A characterization of middle graphs and a matroid associated with middle graphs of hypergraphs

by

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Abstract. The characterization of a middle graph of a graph is given by Akiyama, Hamada and Yoshimura [1]. Some other properties of middle graphs are presented in [2] and [3]. In a similar way we introduce a middle graph of a hypergraph and we give a characterization of those graphs. With any middle graph G we associate a matroid $\mathcal{M}_G$ and we prove that it is graphic.

1. Definitions and notation. Let X be a finite set and let $\mathcal{E} = \{E_i\colon i \in I\}$ be a family of subsets of X. The pair $H = (X, \mathcal{E})$ is called a *hypergraph*, which will be denoted as the pair $H = \big(V(H), E(H)\big)$. The hypergraph is said to be *simple* if the edges E_i are all distinct. If $|E_i| \leqslant 2$ for all $i \in I$, then H is a *multigraph*. Now, if $|E_i| = 2$ for $i \in I$ and H is simple, then H is a *graph*.

We define the middle graph of the hypergraph $H = (X, \mathcal{E})$, denoted by $M(H)$, as an intersection graph $\Omega(F)$, where

$$F = X' \cup \mathcal{E}, \quad X = \{x_1, \ldots, x_n\}, \quad X' = \{\{x_1\}, \ldots, \{x_n\}\}.$$

A graph G is called a *middle graph* if it is isomorphic to the middle graph $M(H)$ of a hypergraph H.

If H is a hypergraph and $x \in V(H)$, then let us denote by $N(x)$ and $N[x]$ the open and the closed neighbourhood of the vertex x in the hypergraph H, respectively, i.e. $x' \in N(x)$ if and only if $x' \neq x$ and there exists an edge E of H such that $\{x', x\} \subset E$. Obviously, $N[x] = N(x) \cup \{x\}$.

Let G be a graph. The set $\{C_i\colon i = 1, \ldots, m\}$ of the cliques of G is defined as a C-cover of G, if $\bigcup_{i=1}^{m} V(C_i) = V(G)$ and $\bigcup_{i=1}^{m} E(C_i) = E(G)$.

If in the graph G there exists a stable set S such that the collection (set) $\{\langle N[x]\rangle\colon x \in S\}$ is a C-cover of G, then the set S is called *C-stable*, where $\langle A\rangle$ denotes a subgraph of G induced by $A \subset V(G)$.

A matroid $\mathcal{M}$ is a pair $(Q, \boldsymbol{B})$ where Q is a non-empty finite set and $\boldsymbol{B}$ is a non-empty collection of subsets of Q (called bases) satisfying the following properties:

(B1) no base properly contains another base, (B2) if B_1 and B_2 are bases and if q is any element of B_1, then there is an element q' of B_2 with the property that $(B_1 \setminus \{q\}) \cup \{q'\}$ is also a base.

Throughout, the terminology of Wilson [4] is used.

2. Theorems.

THEOREM 1. *A graph G is a middle graph if and only if there exists a maximal stable set $S = \{x_1, \ldots, x_k\} \subset V(G)$ such that the collection $\{\langle N[x_i]\rangle: i = 1, \ldots, k\}$ is a C-cover of G.*

Proof. Let us assume that G is a middle graph of a hypergraph H. Now, we consider the set $S = \{\{x_1\}, \ldots, \{x_n\}\}$ and the collection

$$\{\langle N[\{x_i\}]\rangle: i = 1, \ldots, n\}.$$

From the definition of the middle graph of the hypergraph H, the set S is stable and it is maximal. Moreover, any two elements of $N(\{x_i\})$ have a non-empty intersection; therefore $\langle N(\{x_i\})\rangle$ is a clique of G for all $i = 1, \ldots, n$. Obviously, $\langle N[\{x_i\}]\rangle$ is also a clique of G, and the collection $\{\langle N[\{x_i\}]\rangle: i = 1, \ldots, n\}$ is a C-cover of G.

Now, assume that the collection $\{\langle N[x_i]\rangle: i = 1, \ldots, k\}$ is a C-cover of G and $S = \{x_1, \ldots, x_k\}$ is a maximal stable set of G. A hypergraph whose middle graph is isomorphic to G may be obtained in the following way:

Let $V(H) = S$ and let $V(G) \setminus S = \{e_1, \ldots, e_m\}$. We denote the family of edges of our hypergraph H by $\{E_i: i = 1, \ldots, m\}$, where $E_i = \{x_j: x_j \in S \text{ and } e_i \in N[x_j]\}$, for $j = 1, \ldots, k$ and $i = 1, \ldots, m$. It is easy to see that $M(H) \simeq G$, and the proof is complete.

Let G be a graph and let $\mathbf{B}_G$ be the collection

$$\{B: B \subset V(G) \text{ and } B \text{ is a } C\text{-stable set of } G\}.$$

EXAMPLES 1. Let $G = K_n$, $V(K_n) = \{x_1, \ldots, x_n\}$, then $\mathbf{B}_G = \{\{x_i\}: i = 1, \ldots, n\}$.

2. If $G = K_{1,n}$, $V(G) = \{y, x_1, \ldots, x_n\}$, then $\mathbf{B}_G = \{\{x_1, \ldots, x_n\}\}$, $n \geq 2$.

3. If $G = P_n$, $V(P_n) = \{x_1, \ldots, x_n\}$, $n \geq 4$, then $\mathbf{B}_G = \emptyset$.

THEOREM 2. *Suppose that $\mathbf{B}_G \neq \emptyset$. Then the pair $\mathcal{M}_G = (V(G), \mathbf{B}_G)$ is a matroid.*

Proof. Let G be a middle graph. We wish to prove properties (B1), (B2). Clearly (B1) is trivial. To prove (B2), we let $B_1, B_2 \in \mathbf{B}_G$ and $q \in B_1$. If $q \in B_1 \cap B_2$, then $q' = q$ and (B2) is true. Suppose that $q \in B_1 \setminus B_2$. Obviously, $B_2 \setminus B_1$ is not empty. Since B_1 is C-stable, we have $N(q) \cap (B_2 \setminus B_1) \neq \emptyset$ for every $q \in B_1 \setminus B_2$.

Moreover, $|N(q) \cap (B_2 \setminus B_1)| = 1$. If it were not so, the induced subgraph $\langle N[q]\rangle$ would not be a clique and $B_1 \notin \mathbf{B}_G$, in contradiction with the assumption. Let $N(q) \cap (B_2 \setminus B_1) = \{q'\}$. In a similar way, we obtain $N(q') \cap (B_1 \setminus B_2) = \{q\}$ for $q' \in B_2 \setminus B_1$. Hence, there exists a bijection $f: (B_1 \setminus B_2) \to (B_2 \setminus B_1)$ such that $(B_1 \setminus \{q\}) \cup \{f(q)\}$ is C-stable, i.e. it is an element of $\mathbf{B}_G$. Thus $(V(G), \mathbf{B}_G)$ is a matroid.

From the above and from the properties of matroids it is easy to verify the facts described in the next theorem. Other properties of the middle graphs, including the algorithm to verify whether a given graph G is a middle graph or not, are presented in paper [2].

THEOREM 3. *If G is a middle graph and $\mathcal{M}_G$ is its matroid, then:*

(a) The rank $r(\mathcal{M}_G)$ of $\mathcal{M}_G$ is equal to the stability number $\beta(G)$ of G.

(b) If S is a stable set and $|S| = \beta(G)$, then $S \in \mathbf{B}_G$.

(c) The hypergraph H is uniquely determined up to an isomorphism by its middle graph $M(H)$.

It is a reasonable question to ask whether a given matroid $\mathcal{M}_G$ is the circuit matroid of some multigraph; in other words: whether there exists a multigraph G' such that $\mathcal{M}_G$ is isomorphic to the circuit matroid $\mathcal{M}(G')$. The answer to this question is obtained in the next theorem. Moreover, we give the construction of such multigraphs.

Suppose we are given the middle graph $G = M(H)$ of a hypergraph H and the matroid $\mathcal{M}_G = (V(G), \mathbf{B}_G)$ with rank function r, and let $A = \bigcup_{B \in \mathbf{B}_G} B$.

Obviously, $A \subset V(G)$ and A does not contain the loops of $\mathcal{M}_G$. Note that the set A contains only those elements of G which correspond to the vertices and loops of H; if it were not so, the collection $\mathbf{B}_G$ would not satisfy axiom (B2). These facts imply that the matroid $\mathcal{M}_G$ does not have a circuit (a minimal dependent set) of size greater than two. We define on the set A a relation R in the following way:

$$xRy \text{ if and only if } r(\{x, y\}) = 1. \tag{1}$$

Note that x and y form a pair of parallel elements of $\mathcal{M}_G$.

Above considerations give the following

LEMMA. *The relation R defined above is the equivalence relation on the set A. The matroid $\mathcal{M}_G$ does not contain circuits of size greater than two.*

THEOREM 4. *Suppose we are given a matroid $\mathcal{M}_G = (V(G), \mathbf{B}_G)$. Then there exists a connected multigraph G' such that $\mathcal{M}(G') \simeq \mathcal{M}_G$.*

Proof. Let $A = \bigcup_{B \in \mathbf{B}_G} B$ and let R be the relation defined by (1). Let us denote by

$$A/R = \{A_1, \ldots, A_k\}$$

the factor set of A with respect to R.

Now, with every set A_i let us associate a multigraph G_i with two vertices and $|A_i|$ parallel edges joining these vertices, and let H_1 be a multigraph with one vertex and $|V(G) \setminus A|$ loops. By the above and by the lemma it is easy to see that the circuit matroid of the multigraph

$$G' = (\bigoplus_{i=1}^{k} G_i) \oplus H_1,$$

where the operation $\oplus$ is a direct sum operation (i.e. it is a multigraph obtained by the coalescence of a vertex of G_1 with a vertex of G_2 and then of a vertex of $G_1 \oplus G_2$ with a vertex of G_3 and so on) satisfies the required isomorphism.

Note that the size of the collection $\mathbf{B}_G$ is equal to $\prod_{i=1}^{k} |A_i|$.

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Powers of spaces of non-stationary ultrafilters

by

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Abstract. Let X denote the space of all non-stationary ultrafilters on a regular uncountable cardinal $\varkappa$ (or more generally, the space associated with a normal ideal on $\varkappa$). These spaces were recently introduced by Eric van Douwen, who showed that X is strongly $\varkappa$-compact but not $\varkappa$-bounded. We show in this paper that X^{ω} is not strongly $\varkappa$-compact, $X^{2^{\varkappa}}$ is not totally initially $\varkappa$-compact and X^{μ} (assuming GCH) is initially $\varkappa$-compact for all cardinals μ. These results answer two basic questions concerning these compactness-like properties.

1. Introduction. The theory of products of countably compact and related spaces is extensive, but the generalization of this theory to higher cardinals is not as well developed. There are some very basic questions which have been answered in the countable case but not in the uncountable case. Two of these questions are concerned with the notions of strong $\varkappa$-compactness and TI-$\varkappa$-compactness.

A space X is said to be *strongly $\varkappa$-compact* provided that for every filter base $\mathscr{F}$ on X of cardinality $\leqslant\varkappa$, there exists a compact set $K\subset X$ such that $F\cap K\neq\emptyset$ for all F in $\mathscr{F}$. A T_3-space X is TI-*$\varkappa$-compact* provided that for every filter base $\mathscr{F}$ on X of cardinality $\leqslant\varkappa$, there exist a compact set $K\subset X$ and a filter base $\mathscr{G}$ of cardinality $\leqslant\varkappa$ such that $\mathscr{G}$ is finer than $\mathscr{F}$ (i.e., every member of $\mathscr{F}$ contains a member of $\mathscr{G}$) and $\mathscr{G}$ converges to K in the sense that every open set containing K also contains a member of $\mathscr{G}$ (see § 2 for the definition of TI-$\varkappa$-compactness in general spaces and for all other definitions).

Clearly, every strongly $\varkappa$-compact space is TI-$\varkappa$-compact (take $\mathscr{G}$ to be $\{F\cap K\colon F\in\mathscr{F}\}$), and the converse is true if $\varkappa=\omega$ (in the class of T_3-spaces). The simple proof of the equivalence of these two properties for the case $\varkappa=\omega$ does not extend to higher cardinals; so we have the basic question:

1.1. For $\varkappa>\omega$, is every TI-$\varkappa$-compact space strongly $\varkappa$-compact?

An important property of the class of TI-$\varkappa$-compact spaces is that it is stable under $\varkappa$-fold products (i.e., every product of $\leqslant\varkappa$ TI-$\varkappa$-compact spaces is TI-$\varkappa$-compact). The proof of this does not extend to strong $\varkappa$-compactness; so we have a second basic question:

1.2. For $\varkappa>\omega$, is every product of no more than $\varkappa$ strongly $\varkappa$-compact spaces, strongly $\varkappa$-compact?