Close PL involutions of 3-manifolds have close fixed point sets: 1-dimensional components

by

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Abstract. Let M be a closed PL 3-manifold and let ϱ be a metric on M. It is shown that for every PL-involution $f\colon M\to M$, and every $\varepsilon>0$ there exists an $\eta>0$ such that if $g\colon M\to M$ is a PL involution, η-close to f then there exists a homeomorphism $h\colon \mathrm{Fix}^1(f)\xrightarrow{\text{onto}}\mathrm{Fix}^1(g)$ which is ε-close to the inclusion $\mathrm{id}_{\mathrm{Fix}^1(f)}\colon \mathrm{Fix}^1(f)\to M$. $\mathrm{Fix}^1(f)$ denotes the sum of all 1-dimensional components of the fixed point set of f. (1)

1. Introduction. Let M be a closed PL 3-manifold and let $f\colon M\to M$ be a PL involution of M, i.e. $f^2=\mathrm{id}$. Let $g\colon M\to M$ be another PL involution of M. We are interested in the question how close to the fixed point set $\mathrm{Fix}(f)$ of f must be the set $\mathrm{Fix}(g)$ if g is sufficiently close to f. It is obvious that $\mathrm{Fix}(g)$ must be contained in some small neighbourhood of $\mathrm{Fix}(f)$. In this note we consider only 1-dimensional components of $\mathrm{Fix}(f)$. By [2], p. 76 each of them is homeomorphic to S^1. For any $f\colon M\to M$, by $\mathrm{Fix}^1(f)$ we denote the sum of all 1-dimensional components of $\mathrm{Fix}(f)$. The aim of this note is to show that, for every PL involution f of M, and every $\varepsilon>0$, there exists an $\eta>0$ such that if g is a PL involution of M, η-close to f, then there is a homeomorphism $h\colon \mathrm{Fix}^1(f)\to\mathrm{Fix}^1(g)$ which is ε-close to $\mathrm{id}_{\mathrm{Fix}^1(f)}$. (We consider M as a metric space with some metric ϱ.) An analogous fact for the components of $\mathrm{Fix}(f)$ of dimension $\neq 1$, will be proved in [5].

The proof of the existence of a homeomorphism $h\colon \mathrm{Fix}(f)\to\mathrm{Fix}(g)$, close to $\mathrm{id}_{\mathrm{Fix}^1(f)}$, is a first step towards the proof of the fact that f and g are conjugate by the homeomorphism of M onto itself close to id_M.

2. Notation and terminology. We assume in the paper that all the manifolds, maps, homeomorphisms, isotopies and actions are PL. Any map which is not PL will be called a *topological map*. Let M be a compact 3-manifold, possibly with boundary. Then $\mathscr{H}_{\mathrm{PL}}(M)$ denotes the space of all PL homeomorphisms of M onto itself. The space of PL actions of a finite group G on M, $\mathscr{A}_{\mathrm{PL}}(G, M)$, is the space of all group homomorphisms $\varphi\colon G\to\mathscr{H}_{\mathrm{PL}}(M)$, $g\mapsto\varphi^g$. In particular $\mathscr{A}_{\mathrm{PL}}(Z_2, M)$ is equal to the space $I(M)$ of all PL-involutions on M. We can identify $\mathscr{A}_{\mathrm{PL}}(Z_2, M)$ and $I(M)$ as follows: if $\varphi\in\mathscr{A}_{\mathrm{PL}}(Z_2, M)$, then $\varphi\mapsto f$ where $f\mapsto\varphi^1$, $1\in Z_2$. All

(1) See "Added in proof" at the end of [5].

the spaces described are considered with compact-open topology. If $h \in \mathscr{H}_{PL}(M)$, define $h*\varphi \in \mathscr{A}_{PL}(G, M)$ by $h\circ\varphi^g\circ h^{-1} = (h*\varphi)^g$ for any $g\in G$ (if $f\in I(M)$ then $h*f = h\circ f\circ h^{-1}$). For any $\varphi\in\mathscr{A}_{PL}(G,M)$ we denote by $\mathrm{Fix}(\varphi)$ the fixed point set of φ, and by $\mathrm{Fix}^1(\varphi)$ the sum of all 1-dimensional components of $\mathrm{Fix}(\varphi)$. We fix on M some metric ϱ, coincident with the topology on M. Then the space of all maps of $K\subset M$ into M is a metric space with the metric ϱ_K defined by

$$\varrho_K(f, g) = \sup_{x\in K}\{\varrho(f(x), g(x))\},$$

and $I(M)$ is a metric subspace of this space, for $K = M$. For any $f\in I(M)$ the quotient space M/f is a PL space (possibly not a manifold) such that the projection map of M onto M/f, which we shall denote by p_f, is PL. If K is a f-invariant PL-submanifold of M, then $I(K,f)$ is the space of all involutions $g\in I(M)$ such that $g|M\setminus K = f|M\setminus K$ and $I'(K,f)$ is the space of all involutions $g\in I(K)$ such that there is a $g'\in I(K,f)$ such that $g'|K = g$. For any space W and any subspace T of W, by $\mathrm{Int}_W T$ and $\mathrm{Cl}_W T$ we denote the topological interior of T in W and a closure of T in W, respectively, and we put $\mathrm{Fr}_W T = \mathrm{Cl}_W T\setminus \mathrm{Int}_W T$.

3. The equivalence of close free actions. The following theorem shows that if φ is a free action of a finite group G on a 3-manifold, then the action of G, close to φ must be conjugate to φ by a homeomorphism of M close to id_M, and, moreover, each of them can be joined with φ by a small G-isotopy.

THEOREM (3.1). *Let M be a compact 3-manifold, $\varphi\in\mathscr{A}_{PL}(G,M)$, and let K be a compact, φ-invariant, 3-dimensional PL submanifold of M, such that $\varphi|K$ is free. Let L be a neighbourhood of K in M, and let V be a neighbourhood of id_M in $\mathscr{H}_{PL}(M)$. Then there is a neighbourhood U of φ in $\mathscr{A}_{PL}(G,M)$, such that if $\psi\in U$ and $\psi|K\cap\partial M = \varphi|K\cap\partial M$, then there is an isotopy h_t, $t\in[0,1]$, such that $h_0 = \mathrm{id}_M$, $h_1*\psi|K = \varphi|K$, $h_t\in V$ for any $t\in[0,1]$, and $h_t|\partial M\cup(M\setminus L) = \mathrm{id}_{\partial M\cup(M\setminus L)}$.*

In the proof of (3.1) we shall need the following lemma, which easily results from [8], Theorems 1 and 3.

LEMMA (3.2). *Let M be a compact 3-manifold, L the 3-dimensional PL submanifold of M, and U a neighbourhood of id_M in $\mathscr{H}_{PL}(M)$. Then there exists a neighbourhood V of id_M in U, such that, for any $h\in V$ satisfying $h|L\cup\partial M = \mathrm{id}_{L\cup\partial M}$, there is an isotopy h_t, $t\in[0,1]$, such that $h_0 = \mathrm{id}_M$, $h_1 = h$, and for every $t\in[0,1]$, $h_t\in U$ and $h_t|L\cup\partial M = \mathrm{id}_{L\cup\partial M}$.*

Proof of (3.1). By Lemma (3.2), there is a neighbourhood V' of id_M in V, such that, for every $h\in V'$ satisfying $h|\partial M\cup(M\setminus L) = \mathrm{id}_{\partial M\cup(M\setminus L)}$, there is an isotopy $h_t\in V$ such that $h_t|\partial M\cup(M\setminus L) = \mathrm{id}_{\partial M\cup(M\setminus L)}$, and h_t joins h and id_M. Then, by Theorem (2.3) of [3], there is a neighbourhood U of φ in $\mathscr{A}_{PL}(G,M)$, such that for every $\psi\in U$ there is a topological homeomorphism $h\in\overline{V}'$ such that $h|\partial M\cup(M\setminus L) = \mathrm{id}_{\partial M\cup(M\setminus L)}$, $h*\psi = \varphi$ on K, and h is PL on K, where $\overline{V}'$ is a closure of V' in the space of all topological homeomorphism of M onto itself. The fact that h may be chosen PL on K follows from the proof of (2.3). Then, by [4], there is a homeomorphism $h'\in V'$, with $h = h'$ on $\partial M\cup(M\setminus L)\cup K$. Then we put $h_1 = h'$, and we find h_t joining h_1 and id_M by the choice of V'.

4. Actions with fixed points. Now we restrict our attention to the case of $G = Z_p$, where p is a prime number. In such a situation $\varphi\in\mathscr{A}_{PL}(Z_p,M)$ is free on the complement of $\mathrm{Fix}(\varphi)$. Note, that $\mathrm{Fix}(\varphi)$ is a PL subspace of M, and by [2], p. 76, and [10], p. 280, each of the components of $\mathrm{Fix}(\varphi)$ is a manifold of dimension $\leqslant 2$ (we assume, that φ is not trivial on any component of M).

Note that for any neighbourhood L of $\mathrm{Fix}(\varphi)$ in M, we can find a regular, φ-invariant neighbourhood K of $\mathrm{Fix}(\varphi)$ in M, such that $K\subset L$, and $p_\varphi(K)$ is a regular neighbourhood of $p_\varphi(\mathrm{Fix}(\varphi))$ in M/φ. The next lemma is a direct consequence of (3.1):

LEMMA (4.1). *Let M be a closed 3-manifold, $\varphi\in\mathscr{A}_{PL}(Z_p,M)$, where p is a prime number, and let K be a regular, φ-invariant neighbourhood of $\mathrm{Fix}(\varphi)$ in M. Then for every neighbourhood V of id_M in $\mathscr{H}_{PL}(M)$ there is a neighbourhood U of φ in $\mathscr{A}_{PL}(Z_p,M)$, such that for every $\psi\in U$ there is an isotopy h_t, $t\in[0,1]$, such that $h_0 = \mathrm{id}_M$, $h_1*\psi|M\setminus K = \varphi|M\setminus K$, and, for every $t\in[0,1]$, $h_t\in V$, and $h_t|\mathrm{Fix}(\psi) = \mathrm{id}_{\mathrm{Fix}(\psi)}$.*

5. The proof of the main theorem. The aim of this section is to prove

THEOREM (5.1). *Let M be a closed PL 3-manifold. Then, for every $f\in I(M)$ and every $\varepsilon>0$, there is an $\eta>0$, such that for every $g\in I(M)$ satisfying $\varrho_M(f,g)<\eta$ there exists a homeomorphism $h\colon \mathrm{Fix}^1(f)\to\mathrm{Fix}^1(g)$ such that $\varrho_{\mathrm{Fix}^1(f)}(h, \mathrm{id}_{\mathrm{Fix}^1(f)})<\varepsilon$.*

We begin by proving some lemmas: The first of them is easy, and so we omit its proof.

LEMMA (5.2). *Let M be a closed 3-manifold, and let $f\in I(M)$. Then, for every $\delta>0$ there exists a regular f-invariant neighbourhood K of $\mathrm{Fix}(f)$ in M, such that $p_f(K)$ is a regular neighbourhood of $p_f(\mathrm{Fix}(f))$ in M/f and that the following condition is satisfied:*

(*) *For every component F of $\mathrm{Fix}^1(f)$ the component K_F of K containing F may be considered as a total space of a locally trivial PL fibre bundle $q_F\colon K_F\to F$ such that each fibre $q_F^{-1}(x)$, $x\in F$, is a PL, f-invariant disc of diameter $<\delta$ (in metric ϱ), properly embedded in K_F (i.e. $\partial(q_F^{-1}(x)) = q_F^{-1}(x)\cap\partial K_F$) and such that $q_F^{-1}(x)\cap F = \{x\}$.*

We are going to use all the notations of (5.2) in the whole of section 5. Moreover, we shall use the following notations: Let $g\in I'(K_F,f)$. If K_F is orientable, then we put $\tilde K_F = K_F$ and $\tilde g = g$. If K_F is non-orientable, then we define $\tilde K_F$ as a double covering of K_F, and $\tilde g$ as an involution on $\tilde K_F$, such that $g\circ\pi = \pi\circ\tilde g$ where $\pi\colon\tilde K_F\to K_F$ is a covering projection. In both cases $\tilde K_F$ is a solid torus and $\tilde f'$ is a standard rotation on it, where $f' = f|K_F$. Fixing an orientation on $F\cong S^1$ and on $[0,1]\subset R^1$, we can define a space $\langle x,y\rangle\subset F$, $x,y\in F$, as an arc $s([0,1])$, where $s\colon[0,1]\to F$ is an orientation-preserving embedding with $s(0) = x$, $s(1) = y$. For any $W\subset K_F$, we denote $g[W] = W\cup g(W)$.

In the next two lemmas we assume that $K = K(\delta)$ is the neighbourhood of $\mathrm{Fix}(f)$ in M, found in the previous lemma, which satisfies (*), and that F is some component of $\mathrm{Fix}^1(f)$.

LEMMA (5.3). *Let F be a component of $\mathrm{Fix}^1(f)$, and let K_F be a component of K containing F. Then there exists a $\delta_1>0$ such that, for any $g \in I'(K_F, f)$ satisfying $\varrho_{K_F}(g, f|K_F)<\delta_1$, the set $F_g = \mathrm{Fix}(g)$ is homeomorphic to S^1 and $(q_F|F_g)_*: H_i(F_g, Z_2) \to H_i(F, Z_2)$ is an isomorphism, where $(q_F|F_g)_*$ is a homomorphism induced on the Z_2-homology by the map $q_F|F_g: F_g \to F$.*

Proof. By (4.2) and the proof of (4.3) of [3], given $\varepsilon>0$, there is a $\delta_1>0$, such that for every $g \in I'(K_F, f)$, satisfying $\varrho_{K_F}(g, f|K_F)<\delta_1$, there is an equivariant map $r_g: (K_F, g) \to (K_F, f|K_F)$, $\frac{1}{2}\varepsilon$-close to id_{K_F} and such that $(r_g|F_g)_*: H_i(F_g, Z_2) \to H_i(F_g, Z_2)$ is an isomorphism, where $F_g = \mathrm{Fix}(g)$ (r_g is equivariant, and so $r_g(F_g) \subset F$). Moreover, it is easy to see that if δ_1 is small enough then $\varrho_{F_g}(q_F|F_g, \mathrm{id}_{F_g}) < \frac{1}{2}\varepsilon$. This implies that we can choose δ_1 such that, for any homeomorphism $s: S^1 \to F_g = \mathrm{Fix}(g)$ where g is such that $\varrho_{K_F}(g, f|K_F)<\delta_1$, the compositions $q_F \circ s$ and $r_g \circ s$ are ε-close. But $(r_g|F_g)_*$ is an isomorphism of the Z_2-homology, and F_g is a manifold, so at least one homeomorphism $s: S^1 \to F_g$ must exist. If ε is small enough, then $q_F \circ s$ and $r_g \circ s$ are homotopic, and so they induce the same homomorphism on the Z_2-homology. This implies that $q_F|F_g$ induces an isomorphism on the Z_2-homology as well as $r_g|F_g$.

LEMMA (5.4). *Let $\alpha, \alpha_1, \alpha_2 \in F$, and let $\delta_1>0$ be the number satisfying the hypothesis of (5.3), and let $g \in I'(K_F, f)$ be such that $\varrho_{K_F}(g, f|K_F)<\delta_1$ and $g[q_F^{-1}(a)] \subset L = q_F^{-1}(\langle a_1, a_2\rangle)$. Then, there exists a PL disc $D \subset p_g(L) \subset K_F/g$, properly embedded in K_F/g and such that $\partial D = p_g(q_F^{-1}(a) \cap \partial K_F) \subset \partial(K_F/g)$.*

Proof. By (5.3) $F_g = \mathrm{Fix}(g) \cong S^1$ and $q_F|F_g: F_g \to F$ has degree $\neq 0$. So $q_F^{-1}(a) \cap F_g \neq \emptyset$, and we can find a curve $s: [0, 1] \to q_F^{-1}(a)$ joining some point $s(0) = a' \in q_F^{-1}(a) \cap F_g$ and some point $b = s(1) \in q_F^{-1}(a) \cap \partial K_F$. There exists a PL curve $\alpha: [0, 1] \to q_F^{-1}(a) \cap \partial K_F$ such that $\alpha(0) = b$, $\alpha(1) = g(b)$, and $p_g \circ \alpha$ is a simple, closed curve in $\partial(K_F/g)$ (i.e. $p_g \circ \alpha(0) = p_g \circ \alpha(1)$, and $p_g \circ \alpha|(0, 1)$ is an embedding). Then we take the curve $\alpha': [0, 1] \to g[q_F^{-1}(a)]$ defined by $\alpha'(t) = s(3t)$ for $t \in [0, \frac{1}{3}]$, $\alpha'(t) = \alpha(3t-1)$ for $t \in [\frac{1}{3}, \frac{2}{3}]$, and $\alpha'(t) = g \circ s(-3t+3)$ for $t \in [\frac{2}{3}, 1]$. Of course α' is a closed curve contractible in L. This implies that $p_g \circ \alpha'$ is a closed curve, contractible in $p_g(L)$. But the curves $p_g \circ s$ and $p_g \circ g \circ s$ are identical, and so it follows that $\bar{\alpha} = p_g \circ \alpha$ is a simple, closed curve contractible in $p_g(L)$, and such that $\bar{\alpha}([0, 1]) = p_g(q_F^{-1}(a) \cap \partial K_F)$.

Now we can use the Dehn lemma (see [7], p. 101) to find the required disc D (we use the Dehn lemma to the manifold $p_g(\mathrm{Int}_{K_F} L)$, because $p_g(L)$ may not to be a manifold).

The most complicated part of the proof is contained in the following:

LEMMA (5.5). *Let M be a closed 3-manifold $f \in I(M)$ and let $K = K(\delta)$ be a regular f-invariant neighbourhood of $\mathrm{Fix}(f)$ in M satisfying condition (*) of (5.2). Then for every $\varepsilon_1>0$ there exists a $\delta_1>0$, such that, for every component F of $\mathrm{Fix}^1(f)$, and every $g \in I'(K_F, f)$ such that $\varrho_{K_F}(f|K_F, g)<\delta_1$, the space $\mathrm{Fix}(g)$ is homeomorphic to S^1, and the following condition is satisfied:*

(**) *For every $b \in F$ and every $c, d \in \mathrm{Fix}(g) \cap q_F^{-1}(b)$, there is a component m of the space $\mathrm{Fix}(g) \setminus \{c, d\}$, such that $\mathrm{diam}(q_F(m))<\varepsilon_1$.*

Proof. Suppose that, on the contrary, there is an $\varepsilon_1>0$ such that for every $\delta_1>0$ there exists a component F of $\mathrm{Fix}^1(f)$ and $g \in I'(K_F, f)$ such that $\varrho_{K_F}(f|K_F, g)>\delta_1$, or $\mathrm{Fix}(g)$ is not homeomorphic to S^1, or (**) is not satisfied. This fact and Lemma (5.3) imply that for our ε_1 we can choose some particular δ_1, a component F of $\mathrm{Fix}^1(f)$, and $g \in I'(K_F, f)$ such that $\varrho_{K_F}(f|K_F, g)<\delta_1$, $\mathrm{Fix}(g) \cong S^1$, and $(q_F|\mathrm{Fix}(g))_*: H_i(\mathrm{Fix}(g), Z_2) \to H_i(F, Z_2)$ is an isomorphism, but (**) does not hold. Moreover, we can claim that our δ_1 is chosen so small that the following condition holds:

(***) For our chosen δ_1 and g, the condition $\varrho_{K_F}(f|K_F, g)<\delta_1$ implies that for every two points $a_1, a_2 \in F$, with $\varrho(a_1, a_2)>\varepsilon_1$, there are three points $b_0, b_1, b_2 \in \langle a_1, a_2\rangle \subset F$, such that $b_i \neq b_j$ for $i \neq j$, $b_0 \in \langle b_1, b_2\rangle$, and the neighbourhood L_i of $q_F^{-1}(b_i)$ in K_F for $i = 1, 2$, such that

$$g[q_F^{-1}(b_i)] \subset L_i \subset g[L_i] \subset q_F^{-1}(\langle a_1, a_2\rangle), \qquad g[L_i] \cap q_F^{-1}(b_0) = \emptyset,$$

and $L_i = q_F^{-1}(n_i)$, where n_i is some arc contained in $\langle a_1, a_2\rangle$.

Now, in the described situation we want to obtain a contradiction with our assumptions. We shall get it in several steps. In all of them we shall denote $F_g = \mathrm{Fix}(g)$.

The construction of D_0, D_1, m_1 and m_2. The degree of the map $(q_F|F_g): F_g \to F$ is not 0; so, for $b \in F$ and $c, d \in q_F^{-1}(b) \cap F_g$, there is a component m of $F_g \setminus \{c, d\}$ such that $q_F(\overline{m}) = F$, where $\overline{m} = \mathrm{Cl}_{F_g}(m)$. Condition (**) does not hold, and so we can choose points $b \in F$, and $c, d \in q_F^{-1}(b) \cap F_g$, so that the sets $q_F(\overline{m}_1)$ and $q_F(\overline{m}_2)$ both have diameters $>\varepsilon_1$, where $\overline{m}_1$ and $\overline{m}_2$ are the closures of the components m_1, m_2 of $F_g \setminus \{c, d\}$. Then $q_F(\overline{m}_1) \cap q_F(\overline{m}_2) = q_F(\overline{m}_i)$, where $i = 1$, or 2, and so there is an arc $n = \langle a_1, a_2\rangle \subset q_F(\overline{m}_1) \cap q_F(\overline{m}_2)$ such that a_1, a_2 are the points of F, and $\varrho(a_1, a_2)>\varepsilon_1$. Having chosen a_1, a_2, we can find the points b_0, b_1, b_2 and the neighbourhoods L_i of $q_F^{-1}(b_i)$ for $i = 1, 2$, guaranteed by (***). We consider the projection $p_g: K_F \to K_F/g$.

The properties of the neighbourhoods L_i described in (***) and Lemma (5.4) imply that there are two discs, D_0 and D_1, contained in $p_g(L_1)$ and $p_g(L_2)$, respectively, properly embedded in K_F/g and such that $p_g(q_F^{-1}(b_i) \cap \partial K_F) = \partial D_{i-1}$, for $i = 1, 2$.

Additionally, by the standard general position arguments, we may require that, for each $i = 0, 1$, the sets D_i and $p_g(F_g)$ should be in general position; we mean by this that for any point $a \in D_i \cap p_g(F_g)$ there is a PL homeomorphism of some neighbourhood T of a in K_F/g onto $[-1, 1]^3$ which takes $T \cap D_i$ onto $\{0\} \times [-1,1]^2$ and $T \cap p_g(F_g)$ onto $[-1, 1] \times \{(0, 0)\}$).

The proof that $\tilde{K}_F/\tilde{g}$ is a solid torus. We use the notation of $\tilde{K}_F$ and $\tilde{g}$ introduced below Lemma (5.2).

We may use the famous conjecture of P. A. Smith, proved for involutions by Waldhausen [9], to prove that $\tilde{K}_F/\tilde{g}$ is a solid torus. Actually, $\tilde{g}|\partial\tilde{K}_F = \tilde{f}'|\partial\tilde{K}_F$, $\partial\tilde{K}_F \cong S^1\times S^1$, where $f' = f|K_F$, and $\tilde{f}'|\partial\tilde{K}_F$ is equivalent to the rotation of one of the S^1-factors, so $\tilde{K}_F$ can be contained in some 3-sphere N as a solid torus unknotted in N, and $\tilde{g}$ can be extended to some involution $\bar{g}$ on N, such that $\bar{g}|\tilde{K}_F = \tilde{g}$ and $\mathrm{Fix}(\bar{g}) = \tilde{F}_g = \pi^{-1}(F_g)$, where π: $\tilde{K}_F \to K_F$ is a covering projection when $\tilde{K}_F$ is a double covering of K_F, and $\pi = \mathrm{id}_{\tilde{K}_F}$ if $\tilde{K}_F = K_F$. Then, by the Smith conjecture, $\tilde{F}_g$ is unknotted in N, and by [6] $N/\bar{g}$ is a 3-sphere. $p_{\bar{g}}(\partial\tilde{K}_F)$ is a torus in $N/\bar{g}$, and $\tilde{K}_F/\tilde{g}$ is a closure of one of the components of $(N/\bar{g})\setminus p_{\bar{g}}(\partial\tilde{K}_F)$. Let us note that we have found the discs D_0 and D_1, properly embedded in K_F/g. We can use one of the components of $\pi^{-1}(D_0)$, which is a disc in $\tilde{K}_F/\tilde{g}$, as in [7], p. 107, to prove that $\tilde{K}_F/\tilde{g}$ is a solid torus.

$\tilde{K}_F/\tilde{g}$ is a double covering over K_F/g, or $\tilde{K}_F/\tilde{g} = K_F/g$, and $\tilde{K}_F/\tilde{g}$ is a solid torus; so K_F/ψ is a D^2-bundle over S^1 (D^2 is a disc), and thus the Schoenflies theorem implies that there is a 3-cell $C\subset K_F/g$ such that C is a closure of one of the components of $(K_F/g)\setminus(D_0\cup D_1)$ and that $q_F(p_g^{-1}(C))\subset\langle a_1, a_2\rangle$. Then there are PL maps q_1: $C\to[0,1]$ and q_2: $C\to D^2$, such that the map $(q_1, q_2$: $C\to[0,1]\times D^2$ defined by $(q_1, q_2)(x) = (q_1(x), q_2(x))$ is a homeomorphism, and $q_1^{-1}(0) = D_0$ and $q_1^{-1}(1) = D_1$.

The existence of u_1, u_2 and u_3. The fact that $\langle a_1, a_2\rangle\subset q_F(\overline{m}_1)\cap q_F(\overline{m}_2)$ implies that there are three arcs $u_1, u_2, u_3\subset p_g(F_g)\cap C$, such that $u_i\cap u_j = \emptyset$ for $i\neq j$, and that each u_i is bounded by two points, v_{i0} and v_{i1}, such that $v_{ij} = u_i\cap D_j$ for $j = 0, 1$.

Actually, the quotient spaces $F_g/\overline{m}_1$ and $F_g/\overline{m}_2$ are both homeomorphic to S^1, and there are two maps π_i: $F_g/\overline{m}_i\to F$, $i = 1, 2$, defined by $\pi_i(\overline{m}_i) = b$, and $\pi_i(x) = q_F(x)$ for $x\in F_g\setminus\overline{m}_i$. By Lemma (5.3) $q_F|F_g$ has degree $2d+1$ for some integer d, and so one of the maps π_1, π_2, say π_1, has degree $2e+1$, and other one, (π_2), has degree $2h$, $e+h = d$. It is then easy to prove that $p_g(\overline{m}_1)$ must contain two disjoint arcs u_1, u_2 with the required properties, and $p_g(\overline{m}_2)$ must contain the third arc u_3.

Note that only when $h = 0$ we have to use the fact that $q_F(\overline{m}_1)\supset\langle a_1, a_2\rangle$ to prove the existence of u_1 and u_2. Let u_i^*: $[0,1]\to C$ be a PL simple curve such that $u_i^*([0,1]) = u_i$ and $u_i^*(j) = v_{ij}$ for $i = 1, 2, 3$, $j = 0, 1$.

The description of C_1, S_0 and S_1. Let $C_1 = p_g^{-1}(C)$ and let q: $C_1\to[0,1]$ be defined by $q = q_1\circ p_g$, and let $S_i = q^{-1}(i) = p_g^{-1}(D_i)$ for $i = 0, 1$. Then $p_g|S_i$: $S_i\to D_i$ is a branched covering with a branch set $S_i\cap F_g$, i.e. $p_g|S_i\setminus F_g$: $S_i\setminus F_g\to D_i\setminus p_g(F_g)$ is a covering map and $p_g|S_i\cap F_g$ is a homeomorphism of $S_i\cap F_g$ onto $D_i\cap p_g(F_g)$. It is easy to prove, using the fact that D_i's are in general position with $p_g(F_g)$, that S_0 and S_1 are compact surfaces, and so C_1 is a 3-manifold.

The description of γ_1, γ_2, $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$. Let γ_1, γ_2: $[0,1]\to D_1$ be two simple PL curves such that $\gamma_1'(0) = v_{11}$, $\gamma_1'(1) = v_{21}$, $\gamma_2'(0) = v_{21}$, $\gamma_2'(1) = v_{31}$ and $\gamma_1'([0,1))\cap\gamma_2'((0,1]) = \emptyset$, and $\big(\gamma_1'((0,1))\cup\gamma_2'((0,1))\big)\cap p_g(F_g) = \emptyset$, and let $\gamma_i = \gamma_i'([0,1])$ for $i = 1, 2$. Then let $\tilde{\gamma}_1 = p_g^{-1}(\gamma_1)$, $\tilde{\gamma}_2 = p_g^{-1}(\gamma_2)$. For $i = 1, 2$ we can find two curves γ_{ij}': $[0,1]\to\tilde{\gamma}_i$ and two arcs $\gamma_{ij} = \gamma_{ij}'([0,1])$, $j = 1, 2$, such that $\tilde{\gamma}_i = \gamma_{i1}\cup\gamma_{i2}$, $p_g\circ\gamma_{ij}' = \gamma_i'$, $\gamma_{i1}\cap\gamma_{i2} = p_g^{-1}(\{v_{i1}, v_{i2}\}) = \tilde{\gamma}_i\cap F_g$ and that $\gamma_{i1} = g(\gamma_{i2})$.

It is not difficult to prove, considering some small g-invariant neighbourhood of the unique intersection point $p_g^{-1}(v_{21})$ of $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$, that $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ intersect transversally, and so they have the intersection index 1 in S_1.

The proof that either $\tilde{\gamma}_1$ or $\tilde{\gamma}_2$ is not homologous to 0 in C_1. We make us of the theory of the intersection index of simplicial chains as developed in [1] on page 100. We shall assume, that all the chains and cycles are considered with Z_2-coefficients (not Z as in [1]), and so the intersection index of chains takes a value in Z_2, and all the chains are in some triangulated manifold, not in R^n as in [1]. All the arguments of [1] remain valid in this case. For a given complex D in a given manifold, we shall denote by $z_K(D)$ the k-chain $\sum_{j\leqslant n_D} 1\cdot\sigma_j$ with Z_2-coefficients, where σ_j, $j\leqslant n_D$, are all the k-simplexes of D.

Let us consider some triangulation of C_1 in which $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are subcomplexes of C_1. Then each $\tilde{\gamma}_i$ supports a simplicial 1-cycle, $z_i = z_1(\tilde{\gamma}_i)$, which determines a homology class $[z_i]\in H_1(\partial C_1, Z_2)$, $i = 1, 2$. Of course the fact that $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ intersect transversally at a single point implies that z_1 and z_2 have the intersection index 1 in ∂C_1, and so $[z_1]\neq 0$ and $[z_2]\neq 0$. Suppose that $i_*([z_1]) = 0$, where i_*: $H_1(\partial C_1, Z_2)\to H_1(C_1, Z_2)$ is induced by the inclusion i: $\partial C_1\to C_1$. Then there is a 2-dimensional subcomplex D od C_1 with $D\cap\partial C_1 = \tilde{\gamma}_1$ (we may assume that the triangulation of C_1 is fine enough) and such that D supports a chain $v = z_2(D)$ with $\partial v = i(z_1)$ where i: $\partial C_1\to C_1$ is an inclusion as before. Then let $\overline{C} = C_1^1\cup C_1^2$ be the sum of two copies of C_1 with the boundary $\partial C_1 = \partial C_1^1 = \partial C_1^2$ identified by the identity map (i.e. $\partial C_1^1 = \partial C_1^2 = C_1^1\cap C_1^2$). Then each C_1^i contains a copy D_i of D which supports a 2-chain $v_i = z_2(D_i)$ with the boundary $\partial v_i = i(z_1)$. Then, we have $D_1\cap D_2 = \tilde{\gamma}_1\subset\partial C_1^1$, and the complex $\overline{D} = D_1\cup D_2$ supports a 2-cycle $\bar{v} = z_2(\overline{D}) = v_1+v_2$. The fact that the intersection index of z_1 and z_2 in ∂C_1 is 1 implies that the intersection index of $\bar{v}$ and $i'(z_2)$ in $\overline{C}$ is 1, where i': $\partial C_1\to\overline{C}$ is the inclusion. This implies that the homology class $i_*'([z_2])\in H_1(\overline{C}, Z_2)$ is non-trivial. This implies that $0\neq i_*([z_2])\in H_1(C_1, Z_2)$. So one of the 1-complexes $\tilde{\gamma}_1$ or $\tilde{\gamma}_2$ supports a cycle which is not homologous to 0. We shall assume that $\tilde{\gamma}_1$ supports such a cycle, and so we are no longer interested in γ_2 and u_3.

The final contradiction. The fact that C is a 3-cell implies that there is a PL map F: $[0,1]\times[0,1]\to C$ such that $F|\{1\}\times[0,1]$ is an embedding of $\{1\}\times[0,1]$ onto γ_1, $F|[0,1]\times\{i\}$ is an embedding of $[0,1]\times\{i\}$ onto u_i, $i = 1, 2$, and $F|\{0\}\times[0,1]$ is an embedding of $\{0\}\times[0,1]$ into D_0. Moreover, we can assume that $\dim(S(F))\leqslant 1$, where $S(F) = \{x\in[0,1]\times[0,1]$: $\mathrm{card}\{F^{-1}(F(x))\}>1$ and that $\dim\big(A\cap(p_g(F_g\setminus u_1\setminus u_2))\leqslant 0$, where $A = F([0,1]\times[0,1])$. A is a subcomplex of C in some triangulation of C, and the fact that $\dim(S(F))\leqslant 1$ implies that each

1-simplex in $A\setminus u_1\setminus u_2\setminus\tilde{\gamma}_1\setminus F(\{0\}\times[0,1])$ is a face of an even number of 2-simplexes in A. This implies that the chain $v = z_2(A)$ supported by A has the boundary $s = \partial v = z_1(u_1 \cup u_2 \cup \gamma_1 \cup F(\{0\}\times[0,1]))$. Then, $p_g^{-1}(A)$ is a complex in C_1 which supports a chain $\tilde{v} = z_2(p_g^{-1}(A))$. The fact that $p_g\colon C_1 \to C$ is a double branched covering with the branch set $C \cap p_g(F_g)$ and that $\dim(A \cap (p_g(F_g)\setminus u_1\setminus u_2)) \leqslant 0$ imply that $\partial\tilde{v} = z_1 + z_1'$, where $z_1 = z_1(\tilde{\gamma}_1)$ and $z_1' = z_1(F(\{0\}\times[0,1]))$. By our previous assumption $[z_1]$ is a non-trivial element of $H_1(C_1, Z_2)$ supported by the subcomplex of $S_1 = p_g^{-1}(D_1)$ and z_1' is a cycle homologous to z_1 supported by the subcomplex of $S_0 = p_g^{-1}(D_0)$. Now, $C_1 = N_0 \cup N_1$, where each N_i is a 3-manifold which contains S_i, $i = 0, 1$, and $N_0 \cap N_1 = q_F^{-1}(b_0)$ (see condition (***)). By the Mayer–Vietoris argument there is an isomorphism $h\colon H_1(C_1, Z_2) \to H_1(N_0, Z_2)\oplus H_1(N_1, Z_2)$ such that $h([z_1']) \in H_1(N_0, Z_2)$ and $h([z_1]) \in H_1(N_1, Z_2)$. This implies, that $h(0) = h([z_1]-[z_1']) = h([z_1]) - h([z_1']) \neq 0$. This is a contradiction, which finishes the proof of the lemma.

The proof of Theorem (5.1). Let $f \in I(M)$, $\varepsilon > 0$ be as in (5.1). Then we find $\delta < \frac{1}{4}\varepsilon$ and a neighbourhood $K = K(\delta)$ of $\mathrm{Fix}(f)$ in M, guaranteed by Lemma (5.2), which satisfies condition (*) of (5.2). Let F be a component of $\mathrm{Fix}^1(f)$. Let $\{b_1, b_2, \ldots, b_l\}$, $l \geqslant 2$, be a finite set of points in F such that $b_i \neq b_j$ for $i \neq j$, $\langle b_i, b_{i+1}\rangle$ contains no points b_k, for $k \neq i, i+1$, and the diameter of $\langle b_i, b_{i+1}\rangle$ is smaller than $\frac{1}{4}\varepsilon$ for every $i \leqslant l$; here and in all the proof of (5.1) we consider the index i of b_i modulo l, i.e. we put $l+1 = 1$, and by $\langle x, y\rangle \in F$, we denote the arc in F bounded by x and y, as in Lemmas (5.3)–(5.5). Let $\varepsilon_1 = \varepsilon_1(F)$ be a positive number smaller than $\frac{1}{4}\min_{1\leqslant i,j\leqslant l}\{\varrho(b_i, b_j)\}$. Then, by Lemma (5.5) we can find a number $\delta_1 = \delta_1(F)$ for our $K(\delta)$ and ε_1, such that for every $g \in I'(K_F, f)$ satisfying $\varrho_{K_F}(f|K_F, g) < \delta_1$ condition (**) of (5.5) holds. Let $g \in I'(K_F, f)$ be such that $\varrho_{K_F}(f|K_F, g) < \delta_1$, and let $F_g = \mathrm{Fix}(g)$. Making δ_1 smaller if necessary, we can assume by Lemma (5.3) that $F_g \cong S^1$ and that the degree of $q_F|F_g\colon F_g \to F$ is $2d+1$ for some integer d. Then condition (**) of (5.5) easily implies that $\deg(q_F|F_g) = 1$. This implies that the space $F_g\setminus q_F^{-1}(b_i)$ for any $i \leqslant l$ has a component m such that $q_F(\overline{m}) = F$ where $\overline{m} = \mathrm{Cl}_{F_g}(m)$. Then the condition (**) of (5.5) implies that $\mathrm{diam}(q_F(F_g\setminus m)) < 2\varepsilon_1 < \mathrm{diam}(F)$. This implies that for every $i \leqslant l$ there is exactly one component m_i of $F_g\setminus q_F^{-1}(b_i)$ such that $q_F(\overline{m}_i) = F$. Let us put $n_i = F_g\setminus m_i$. By (**) $\mathrm{diam}(q_F(n_i)) < 2\varepsilon_1$. From this and from the fact that $\varepsilon_1 < \frac{1}{4}\min_{1\leqslant l\leqslant j\leqslant l}\{\varrho(b_i, b_j)\}$ it follows that $n_i \cap n_j = \emptyset$ for $i \neq j$. Then we consider the space $F_g\setminus\bigcup_{i\leqslant l} n_i$. It consists of l components. The fact that $\deg(q_F|F_g) = 1$ implies that $q_F(F_g\setminus\bigcup_{i\leqslant l} n_i) \cap \langle b_j, b_{j+1}\rangle \neq \emptyset$ for every $j \leqslant l$; otherwise we would have a map $s\colon F_g \to F$ homotopic to $q_F|F_g$ and such that $s|(F_g\setminus\bigcup_{i\leqslant l} n_i) = q_F|(F_g\setminus\bigcup_{i\leqslant l} n_i)$ and $s(n_j) = b_j$ for $j \leqslant l$; but this is impossible, because then we would have $s(F_g) \cap \langle b_j, b_{j+1}\rangle = \emptyset$ for some $j \leqslant l$, and $\deg(s) = 1$. From this fact and from the fact that $q_F(F_g\setminus\bigcup_{i\leqslant l} n_i) \cap (\{b_1, b_2, \ldots, b_l\}) = \emptyset$ it follows that for every $j \leqslant l$ there is exactly one component r_j of $F_g\setminus\bigcup_{i\leqslant l} n_i$, such that $r_j \subset q_F^{-1}(\langle b_j, b_{j+1}\rangle)$. Then for every $j \leqslant l$ such that $q_F^{-1}(b_j) \cap F_g$ contains more than one point we find an arc $s_j \subset F$ such that $\mathrm{diam}(s_j) \leqslant \varepsilon_1$ and $b_j \in s_j$ and for every $j \leqslant l$ such that $q_F^{-1}(b_j) \cap F_g$ contains one point, we put $s_j = b_j$. Then $s_i \cap s_j = \emptyset$ for $i \neq j$. $F\setminus\bigcup_{i\leqslant l} s_i$ has l components, and by z_j we denote the unique component of $F\setminus\bigcup_{i\leqslant l} s_i$ such that $z_j \subset \langle b_j, b_{j+1}\rangle$. Then we can easily find a homeomorphism of F onto F_g, such that, for every $j \leqslant l$, s_j is taken onto n_j, and the closure $\bar{z}_j$ of z_j in F is taken onto $\bar{r}_j$ ($\bar{r}_j = \mathrm{CL}_{F_g}(r_j)$). We denote this homeomorphism by $h|F$. It is easy to check that $\varrho_F(h|F, \mathrm{id}_F) < \varepsilon$.

For every component F of $\mathrm{Fix}^1(f)$ we can find the number $\varepsilon_1 = \varepsilon_1(F)$ and then choose a number $\delta_1 = \delta_1(F)$ for it in the way described. Then we take δ' such that $\delta' < \delta_1(F)$ for every component F of $\mathrm{Fix}^1(f)$. Hence, for every $g \in I(K, f)$, there is a homeomorphism $h\colon \mathrm{Fix}^1(f) \to \mathrm{Fix}^1(g)$ such that $\varrho_{\mathrm{Fix}^1(f)}(h, \mathrm{id}_{\mathrm{Fix}^1(f)}) < \varepsilon$. It can be found in the following way: $g|K_F \in I'(K_F, f)$ for every component F of $\mathrm{Fix}^1(f)$ and so for every such F we can find the homeomorphism $h|F$ in the way described and put $h(x) = (h|F)(x)$ for $x \in F$ (note that we have found $K = K(\delta)$ at the beginning of the proof, common for all the components of $\mathrm{Fix}^1(f)$).

The fact that $\delta' < \delta_1(F)$ for every component F of $\mathrm{Fix}^1(f)$ implies that, for h so defined, we have $\varrho_{\mathrm{Fix}^1(f)}(h, \mathrm{id}_{\mathrm{Fix}^1(f)}) < \varepsilon$. Then, by Lemma (4.1), we can find $\eta > 0$ such that for every $g \in I(M)$, satisfying $\varrho_M(f, g) < \eta$ there is $g' \in I(K, f)$ such that $\varrho_M(f, g') < \eta$ and $\mathrm{Fix}(g) = \mathrm{Fix}(g')$. Of course η is the required number, such that for every $g \in I(M)$ with $\varrho_M(f, g) < \eta$ there is a homeomorphism $h\colon \mathrm{Fix}^1(f) \to \mathrm{Fix}^1(g)$ with $\varrho_{\mathrm{Fix}^1(f)}(h, \mathrm{id}_{\mathrm{Fix}^1(f)}) < \varepsilon$. If η and consequently δ' is sufficiently small, then h is onto $\mathrm{Fix}^1(g)$. This can easily be deduced from Theorem (4.3) of [3].

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