said to be *projectible onto* F (denoted $F \leqslant U$) if there exists $f\colon Y \to X$ such that for all $a \subseteq X$, $a \in F$ if and only if $f^{-1}(a) \in U$.

In [3] the following theorem is proved:

THEOREM (Ketonen). ZFC. *Let $\varkappa \leqslant \lambda$ be regular cardinals and F a $\varkappa$-complete, uniform ultrafilter over λ. Then F is $(\varkappa, \lambda)$-regular if and only if there is a weakly normal, fine ultrafilter U over $P_\varkappa\lambda$ projectible onto F.*

Let $\varkappa$ be a regular cardinal and $\alpha < \varkappa$ an ordinal. A set $a \subseteq \varkappa$ is α-closed unbounded in $\varkappa$, if

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(i) the sup of every increasing sequence of length $\varkappa$ from a is in a; and

(ii) if $\beta<\varkappa$, then there is a $\delta\in a$ such that $\beta\leqslant\delta$.

Denote by μ_0 and μ_1 the filters over ω_2 generated by the collections of ω-closed unbounded subsets of ω_2 and ω_1-closed unbounded subsets of ω_2, respectively.

Martin and Paris proved, assuming AD, μ_0 and μ_1 are the only two ω_2-complete, normal ultrafilters over ω_2, see [4].

§ 1. THEOREM 1.1. AD. *Given any uniform, ω_1-complete, weakly normal ultrafilter ν on ω_2, then ν is μ_0 or μ_1.*

Proof. Assume otherwise. Let $A\in\nu$, $E_0\in\mu_0$ be ω-closed, and $E_1\in\mu_1$ be ω_1-closed such that $A\cap(E_0\cup E_1)=\emptyset$. Define $f\colon A\to\omega_2$ by

$$f(\alpha)=\inf\{\sup(E_0\cap\alpha),\ \sup(E_1\cap\alpha)\}\,.$$

So $f(\alpha)<\alpha$ for all $\alpha\in A$. Next, consider $f^{-1}(\gamma)$ for any $\gamma\in f''A\cap\omega_2$. If $\alpha\in f^{-1}(\gamma)$, then $\gamma=\sup(E_0\cap\alpha)$ or $\gamma=\sup(E_1\cap\alpha)$. In either case, since E_0 and E_1 are unbounded there exist $\beta_0\in E_0$ and $\beta_1\in E_1$ such that $\beta_0>\gamma$ and $\beta_1>\gamma$. For all $\alpha'\in A$ such that $\alpha'>\sup(\beta_0,\beta_1)$, $\gamma<\sup(E_0\cap\alpha')$ and $\gamma<\sup(E_1\cap\alpha')$. That is, $f(\alpha')>\gamma$. So $A\cap f^{-1}(\gamma)\subsetneq\sup(\beta_0,\beta_1)+1<\omega_2$. Hence $f^{-1}(\gamma)$ is bounded in A. But $f(\alpha)<\alpha$ for all $\alpha\in A\in\nu$. Weakly normal implies there is a $B\in\nu$, $B\subseteq A$, and $\zeta<\omega_2$ satisfying:

For all $\beta\in B$, $f(\beta)<\zeta$. So $B\subseteq\bigcup_{\gamma<\zeta}f^{-1}(\gamma)\cap A$ and $\overline{\overline{B}}=\omega_2$. But $\zeta<\omega_2$ and $\overline{\overline{f^{-1}(\gamma)\cap A}}<\omega_2$, contradicting the regularity of ω_2. ■

COROLLARY 1.2. AD. *Let F be an ω_1-complete, uniform, weakly normal ultrafilter on ω_2. F is not (ω_1,ω_2)-regular.*

Proof. By Theorem 1.1 F is either u_0 or u_1 and both are ω_2-complete. ■

THEOREM 1.3. AD. *Let U be an ω_1-complete, fine, normal ultrafilter on $P_{\omega_1}\omega_2$. U is projectible onto an ω_1-complete, uniform, weakly normal ultrafilter on ω_2.*

Proof. Define $f\colon P_{\omega_1}\omega_2\to\omega_2$ by $f(x)=\sup x$. Given $a\subseteq\omega_2$ define F on ω_2 by $a\in F$ iff $f^{-1}(a)\in U$.

CLAIM. *F is an ω_1-complete, uniform, weakly normal ultrafilter on ω_2.*

Proof of claim.

(uniformity) (i) Since U is fine and $\overline{\overline{a}}=\omega_2$ for any $a\in F$.

(ω_1-completeness) (ii) Let $\{X_\alpha\colon \alpha<\gamma\}\subseteq F$ for any $\gamma<\omega_1$. $\{f^{-1}(X_\alpha)\colon \alpha<\gamma\}\subseteq U$. And by ω_1-completeness of U, $\bigcap_{\alpha<\gamma}f^{-1}(X_\alpha)\in U$. Let $x\in\bigcap_{\alpha<\gamma}f^{-1}(X_\alpha)$. Then $x\in f^{-1}(X_\alpha)$ for all $\alpha<\gamma$, giving $f(x)\in X_\alpha$ for all $\alpha<\gamma$. That is, $f(x)\in\bigcap_{\alpha<\gamma}X_\alpha$. Hence $x\in f^{-1}(\bigcap_{\alpha<\gamma}X_\alpha)$. So $\bigcap_{\alpha<\gamma}f^{-1}(X_\alpha)\subseteq f^{-1}(\bigcap_{\alpha<\gamma}X_\alpha)$, yielding $\bigcap_{\alpha<\gamma}X_\alpha\in F$.

(weakly normal) (iii) Let $g\colon\omega_2\to\omega_2$ be such that $g(\alpha)<\alpha$ for all $\alpha\in a\in F$. So $f^{-1}(a)\in U$. Define $G\colon P_{\omega_1}\omega_2\to\omega_2$ by

$$G(x)=\begin{cases}\inf(x-g(\sup x)) & \text{for } x\in f^{-1}(a)\,,\\ 0 & \text{otherwise.}\end{cases}$$

For all $x\in f^{-1}(a)\in U$, $G(x)\in x$. Hence, there exists a $\gamma<\omega_2$ and $B\subseteq A$ with $B\in U$ satisfying: $G(x)=\gamma$ for every $x\in B$. Let $b=\{\sup x\colon x\in B\}$. Now $b\subseteq a$. And for $\beta\in b$, $g(\beta)=g(\sup x)$ for some $x\in B$. So $g(\beta)=g(\sup x)<G(x)=\gamma$. But $B\subseteq f^{-1}(b)$. Hence $b\in F$.

(ultrafilter) (iv) $a\notin F$ iff $f^{-1}(a)\notin U$

iff $P_{\omega_1}\omega_2-f^{-1}(a)\in U$

iff $\{x\in P_{\omega_1}\omega_2\colon \sup x\in\omega_2-a\}\in U$

iff $f^{-1}(\omega_2-a)\in U$. ■

COROLLARY 1.4. AD. *Let U be an ω_1-complete, fine, normal ultrafilter on $P_{\omega_1}\omega_2$. U is projectible onto μ_0 on ω_2. (The existence of such a U follows from* AD, *see* [1].)

Proof. Theorem 1.3, Theorem 1.1 and the facts:

$$\{\alpha<\omega_2\colon \mathrm{cf}(\alpha)=\omega\}\in\mu_0\,,$$
$$\{\alpha<\omega_2\colon \mathrm{cf}(\alpha)=\omega_1\}\in\mu_1\,,$$

and U is normal; imply

$$\{x\in P_{\omega_1}\omega_2\colon \sup x \text{ is a limit ordinal}\}\in U\,.\ ■$$

COROLLARY 1.5. AD. *Any normal (hence weakly normal), fine, ω_1-complete ultrafilter over $P_{\omega_1}\omega_2$ is projectible onto an ω_1-complete, uniform ultrafilter over ω_2 which is not (ω_1,ω_2)-regular.*

Proof. By Corollaries 1.4 and 1.2. ■

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