A dense subsemigroup of $S(R)$ generated by two elements

by

S. Subbiah (Amherst, N.Y.)

Abstract. $S(R)$ denotes the topological semigroup of all continuous selfmaps of the real numbers where the binary operation is composition and the topology is the compact-open topology. As the title of the paper indicates, we prove that $S(R)$ contains a dense subsemigroup which is generated by two elements. It is still an open problem to determine whether or not the same is true for $S(R^N)$ for $N>1$ where R^N denotes the Euclidean N-Space.

1. Introduction. $S(X)$ is the semigroup of all continuous selfmaps of the Hausdorff space X. When X is locally compact, $S(X)$ is also a topological semigroup in the compact-open topology. It has been known for some time that various $S(X)$ contain dense subsemigroups which are finitely generated. Schreier and Ulam in 1934 [5] showed that $S(I^N)$ contains a dense subsemigroup generated by five elements, where I^N is the Euclidean N-cell. The same year Sierpiński [6] showed that $S(I)$ contains a dense subsemigroup generated by four elements. In 1935, Jarnik and Knichal [2] produced dense two-generator subsemigroups of $S(I)$. Much later, Cook and Ingram [1] produced a class of spaces containing all the Euclidean N-cells such that for each X in that class, $S(X)$ contains dense subsemigroups generated by two elements. A few years later, the present author independently rediscovered the same result [7]. The number of generators cannot be reduced any further unless X is a singleton. In [7], the present author also proved that $S(R^N)$ contains dense subsemigroups generated by three elements where R^N is the Euclidean N-space. In other words, there exist three functions in $S(R^N)$ so that one can approximate any function in $S(R^N)$, to any desired degree of accuracy, on any compact subset of R^N, by composing only these three functions. For some time we felt that two functions would not suffice and, in fact, this has been conjectured in [4]. However it turns out that the conjecture is false at least in the case $N = 1$. In this paper we prove the following

THEOREM. *$S(R)$ contains a dense subsemigroup which is generated by two elements.*

As we mentioned earlier, dense two-generator subsemigroups of $S(I)$ have been produced before. But, so far, one of the generators has always been injective but not surjective and the other surjective but not injective. In this paper we produce a dense subsemigroup of $S(I)$ generated by two elements, one of the generators is a homeomorphism of I onto I, the other is neither injective nor surjective.

The compact-open topology is the same as the topology of uniform convergence for $S(I)$. For $S(R)$, it is the topology of uniform convergence on compact subsets of R.

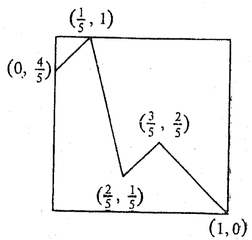
2. The semigroup S_0 of all continuous onto selfmaps of I. In 1938, Knichal [3] proved that there are two increasing homeomorphisms θ_1, θ_2 of I which generate a dense subsemigroup of the group of all increasing homeomorphisms of I. In 1970, Young [8] proved the following

THEOREM (2.1) (Young). *There exists an increasing homeomorphism φ of I such that φ and j generate a dense subsemigroup of the homeomorphism group of I, where $t(x) = 1-x$ for $0 \leqslant x \leqslant 1$.*

In the same paper, Young produced a dense subsemigroup of $S_0(I)$ generated by two elements.

DEFINITION (2.2) (Young). The function f is defined as follows:

$$f(x) = \begin{cases} \frac{4}{5}+x & \text{in } [0, \frac{1}{5}] \\ \frac{9}{5}-4x & \text{in } [\frac{1}{5}, \frac{2}{5}] \\ x-\frac{1}{5} & \text{in } [\frac{2}{5}, \frac{3}{5}] \\ 1-x & \text{in } [\frac{3}{5}, 1] \end{cases} =$$

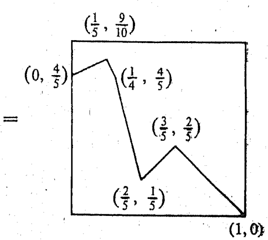


THEOREM (2.3) (Young). *The functions f and φ generate a dense subsemigroup of $S_0(I)$.*

Let us denote the interval $(0, 1)$ by J. Now $f|_J$ is not in $S(J)$. We modify f to a function g such that $g|_J \in S(J)$ since eventually we want to study $S(J)$. This function g, together with φ will still be able to approximate all the elements of $S_0(I)$.

DEFINITION (2.4). The function g is defined as follows:

$$g(x) = \begin{cases} \frac{1}{2}x+\frac{4}{5} & \text{in } [0, \frac{1}{5}] \\ -2x+\frac{13}{10} & \text{in } [\frac{1}{5}, \frac{1}{4}] \\ f(x) & \text{in } [\frac{1}{4}, 1] \end{cases} =$$



LEMMA (2.5). *The homeomorphism j can be approximated by finite compositions of φ and g.*

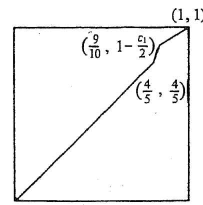
Proof. The proof of Lemma (3.5) in [8] goes through with f replaced by g. We invoke Theorem (2.1) and obtain the

COROLLARY (2.6). *Every homeomorphism of I can be approximated by φ and g.*

PROPOSITION (2.7). *φ and g generate a subsemigroup of $S(I)$ whose closure contains $S_0(I)$.*

Proof. We let $\langle\varphi, g\rangle$ denote the semigroup generated by φ and g. Given $\varepsilon > 0$, we define h as follows

$$h(x) = \begin{cases} x & \text{in } [0, \frac{4}{5}] \\ \text{linear} & \text{in } [\frac{4}{5}, \frac{9}{10}] \\ 1-\frac{1}{2}\varepsilon_1 & \text{for } x = \frac{9}{10} \\ \text{linear} & \text{in } [\frac{9}{10}, 1] \\ 1 & \text{for } x = 1 \end{cases} =$$



where $\varepsilon_1 = \min[\varepsilon, \frac{1}{5}]$. Then h is a homeomorphism of I and so $h \in \overline{\langle\varphi, g\rangle}$ by Corollary (2.6). It is easily verified that

$$|(hg)(x)-f(x)| \leqslant \tfrac{1}{2}\varepsilon_1 < \varepsilon \quad \text{for all } x \text{ in } I.$$

Hence $f \in \overline{\langle\varphi, g\rangle}$.

Since f and φ generate a dense subsemigroup of $S_0(I)$, we see that $S_0(I) \subset \overline{\langle\varphi, g\rangle}$.

We now consider the continuous selfmaps of J. We let $S_0(J)$ be the semigroup of continuous onto selfmaps of J. The compact-open topology for $S(J)$ and $S_0(J)$ coincides with the topology of uniform convergence on compact subsets of J.

3. A subsemigroup of $S_0(J)$ which is dense in $S(J)$. Let

$$S_0(J) = \{l \in S(J)\colon \exists [a, b] \subset J \text{ such that } l(x) = x \text{ for all } x \notin [a, b]\}.$$

LEMMA (3.1). *$S_1(J)$ is dense in $S(J)$.*

Proof. Let $u \in S(J)$. Given any $\varepsilon > 0$ and any compact subset K of J first choose c, d in J such that $K \subset [c, d]$. Define l as follows:

$$l(x) = \begin{cases} x & \text{in } (0, \frac{1}{2}c) \\ \text{linear} & \text{in } [\frac{1}{2}c, c] \\ u(x) & \text{in } [c, d] \\ \text{linear} & \text{in } \left[d, \dfrac{d+1}{2}\right] \\ x & \text{in } \left[\dfrac{d+1}{2}, 1\right] \end{cases}$$

Clearly $l \in S_1(J)$ and $|u(x)-l(x)| = 0 < \varepsilon$ for all x in $[c, d]$ and hence for all x in K.

This shows that $S_1(J)$ is dense in $S(J)$.

4. A dense subsemigroup of $S(J)$ with two generators. Let $\hat{g} = g|_J$ and $\hat{\varphi} = \varphi|_J$. Then both $\hat{\varphi}$ and $\hat{g}$ are in $S(J)$. We proceed to show that $\hat{\varphi}$ and $\hat{g}$ generate a dense subsemigroup of $S(J)$.

LEMMA (4.1). *The semigroup generated by $\hat{\varphi}$ and $\hat{g}$ is dense in a subsemigroup of $S(J)$ containing $S_1(J)$.*

Proof. Let $l \in S_1(J)$. Given any compact subset K of J and any $\varepsilon > 0$ we define

$$k(x) = \begin{cases} 0 & \text{at } 0, \\ l(x) & \text{in } (0, 1), \\ 1 & \text{at } 1 \end{cases}$$

so that $k \in S_0(I)$ and $k|_J \in S_1(J)$. Proposition (2.7) guarantees that there is a sequence $f_1, f_2, \ldots, f_n$ such that

$$|l(x) - (f_1 f_2 \ldots f_n)(x)| < \varepsilon \quad \text{for all } x \text{ in } I$$

where each $f_i \in \{g, \varphi\}$ and so

$$|l(x) - (\hat{f}_1 \hat{f}_2 \ldots \hat{f}_n)(x)| < \varepsilon \quad \text{for all } x \text{ in } K$$

where $\hat{f}_i = f_i|_J \in \{\hat{\varphi}, \hat{g}\}$ for each i. This proves the lemma.

Since $S_1(J)$ is dense in $S(J)$ we have shown that $\{\hat{\varphi}, \hat{g}\}$ generate a dense subsemigroup of $S(J)$. This completes the proof of our main theorem since J is homeomorphic to R.

5. A dense subsemigroup of $S(I)$ generated by two functions one of which is a homeomorphism of I. In Section 2 we showed that $S_0(I) \subset \overline{\langle \varphi, g \rangle}$. Actually we have the following

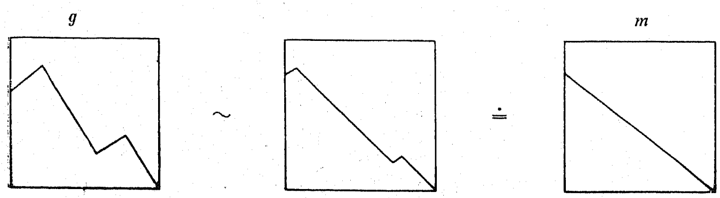
THEOREM (5.1). *The homeomorphism φ and the function g generate a dense subsemigroup of $S(I)$.*

Proof. Let $D = \overline{\langle \varphi, g \rangle}$. We use the following notation:

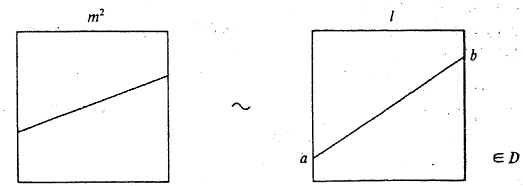
$\theta_1 \sim \theta_2$ iff $\theta_2 = h_1 \theta_1 h_2$ for some homeomorphism h_1, h_2 of I.

$\theta_1 \doteq \theta_2$ iff θ_1 approximates θ_2.

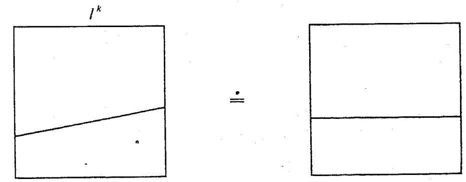
In either case $\theta_1 \in D$ implies $\theta_2 \in D$. We have



and so $m \in D$. This implies $m^2 \in D$ and so

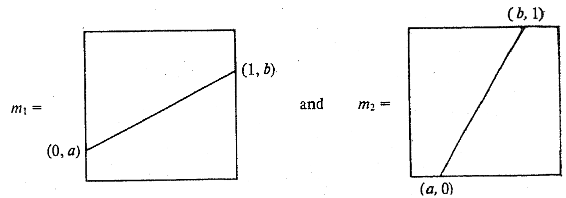


for any $[a, b] \subset I$. Moreover, for sufficiently large k we have



a constant function. Hence D contains all the constant functions also. We look at the non-constant functions.

Let $\theta \in S(I)$ where $\operatorname{Ran}\theta = [a, b]$, $0 \leqslant a < b \leqslant 1$. Then $\theta = m_1 m_2 \theta$ where



We know $m_1 \in D$. Now $m_2\theta \in S_0(I) \subset D$. Hence $\theta \in D$. We have shown that φ and g generate a dense subsemigroup of $S(I)$.

6. Conclusion. In [7] we showed that $S(R^N)$ has dense subsemigroups generated by three elements. We have now shown that in the special case when $N = 1$ the number of generators needed can be reduced to two. It is still an open problem whether or not the number of generators needed can be reduced from three to two when $N > 1$.

References

[1] H. Cook and W. T. Ingram, *Obtaining AR-like continua as inverse limits with only two bonding maps*, Glasnik Matematicki, Tom 4, 24 (2), (1969), pp. 309–311.

[2] V. Jarnik and V. Knichal, *Sur l'approximation des fonctions continues par les superpositions de deux fonctions*, Fund. Math. 24 (1935), pp. 206–208.

[3] V. Knichal, *Sur les superpositions des automorphies continues d'un intervalle fermé*, Fund. Math. 31 (1938), pp. 79–83.

[4] K. D. Magill, Jr., *A survey of semigroups of continuous selfmaps*, Semigroup Forum 11 (3) (1975/76), pp. 189–282.

[5] J. Schreier and S. Ulam, *Über topologische Abbildungen der euklidischen Sphären*, Fund. Math. 23 (1934), pp. 102–118.

[6] W. Sierpiński, *Sur l'approximation des fonctions continues par les superpositions de quatre fonctions*, Fund. Math. 23 (1934), pp. 119–120.

[7] S. Subbiah, *Some finitely generated subsemigroups of $S(X)$*, Fund. Math. 86 (1975), pp. 221–231.

[8] S. W. Young, *Finitely generated semigroups of continuous functions on* [0, 1], Fund. Math. 68 (1970), pp. 297–305.

DAEMEN COLLEGE
Amherst, N.Y.

Accepté par la Rédaction le 29. 9. 1980

The axiom of determinateness implies ω_2 has precisely two countably complete, uniform, weakly normal ultrafilters

by

Robert J. Mignone * (Charleston, S. C.)

Abstract. A well known consequence of the axiom of determinateness is that over ω_2 there are precisely two ω_2-complete, normal ultrafilters. This can be strengthened to precisely two ω_1-complete, uniform, weakly normal ultrafilters exist over ω_2. The proof is a modification of the original proof of Martin and Paris. One application of this result shows that a theorem of ZFC, due to Ketonen, fails in ZF+AD.

§ 0. Notation and preliminaries. A familiarity with set theory is assumed. For the definitions of ultrafilter, $\varkappa$-complete, normal, fine, and other common set theoretic notions the reader is referred to [2].

If A is a set, then $\bar{\bar{A}}$ denotes the cardinality of A.

If $\varkappa \leqslant \lambda$ are cardinals, $P_\varkappa \lambda = \{a \subseteq \lambda \colon \bar{\bar{a}} < \varkappa\}$.

A filter U over $P_\varkappa\lambda$ is called *weakly normal* if given any $f\colon P_\varkappa\lambda \to \lambda$ such that $\{a \in P_\varkappa\lambda\colon f(a) \in a\} \in U$, then there exists a $\gamma < \lambda$ such that $\{a \in P_\varkappa\lambda\colon f(a) < \gamma\} \in U$. Likewise, a filter F over $\varkappa$ is called *weakly normal* if given any $f\colon \varkappa \to \varkappa$ such that $\{\alpha \in \varkappa\colon f(\alpha) < \alpha\} \in F$, then there exists $\gamma < \varkappa$ such that $\{\alpha \in \varkappa\colon f(\alpha) < \gamma\} \in F$.

A filter F over $\varkappa$ is *uniform* if $a \in F$ implies $\bar{\bar{a}} = \varkappa$.

Let $\varkappa \leqslant \mu$ be regular cardinals. F an ultrafilter over λ is $(\varkappa, \mu)$-*regular* if there exists $\{X_\alpha\colon \alpha \in \mu\} \subseteq F$ such that for every $a \subseteq \varkappa$ with $\bar{\bar{a}} = \varkappa$, then $\bigcap_{\alpha \in a} X_\alpha = \emptyset$.

Let X, Y be sets and F, U ultrafilters over X, Y respectively. U is said to be *projectible onto* F (denoted $F \leqslant U$) if there exists $f\colon Y \to X$ such that for all $a \subseteq X$, $a \in F$ if and only if $f^{-1}(a) \in U$.

In [3] the following theorem is proved:

THEOREM (Ketonen). ZFC. *Let $\varkappa \leqslant \lambda$ be regular cardinals and F a $\varkappa$-complete, uniform ultrafilter over λ. Then F is $(\varkappa, \lambda)$-regular if and only if there is a weakly normal, fine ultrafilter U over $P_\varkappa\lambda$ projectible onto F.*

Let $\varkappa$ be a regular cardinal and $\alpha < \varkappa$ an ordinal. A set $a \subseteq \varkappa$ is α-closed unbounded in $\varkappa$, if

* This paper represents Chapter One of my Ph. D. Thesis (The Pennsylvania State University, 1979) written under the supervision of Professor Thomas Jech, to whom I am grateful for his guidance and knowledge of set theory.