Validity up to complementation in graph theory

by

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Abstract. A graph-theoretic property is called "semivalid" exactly when, for each graph $G$, the property must hold either in $G$ or in its complement $\overline{G}$. Examples include connectivity and the "diagonal" cases of Ramsey's theorem. A formal logical characterization is given, from which further features of semivalidity are deduced. We also consider a very restricted subclass of the semivalid properties, which is broad enough to include the various examples which prompted this investigation, and which is also deep enough that the semivalidity of this restricted subclass insures the same of all other semivalid properties.

1. The notion of semivalidity. In a sense, the fundamental activity in graph theory is determining those properties which are valid; that is, which are true of all graphs. Also, there is a simple yet intimate relationship between each graph and its complement—indeed it is sometimes rather arbitrary which is "the graph" and which is "the complement." With this in mind, we propose the study of the "semivalidity" of graph-theoretic statements (or properties expressed as statements), where a statement is said to be semivalid if and only if, for each graph $G$, the statement is necessarily true in either $G$ or its complement $\overline{G}$. (Our graph-theoretic notation and terminology follows that of [2].) Notice that we cannot require truth in exactly one of $G$ and $\overline{G}$, since the existence of selfcomplementary graphs rules out the possibility of being true in only one of each complementary pair.

Probably the best known example of semivalidity [2, page 26] is connectivity: Every graph either is connected or has a connected complement. Near examples are provided by any of the "diagonal" instances of Ramsey's theorem [2, page 284]. For instance, "containing a triangle" must be true of each graph or its complement, if only graphs of order at least six are considered. This leads to the somewhat awkward semivalid statement "has order less than six or contains a triangle." (We could call containment of a triangle "eventually semivalid"). A similar example is the semivalidity of "has order less than nine or is nonplanar" [1]. A recent and sophisticated example [5] is "has order less than six or is both connected and has a pacyclic line graph."

In this section we shall discuss several properties of semivalidity which are rather independent of our choice of a particular formal language. However, to make our arguments specific, we shall introduce a certain language $\mathcal{L}$ for graph
theory and shall use only simple logical manipulations with it such as are found in any elementary text on mathematical logic (for instance, [4]). Although many simple concepts (connectivity, for one) cannot be directly expressed in $L$, they can be expressed in various extensions of $L$ (for instance, by allowing disjunction and conjunction over countable sets of formulas), and the results of this section could be very naturally modified for these extended but somewhat more involved languages.

The variables of $L$ are $x$, $y$, and $z$ (possibly primed or subscripted) and will be interpreted as vertices. The atomic formulas are of the forms $x = y$, $x \neq y$, $xy$, and $\neg x$, interpreted respectively as equality, distinctness, adjacency and non-adjacency of the vertices $x$ and $y$. The formulas of $L$ are built from the atomic formulas using the usual logical connectives $\land$, $\lor$, and $\neg$ (for nagation, conjunction and disjunction) and the universal and existential quantifiers $\forall$ and $\exists$. Formulas having no free (= unquantified) occurrences of variables are called sentences of $L$.

For any formula $\varphi$ of $L$, define $\varphi^*$ to be obtained from $\varphi$ by replacing all occurrences of adjacencies by the corresponding nonadjacencies, and vice versa. Also, $\varphi^*$ is obtained from $\varphi$ by interchanging all occurrences of $\forall$ and $\exists$, of $V$ and $\exists$, and of $\sigma$ and $\varphi$. Note that $\varphi^*$ is not quite the negation of $\varphi$ since (non) adjacencies are left untouched. But applying both the $*$ and $\hat{\imath}$ transformations to $\varphi$ produces precisely $\neg \varphi$. In fact it is easy to see that the three transformations $\hat{\imath}$, $\hat{\sigma}$, and $\neg$ constitute the nonidentity elements of a Klein four group under composition. This is the content of the following lemma, in which "equivalent" means "are true in exactly the same graphs."

**Lemma 1.1.** For each formula $\varphi$ of $L$, $\varphi^*$, $\neg \varphi$, and $\varphi^*$ are equivalent to, respectively, $\varphi$, $\exists \varphi$, and $\varphi^*$.

It is now convenient to define an additional connective for $L$: the conditional symbol $\varphi \rightarrow \psi$, where $\varphi \rightarrow \psi$ is defined to be $\neg \varphi \lor \psi$ (or, equivalently, $\psi \land \neg \varphi$).

**Theorem 1.4.** The set of semivalid sentences is closed under consequence; that is, if $\sigma$ is semivalid and if $\varphi \rightarrow \tau$, then $\tau$ is semivalid.

**Proof.** $\varphi \rightarrow \tau$ means $\neg \varphi \lor \tau$, and so $\varphi \rightarrow \tau$ is semivalid if $\varphi \lor \tau$ is semivalid, which by Lemma 1.1 is $\varphi^* \lor \tau$ and so $\varphi^* \rightarrow \tau$. But Theorem 1.2 gives $\varphi^* \rightarrow \sigma$, and by assumption we have $\varphi \rightarrow \tau$. Transitivity of $\Rightarrow$ shows that $\varphi \rightarrow \tau$, as required for Theorem 1.2.

**Theorem 1.4** suggests searching for stronger and stronger semivalid sentences, so that proving the semivalidity of one such sentence will insure that of all its consequences. For instance the usual proof of connectivity's semivalidity [2, p. 26] really proves the semivalidity of the stronger property of having diameter at most three. In the next section we shall consider this example in more detail as we examine a very restricted family of "special" semivalid sentences whose consequences include all other semivalid sentences.

We close this section with a comment about selfcomplementary graphs: Not only must they satisfy each semivalid sentence, but the converse holds as well.

**Theorem 1.5.** A graph is selfcomplementary if and only if it satisfies all semivalid sentences.

**Proof.** Suppose rather, towards a contradiction, there were a nonselfcomplementary graph $G$ satisfying all semivalid sentences. Since $G$ is finite, there would be a sentence $\sigma$ of $L$ characterizing $G$ up to isomorphism—that is, asserting the existence of the proper number of distinct vertices (each vertex equal to one of them) with the proper pairs adjacent and the rest nonadjacent. Since $G$ was chosen to be nonselfcomplementary, $\neg \sigma$ would be semivalid. But we would then have $G$ satisfying $\neg \sigma$, contradicting the choice of $\sigma$.

Selfcomplementary graphs also satisfy certain nonsimvalid sentences; conjunctions of semivalid sentences, for instance. In fact for (any "first-order" extension of) $L$, the Compactness Theorem [4, p. 312] of elementary model theory can be used to show that the theorems of the theory of selfcomplementary graphs consist precisely of all conjunctions of semivalid sentences.

2. Special sentences and semivalidity. We now consider a family of sentences which contains many natural examples of semivalid sentences and which can be shown to be intimately related to the totality of semivalid sentences. We shall call a sentence of $L$ special if it is equivalent to one of the form

$$(\forall x_1)(\exists y_1) \ldots (\forall x_n)(\exists y_n) \varphi(x_1, y_1, \ldots, x_n, y_n)$$

where $\varphi(x_1, y_1, \ldots, x_n, y_n)$ is a formula of $L$ all of whose free variables are included among $x_1, y_1, \ldots, x_n$ and $y_n$ such that $\varphi(x_1, y_1, \ldots, y_n, y_n) \Rightarrow \varphi(x_1, y_1, \ldots, x_n, y_n)$, where
this last assumption means that the conditional formula \( \phi^a \Rightarrow \phi \) is true for all assignments of vertices to the variables in all graphs. (This is a generalization of the characterization of Theorem 1.2 of semivoidality of sentences to formulas with paired free variables.)

**Theorem 2.1.** Each special sentence is semivalid.

**Proof.** Suppose \( \sigma \) is of the form \((*)\) and, towards applying Theorem 1.2, that \( G \) is a graph satisfying \((3x_1)(\forall y_1) \ldots (3x_n)(\forall y_n)p(x_1, y_1 \ldots x_n, y_n)\), which is \( \sigma^* \). Relabeling the bound variables of \( \sigma^* \) shows that \( G \) also satisfies \( \sigma \).

In order to make \( \mathcal{S} \) more colloquial, we shall introduce two "enriched" quantifiers. For any formula \( \theta(z) \) of \( \mathcal{S} \), define \((Y: \theta(z))p(z)\) to be \((Y: \theta(z) \Rightarrow \phi(z))\) and \((3z: \theta(z))p(z)\) to be \((3z: \theta(z) \Rightarrow \phi(z))\). Since \( \theta(z) \) could be taken as something harmless such as \( z = z \), all quantifiers of \( \mathcal{S} \) can be considered to be of this form. We also define \((P: \theta(z) \cdot z = z_0)p(z_1 \ldots z_n)\) to be \((P: \theta(z) \cdot z = z_0)(Y: z = z_0)p(z_1 \ldots z_n)\) and \((3z_1 \ldots z_n: \theta(z_1 \ldots z_n))p(z_1 \ldots z_n)\) to be \((3z_1 \ldots z_n): \theta(z_1 \ldots z_n) \Rightarrow \phi(z_1 \ldots z_n)\). For \( \sigma = (\exists x_1 \ldots x_n: \theta(x_1 \ldots x_n))p(x_1 \ldots x_n, y_1 \ldots y_n) \) and \( \phi(x_1 \ldots x_n) \), then \( \phi(x_1 \ldots x_n) \).

**Corollary 2.2.** Suppose \( \phi(x_1 \ldots x_n) \Rightarrow \phi(x_1 \ldots x_n) \).

(a) \((Y_1: \theta_1(y_1))(3_2: \theta_2(y_1, y_2))(Y_3: \theta_3(y_1, y_2, y_3)) \ldots \) (b) \((P: \theta(z_1))p(z_1 \ldots z_n)\).

**(a)** is special (and so, semivalid). Then \((a) (P: \theta(z_1))p(z_1 \ldots z_n)\) is special (and so, semivalid).

**Proof.** (a) can be argued by induction on \( n \); we shall show the \( n = 1 \) case. \((Y: \theta(y))p(y, x)\) is defined to be \((Y: \theta(y) \Rightarrow \phi(y, x))\) and is equivalent to \((Y: \theta(y))[\neg \phi(y, x) \lor \theta(y) \land \phi(y, x)]\). To show this special, note that \([\neg \phi(y, x) \lor \phi(y, x)]\) is equivalent to \( \theta(y) \land \phi(y, x) \lor \phi(y, x) \).

Part (b) follows from (a) by taking \( \theta(z_1) \) to be \( z_1 = z_1 \ldots z_n = z_1 \) for \( i < n \), and \( \theta(z_1) \) to be \( z_1 = z_1 \ldots z_n = z_1 \).

As an example, connectivity is a consequence of having diameter at most three, and the latter can be expressed as \((**)(*)\) (by abbreviating the quantifier part as \( \phi(x_1, y_1, y_2) \)). It can be checked that \( \phi^a(x_1, y_1, y_2) \Rightarrow \phi(x_1, y_1, y_2) \).

While the \( \theta \)'s in the enriched quantifiers can be any formulas of \( \mathcal{S} \), it can be shown that nothing is gained by their involving quantifiers; quantifiers within the \( \theta \)'s can be replaced by additional pairs of quantifiers within the sentence itself. But it is worthwhile to allow constants in the \( \theta \)'s. For instance, if \( \theta_1(x_1, y_2) = z_1, z_2 \) in \((**)\) is replaced by \( x_1, x_2 \), then the resulting special sentence expresses the fact that \( G = \{a\} \) is at most three and hence implies the semivalidity of "vertex \( a \) is not a cutpoint" [3].

It is interesting to note that each diagonal instance of Ramsey's theorem can be expressed as a special sentence. For instance, "each graph of order at least six contains a triangle" can be expressed

\[ (\forall x_1 \ldots x_6)(\exists y_1 \ldots y_6)(\exists z_1 \ldots z_6)[(\forall x_1 \ldots x_6)\phi(x_1 \ldots x_6)] \]

where \( \phi \) (and \( \delta \)) asserts the pairwise distinctness of the six variables and \( \delta \) asserts the existence of a triangle (that is, the disjoint union of twenty edges each of the form \( x_1 \cdot x_2 \cdot x_3 \cdot x_4 \cdot x_5 \cdot x_6 \)). The universal quantification causes the sentence to be vacuously true for orders less than six. Finally, the sentence is special since it can be checked that \( [(\forall x_1 \ldots x_6)\phi(\alpha x_1 \ldots x_6)] \) (and this fails for the corresponding formulas with fewer than six variables).

**Theorem 2.3.** Each semivalid sentence \( \sigma \) is a consequence of some special sentence \( \sigma^* \).

**Proof.** Suppose \( \sigma \) is of the form \((\exists x_1 \ldots x_n)(\forall y_1 \ldots y_n)\phi(x_1 \ldots x_n, y_1 \ldots y_n)\); the verification of any other form is similar. Take \( \sigma^* \) to be \((\exists x_1 \ldots x_n)\phi(x_1 \ldots x_n)\).

Then \( \sigma^* \) is equivalent to \((\exists x_1 \ldots x_n)\phi(x_1 \ldots x_n)\).

Relabeling the bound variables in the second term shows \( \sigma^* \) equivalent to \( \sigma \lor \sigma^* \).

Since \( \sigma \) is semivalid, Theorem 1.4 shows that \( \sigma \) implies \( \sigma \lor \sigma^* \) and so \( \sigma \lor \sigma^* \) is special, as claimed.

While Theorem 2.3 shows a sense in which special sentences are fundamental or primitive semivalid sentences, we should not expect them to be maximally strong semivalid sentences. For instance, even strengthening \((**)\) by reexpressing it as \((\exists x_1 \ldots x_n)(\exists y_1 \ldots y_n)\phi(x_1 \ldots x_n, y_1 \ldots y_n)\) is not maximally strong. The quantifier part can be strengthened by adding either one of the conjuncts \( y_1 \lor y_2 \), ..., \( y_1 \lor y_n \) and the resulting sentences are still semivalid (and special). (Furthermore, these resulting quantifierless formulas can be shown not to be consequences of a stronger such formula resulting in a special sentence.)

We must also warn that despite the tools given in Corollary 1.3 and Theorem 1.4, we have given no test for determining whether or not a sentence is semivalid.
and it is easy to see why we cannot. For any sentence $\sigma$, $\delta$ will be valid exactly when $\sigma \& \delta$ is semivvalid. Thus determining semivvalidity is at least as hard (and, in fact, is exactly as hard) as determining validity.

Finally, it should be noted that the semivvalid sentences based upon Ramsey theory or being nonplanar or the like are going to be quite complicated if for no other reason than the requirements of sufficiently large order. And because of the inherent lack of decision procedures for semivvalidity, this approach cannot be expected to help in answering extremal questions such as the determination of Ramsey numbers.

References


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Compactness = JEP in any logic

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Abstract. $L$-elementary embeddings are for logic $L$ what elementary embeddings are for first-order logic. If the Joint Embedding Property holds for $L$-elementary embeddings (for short: $L$ has JEP), then the latter become a fundamental model, as well as an arrow-theoretical feature of $L$. Assuming Constructibility, "$\in^P"$, or even "$\in^L"$, we prove that in any small extension of first-order logic JEP is equivalent to compactness. We further give a characterization of Craig's interpolation along the same lines, by making use of a strong notion of amalgamation.

Preliminaries. The reader is referred to [MSS] for everything explained here; following [Fe2], for $\tau$ a (similarity) type, $Str(\tau)$ is the class of all structures of type $\tau$; if $L$ is a (many-sorted) logic then $Str_c(\tau)$ is the class of all sentences of $L$ of type $\tau$; given $\mathfrak{B}, \mathfrak{B} \in Str(\tau)$ we let $\text{th} \mathfrak{B} = \{ \phi \in Str_c(\tau) | \mathfrak{B} \models \phi \}$

and we let $\text{th} \mathfrak{B} = \{ \phi \in Str_c(\tau) | \mathfrak{B} \models \phi \}$

For $\Gamma \subseteq Str_c(\tau)$ we let $\text{mod} \Gamma = \{ \mathfrak{B} \in Str(\tau) | \mathfrak{B} \models \Gamma \}$. In logic $L$ we allow relativization, e.g., relativization of formula $\psi$ to formula $\psi(\phi(x, \upsilon_1, ..., \upsilon_n)$ where $\upsilon_1, ..., \upsilon_n$ act as parameters, and we write $\psi(\phi(x, \upsilon_1, ..., \upsilon_n))$

$\psi(x, \upsilon_1, ..., \upsilon_n)$

to denote the formula obtained by this process. If $\mathfrak{B} \in Str(\tau)$ and $\mathfrak{B}' \subseteq \mathfrak{B}$ (with $\mathfrak{B}$ the universe of $\mathfrak{B}$) then $\mathfrak{B}|\mathfrak{B}'$ is the substructure of $\mathfrak{B}$ generated by $\mathfrak{B}'$, see [FI]. For the definition of $(\lambda, \omega)$-compactness, see [MSS] or [MS]. (Full) compactness is $(\lambda, \omega)$-compactness for all $\lambda \geq \omega$; an important related notion is given by the following (see [MS]):

DEFINITION. Logic $L$ is $\mu$-relatively compact (for short: $\mu$-r.c.) with $\mu \geq \omega$, if for all classes of sentences $\Sigma$, $\Gamma$ with $|\Sigma| = \mu$, if for each $\Sigma' \subseteq \Sigma$ with $|\Sigma'| < \mu$, $\Sigma' \cup \Gamma$ is consistent, then $\Sigma \cup \Gamma$ is consistent.

For the definition of $L$ having Craig's interpolation property (or theorem), see [Fe1], [Be], [MSS]. An important related notion is given by the following:

DEFINITION. We say that in $L$ Robinson's consistency theorem holds (or: $L$ has the Robinson property) iff for given any types $\tau_1$ and $\tau_2$ and classes of sentences $T, T_1$ and $T_2$, if $T$ is complete in $\tau$ and $T_1, T_2$ are consistent extensions of $T$ in type $\tau_1$ and $\tau_2$ respectively, with $\tau = \tau_1 \land \tau_2$, then $T_1 \cup T_2$ is consistent.