Validity up to complementation in graph theory

by

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Abstract. A graph-theoretic property is called "semivalid" exactly when, for each graph G, the property must hold either in G or in its complement $\bar{G}$. Examples include connectivity and the "diagonal" cases of Ramsey's theorem. A formal logical characterization is given, from which further features of semivalidity are deduced. We also consider a very restricted subclass of the semivalid properties, which is broad enough to include the various examples which prompted this investigation, and which is also deep enough that the semivalidity of this restricted subclass insures the same of all other semivalid properties.

1. The notion of semivalidity. In a sense, the fundamental activity in graph theory is determining those properties which are *valid*; that is, which are true of all graphs. Also, there is a simple yet intimate relationship between each graph and its complement—indeed it is sometimes rather arbitrary which is "the graph" and which is "the complement." With this in mind, we propose the study of the "semivalidity" of graph-theoretic statements (or properties expressed as statements), where a statement is said to be *semivalid* if and only if, for each graph G, the statement is necessarily true in either G or its complement $\bar{G}$. (Our graph-theoretic notation and terminology follows that of [2].) Notice that we cannot require truth in exactly one of G and $\bar{G}$, since the existence of selfcomplementary graphs rules out the possibility of being true in only one of each complementary pair.

Probably the best known example of semivalidity [2, page 26] is connectivity: Every graph either is connected or has a connected complement. Near examples are provided by any of the "diagonal" instances of Ramsey's theorem [2, page 284]. For instance, "containing a triangle" must be true of each graph or its complement, if only graphs of order at least six are considered. This leads to the somewhat awkward semivalid statement "has order less than six or contains a triangle." (We could call containment of a triangle "eventually semivalid.") A similar example is the semivalidity of "has order less than nine or is nonplanar" [1]. A recent and sophisticated example [5] is "has order less than six or is both connected and has a pancyclic line graph."

In this section we shall discuss several properties of semivalidity which are rather independent of our choice of a particular formal language. However, to make our arguments specific, we shall introduce a certain language $\mathscr{L}$ for graph theory and shall use only simple logical manipulations with it such as are found in any elementary text on mathematical logic (for instance, [4]). Although many simple concepts (connectivity, for one) cannot be directly expressed in $\mathscr{L}$, they can be expressed in various extensions of $\mathscr{L}$ (for instance, by allowing disjunction and conjunction over countable sets of formulas), and the results of this section could be very naturally modified for these extended but somewhat more involved languages.

The variables of $\mathscr{L}$ are x, y, and z (possibly primed or subscripted) and will be interpreted as vertices. The atomic formulas are of the forms $x = y$, $x \neq y$, xy, and $\overline{xy}$, interpreted respectively as equality, distinctness, adjacency and non-adjacency of the vertices x and y. The formulas of $\mathscr{L}$ are built from the atomic formulas using the usual logical connectives $\neg$, &, and $\vee$ (for negation, conjunction and disjunction) and the universal and existential quantifiers $\forall$ and $\exists$. Formulas having no free (= unquantified) occurrences of variables are called *sentences* of $\mathscr{L}$.

For any formula φ of $\mathscr{L}$, define $\bar{\varphi}$ to be obtained from φ by replacing all occurrences of adjacencies by the corresponding nonadjacencies, and vice versa. Also, φ^* is obtained from φ by interchanging all occurrences of & and $\vee$, of $\forall$ and $\exists$, and of $=$ and $\neq$. Note that φ^* is not quite the negation of φ since (non) adjacencies are left untouched. But applying both the $^-$ and * transformations to φ produces precisely $\neg\varphi$. In fact it is easy to see that the three transformations $^-$, *, and $\neg$ constitute the nonidentity elements of a Klein four group under composition. This is the content of the following lemma, in which "equivalent" means "are true in exactly the same graphs."

LEMMA 1.1. *For each formula φ of $\mathscr{L}$, $\bar{\varphi}^*$, $\neg\varphi^*$, and $\neg\bar{\varphi}$ are equivalent to, respectively, $\neg\varphi$, $\bar{\varphi}$ and φ^*.*

It is now convenient to define an additional connective for $\mathscr{L}$: the conditional symbol $\Rightarrow$, where $\varphi \Rightarrow \psi$ is defined to be $(\neg\varphi) \vee \psi$ (or, equivalently, $\psi \vee (\neg\varphi)$). (Note that this definition must be used to eliminate all occurrences of $\Rightarrow$ from a formula before the $*$ transformation is applied.) One final piece of logical notation is useful: for each sentence σ, $\models \sigma$ will mean that σ is true of all graphs (that is, σ is valid). Note that σ is true in a graph G exactly when $\bar{\sigma}$ is true in $\bar{G}$; also that $\models \sigma$ if and only if $\models \bar{\sigma}$.

THEOREM 1.2. *A sentence σ of $\mathscr{L}$ is semivalid if and only if $\models \sigma^* \Rightarrow \sigma$.*

Proof. $\sigma^* \Rightarrow \sigma$ means $\neg\sigma^* \vee \sigma$ and so is equivalent to $\bar{\sigma} \vee \sigma$ by Lemma 1.1. It is immediate that σ is semivalid if and only if $\models \bar{\sigma} \vee \sigma$.

COROLLARY 1.3. *Suppose σ and τ are sentences of $\mathscr{L}$. If σ is semivalid and $\tau^* \Rightarrow \tau$ is true in all graphs which satisfy σ, then τ is semivalid.*

Proof. Note that since $\tau^* \Rightarrow \tau$ is true in all graphs satisfying σ, then $\bar{\tau}^* \Rightarrow \bar{\tau}$ will be true in all graphs satisfying $\bar{\sigma}$. But $\bar{\tau}^* \Rightarrow \bar{\tau}$ means $\neg\bar{\tau}^* \vee \bar{\tau}$ and so is equivalent by Lemma 1.1 to $\tau \vee \neg\tau^*$ and so finally to $\tau^* \Rightarrow \tau$. Thus we have that $\tau^* \Rightarrow \tau$ is true in all graphs satisfying σ or $\bar{\sigma}$, and so (by σ's semivalidity) in all graphs, as required in Theorem 1.2.

Corollary 1.3 asserts that to prove a sentence τ semivalid, one need prove $\tau^* \Rightarrow \tau$ only for connected graphs, or only for graphs which are nonplanar (or of order less than nine), or only for graphs satisfying any other semivalid sentence. The next theorem shows we can also prove a sentence semivalid by showing that it follows from any other sentence previously known to be semivalid.

THEOREM 1.4. *The set of semivalid sentences is closed under consequence; that is, if σ is semivalid and if $\models \sigma \Rightarrow \tau$, then τ is semivalid.*

Proof. $\models \sigma \Rightarrow \tau$ means $\models \neg\sigma \vee \tau$, and so $\models \neg\bar{\sigma} \vee \bar{\tau}$, which by Lemma 1.1 is $\models \sigma^* \vee \neg\tau^*$ and so $\models \tau^* \Rightarrow \sigma^*$. But Theorem 1.2 gives $\models \sigma^* \Rightarrow \sigma$, and by assumption we have $\models \sigma \Rightarrow \tau$. Transitivity of $\Rightarrow$ shows that $\models \tau^* \Rightarrow \tau$, as required for Theorem 1.2.

Theorem 1.4 suggests searching for stronger and stronger semivalid sentences, so that proving the semivalidity of one such sentence will insure that of all its consequences. For instance the usual proof of connectivity's semivalidity [2, p. 26] really proves the semivalidity of the stronger property of having diameter at most three. In the next section we shall consider this example in more detail as we examine a very restricted family of "special" semivalid sentences whose consequences include all other semivalid sentences.

We close this section with a comment about selfcomplementary graphs: Not only must they satisfy each semivalid sentence, but the converse holds as well.

THEOREM 1.5. *A graph is selfcomplementary if and only if it satisfies all semivalid sentences.*

Proof. Suppose rather, towards a contradiction, there were a nonselfcomplementary graph G satisfying all semivalid sentences. Since G is finite, there would be a sentence σ of $\mathscr{L}$ characterizing G up to isomorphism—that is, asserting the existence of the proper number of distinct vertices (with each vertex equal to one of them) with the proper pairs adjacent and the rest nonadjacent. Since G was chosen to be nonselfcomplementary, $\neg\sigma$ would be semivalid. But we would then have G satisfying $\neg\sigma$, contradicting the choice of σ.

Selfcomplementary graphs also satisfy certain nonsemivalid sentences; conjunctions of semivalid sentences, for instance. In fact for (any "first-order" extension of) $\mathscr{L}$, the Compactness Theorem [4, p. 312] of elementary model theory can be used to show that the theorems of the theory of selfcomplementary graphs consist precisely of all conjunctions of semivalid sentences.

2. Special sentences and semivalidity. We now consider a family of sentences which contains many natural examples of semivalid sentences and which can be shown to be intimately related to the totality of semivalid sentences. We shall call a sentence of $\mathscr{L}$ *special* if it is equivalent to one of the form

$$(*) \qquad (\forall x_1)(\exists y_1) \ldots (\forall x_n)(\exists y_n)\varphi(x_1 y_1 \ldots x_n y_n)$$

where $\varphi(x_1 y_1 \ldots x_n y_n)$ is a formula of $\mathscr{L}$ all of whose free variables are included among $x_1, y_1, \ldots, x_n$ and y_n such that $\models \varphi^*(y_1 x_1 \ldots y_n x_n) \Rightarrow \varphi(x_1 y_1 \ldots x_n y_n)$, where this last assumption means that the conditional formula $\varphi^* \Rightarrow \varphi$ is true for all assignments of vertices to the variables in all graphs. (This is a generalization of the characterization of Theorem 1.2 of semivalidity of sentences to formulas with paired free variables.)

THEOREM 2.1. *Each special sentence is semivalid.*

Proof. Suppose σ is of the form (*) and, towards applying Theorem 1.2, that G is a graph satisfying $(\exists x_1)(\forall y_1)\ldots(\exists x_n)(\forall y_n)\varphi^*(x_1y_1\ldots x_ny_n)$, which is σ^*. Relabeling the bound variables of σ^* shows that G must satisfy

$$(\exists y_1)(\forall x_1)\ldots(\exists y_n)(\forall x_n)\varphi^*(y_1x_1\ldots y_nx_n).$$

By [4, p. 127, #81], this implies $(\forall x_1)(\exists y_1)\ldots(\forall x_n)(\exists y_n)\varphi^*(y_1x_1\ldots y_nx_n)$, and now the assumption on φ in the definition of σ being special shows that G also satisfies σ.

In order to make $\mathscr{L}$ more colloquial, we shall introduce two "enriched" quantifiers. For any formula $\theta(z)$ of $\mathscr{L}$, define $(\forall z\colon \theta(z))\varphi(z)$ to be $(\forall z)[\theta(z)\Rightarrow\varphi(z)]$ and $(\exists z\colon \theta(z))\varphi(z)$ to be $(\exists z)[\theta(z)\,\&\,\varphi(z)]$. Since $\theta(z)$ could be taken as something harmless such as $z = z$, all quantifiers of $\mathscr{L}$ can be considered to be of this form. We also define $(\forall z_1\ldots z_m\colon \theta(z_1\ldots z_m))\varphi(z_1\ldots z_m)$ to be $(\forall z_1)\ldots(\forall z_m)[\theta(z_1\ldots z_m)\Rightarrow\varphi(z_1\ldots z_m)]$ and $(\exists z_1\ldots z_m\colon \theta(z_1\ldots z_m))\varphi(z_1\ldots z_m)$ to be $(\exists z_1)\ldots(\exists z_m)[\theta(z_1\ldots z_m)\,\&\,\varphi(z_1\ldots z_m)]$.

COROLLARY 2.2. *Suppose* $\models\varphi^*(y_1x_1\ldots y_nx_n)\Rightarrow\varphi(x_1y_1\ldots x_ny_n)$. *Then*

(a) $(\forall x_1\colon \bar\theta_1(x_1))(\exists y_1\colon \theta_1(y_1))(\forall x_2\colon \bar\theta_2(x_1x_2))(\exists y_2\colon \theta_2(y_1y_2))\ldots$

$$(\forall x_n\colon \bar\theta_n(x_1\ldots x_n))(\exists y_n\colon \theta_n(y_1\ldots y_n))\varphi(x_1y_1\ldots x_ny_n)$$

is special (and so, semivalid).

(b) $(\forall x_1\ldots x_n\colon \bar\theta(x_1\ldots x_n))(\exists y_1\ldots y_n\colon \theta(y_1\ldots y_n))\varphi(x_1y_1\ldots x_ny_n)$ *is special (and so, semivalid).*

Proof. (a) can be argued by induction on n; we shall show the $n = 1$ case. $(\forall x\colon \bar\theta(x))(\exists y\colon \theta(y))\varphi(x, y)$ is defined to be $(\forall x)(\exists y)[\bar\theta(x)\Rightarrow(\theta(y)\,\&\,\varphi(x, y))]$ and so is equivalent to $(\forall x)(\exists y)[\neg\bar\theta(x)\vee(\theta(y)\,\&\,\varphi(x, y))]$. To show this special, note that $[\neg\bar\theta(y)\vee(\theta(x)\,\&\,\varphi(y, x))]^*$ is equivalent to $\theta(y)\,\&\,(\theta^*(x)\vee\varphi^*(y, x))$, which can be shown (using the hypothesis on φ) to imply $\neg\bar\theta(x)\vee(\theta(y)\,\&\,\varphi(x, y))$. Part (b) follows from (a) by taking $\theta_i(z_1\ldots z_i)$ to be $z_1 = z_1\,\&\ldots\&\,z_i = z_i$ for $i<n$, and $\theta_n(z_1\ldots z_n)$ to be $\theta(z_1\ldots z_n)$.

As an example, connectivity is a consequence of having diameter at most three, and the latter can be expressed as

$$(**)\qquad (\forall x_1, x_2\colon \overline{x_1x_2})(\exists y_1, y_2\colon y_1y_2)[(x_1y_1\vee x_1y_2)\,\&\,(x_2y_1\vee x_2y_2)];$$

by abbreviating the quantifierless part as $\varphi(x_1y_1x_2y_2)$, it can be checked that $\models\varphi^*(y_1x_1y_2x_2)\Rightarrow\varphi(x_1y_1x_2y_2)$. (In fact there are relatively simple combinatorial means of checking this for any formula.) Hence by Corollary 2.2, having diameter at most three is semivalid, and so by Theorem 1.4, so is connectivity.

While the θ's in the enriched quantifiers can be any formulas of $\mathscr{L}$, it can be shown that nothing is gained by their involving quantifiers; quantifiers within the θ's can be replaced by additional pairs of quantifiers within the sentence itself. But it is worthwhile to allow constants in the θ's. For instance, if $\theta(z_1, z_2) = z_1z_2$ in (**) is replaced by $z_1z_2\,\&\,z_1\neq a\,\&\,z_2\neq a$, where the constant a is some vertex, then the resulting special sentence expresses that the diameter of $G-\{a\}$ is at most three and hence implies the semivalidity of "vertex a is not a cutpoint" [3].

It is interesting to note that each diagonal instance of Ramsey's theorem can be expressed as a special sentence. For instance, "each graph of order at least six contains a triangle" can be expressed

$$(\forall x_1\ldots x_6\colon \bar\delta(x_1\ldots x_6))(\exists y_1\ldots y_6\colon \delta(y_1\ldots y_6))[\Delta(x_1\ldots x_6)\vee\Delta(y_1\ldots y_6)],$$

where δ (and $\bar\delta$) asserts the pairwise distinctness of the six variables and Δ asserts the existence of a triangle (that is, the disjunction of twenty terms each of the form $x_ix_j\,\&\,x_ix_k\,\&\,x_jx_k$). The universal quantification causes the sentence to be vacuously true for orders less than six. Finally, the sentence is special since it can be checked that $\models[\Delta^*(y_1\ldots y_6)\,\&\,\Delta^*(x_1\ldots x_6)]\Rightarrow[\Delta(x_1\ldots x_6)\vee\Delta(y_1\ldots y_6)]$ (and this fails for the corresponding Δ formulas with fewer than six variables).

THEOREM 2.3. *Each semivalid sentence* σ *is a consequence of some special sentence* σ^+.

Proof. Suppose σ is of the form $(\forall x_1)(\exists y_2)(\exists y_3)(\forall x_4)\varphi(x_1y_2y_3x_4)$; the verification of any other form is similar. Take σ^+ to be

$$(\forall x_1)(\exists y_1)(\forall x_2)(\exists y_2)(\forall x_3)(\exists y_3)(\forall x_4)(\exists y_4)[\varphi(x_1y_2y_3x_4)\vee\varphi^*(y_1x_2x_3y_4)].$$

Then [4, pp. 127–128] σ^+ is equivalent to

$$(\forall x_1)(\exists y_2)(\exists y_3)(\forall x_4)\varphi(x_1y_2y_3x_4)\vee(\exists y_1)(\forall x_2)(\forall x_3)(\exists y_4)\varphi^*(y_1x_2x_3y_4);$$

relabeling the bound variables in the second term shows σ^+ equivalent to $\sigma\vee\sigma^*$. Since σ is semivalid, Theorem 1.4 shows that σ^+ implies $\sigma\vee\sigma$, and so σ. Since

$$\models[\varphi^*(y_1x_2x_3y_4)\,\&\,\varphi(x_1y_2y_3x_4)]\Rightarrow[\varphi(x_1y_2y_3x_4)\vee\varphi^*(y_1x_2x_3y_4)],$$

σ^+ is special, as claimed.

While Theorem 2.3 shows a sense in which special sentences are fundamental or primitive semivalid sentences, we should not expect them to be maximally strong semivalid sentences. For instance, even strengthening (**) by requantifying it as $(\forall x_1)(\exists y_1)(\forall x_2\colon \overline{x_1x_2})(\exists y_2\colon y_1y_2)\ldots$ is not maximally strong. The quantifierless part can be strengthened by adding either one of the conjuncts $(x_1y_2\vee x_2y_1)$ or $(x_1y_1\vee x_2y_2)$, and the resulting sentences are still semivalid (and special). (Furthermore, these resulting quantifierless formulas can be shown not to be consequences of a stronger such formula resulting in a special sentence.)

We must also warn that despite the tools given in Corollary 1.3 and Theorem 1.4, we have given no test for determining whether or not a sentence is semivalid, and it is easy to see why we cannot. For any sentence σ, σ will be valid exactly when $\sigma \,\&\, \bar{\sigma}$ is semivalid. Thus determining semivalidity is at least as hard (and, in fact, is exactly as hard) as determining validity.

Finally, it should be noted that the semivalid sentences based upon Ramsey theory or being nonplanar or the like are going to be quite complicated if for no other reason than the requirements of sufficiently large order. And because of the inherent lack of decision procedures for semivalidity, this approach cannot be expected to help in answering extremal questions such as the determination of Ramsey numbers.

References

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Accepté par la Rédaction le 2. 6. 1980

Compactness = JEP in any logic

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Abstract. L-elementary embeddings are for logic L what elementary embeddings are for first-order logic. If the Joint Embedding Property holds for L-elementary embeddings (for short: L has JEP), then the latter become a fundamental model, as well as an arrow-theoretical feature of L. Assuming Constructibility, $\neg 0^{\#}$, or even $\neg L^{\mu}$, we prove that in any small extension of first-order logic JEP is equivalent to compactness. We further give a characterization of Craig's interpolation along the same lines, by making use of a strong notion of amalgamation.

Preliminaries. The reader is referred to [MSS] for everything unexplained here; following [Fe2], for τ a (similarity) type, $\mathrm{Str}(\tau)$ is the class of all structures of type τ; if L is a (many-sorted) logic then $\mathrm{Stc}_L(\tau)$ is the class of all sentences of L of type τ; given $\mathfrak{N}, \mathfrak{M} \in \mathrm{Str}(\tau)$ we let

$$\mathrm{th}_L \mathfrak{M} = \{\varphi \in \mathrm{Stc}_L(\tau) | \mathfrak{M} \models \varphi\}$$

and we let $\mathfrak{M} \equiv_L \mathfrak{N}$ mean that $\mathrm{th}_L\mathfrak{M} = \mathrm{th}_L\mathfrak{N}$. For $\Gamma \subseteq \mathrm{Stc}_L(\tau)$ we let $\mathrm{mod}_L\Gamma = \{\mathfrak{U} \in \mathrm{Str}(\tau) | \mathfrak{U} \models \Gamma\}$. In logic L we allow *relativization*, e.g., relativization of formula ψ to formula $\varphi(x, y_1, \ldots, y_q)$ where $y_1, \ldots, y_q$ act as parameters, and we write

$$\psi\{x|\varphi(x, y_1, \ldots, y_q)\}$$

to denote the formula obtained by this process. If $\mathfrak{B} \in \mathrm{Str}(\tau)$ and $B' \subseteq B$ (with B the universe of $\mathfrak{B}$) where B' is nonempty on each sort of τ, then $\mathfrak{B}|B'$ is the substructure of $\mathfrak{B}$ generated by B', see [Fl]. For the definition of (λ, ω)-compactness, see [MSS] or [MS]. *(Full) compactness* is (λ, ω)-compactness for all $\lambda \geqslant \omega$; an important related notion is given by the following (see [MS]):

DEFINITION. Logic L is *μ-relatively compact* (for short: μ-r.c.) with $\mu \geqslant \omega$, iff for any classes of sentences Σ, Γ with $|\Sigma| = \mu$, if for each $\Sigma' \subseteq \Sigma$ with $|\Sigma'| < \mu$, $\Sigma' \cup \Gamma$ is consistent, then $\Sigma \cup \Gamma$ is consistent.

For the definition of L having *Craig's interpolation property* (or theorem), see [Fe1], [Ba], [MSS]. An important related notion is given by the following:

DEFINITION. We say that in L *Robinson's consistency theorem* holds (or: L has the *Robinson property*) iff given any types τ, τ_1 and τ_2 and classes of sentences T, T_1 and T_2, if T is complete in τ and T_1, T_2 are consistent extensions of T in type τ_1 and τ_2 respectively, with $\tau = \tau_1 \cap \tau_2$, then $T_1 \cup T_2$ is consistent.