# Close PL involutions of 3-manifolds which are conjugate by a small homeomorphism

by

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**Abstract.** Let $M$ be a closed PL 3-manifold and let $\varrho$ be a metric on $M$. It is shown that for every PL involution $f\colon M \to M$, and every $\varepsilon > 0$ there exists an $\eta > 0$, such that if $g\colon M \to M$ is a PL involution, $\eta$-close to $f$, then there exists a homeomorphism $h\colon \mathrm{Fix}(f) \xrightarrow{\text{onto}} \mathrm{Fix}(g)$ which is $\varepsilon$-close to the inclusion $\mathrm{id}_{\mathrm{Fix}(f)}\colon \mathrm{Fix}(f) \to M$. $\mathrm{Fix}(f)$ denotes here the fixed point set of $f$. In the case where $f$ is a PL-involution of $M$ and $\mathrm{Fix}(f)$ contains no components of dimension 1 it is shown that every PL involution of $M$ which is sufficiently close to $f$ can be joined with $f$ by a PL isotopy consisting of involutions close to $f$. [1]

**1. Introduction.** Let $M$ be a closed PL 3-manifold and $\varrho$ a metric on $M$. We prove the following

THEOREM (1.1). *Let $f\colon M \to M$ be any* PL *involution of a closed,* PL 3-*manifold $M$, i.e. $f^2 = \mathrm{id}$. Then for every $\varepsilon > 0$ there exists an $\eta > 0$ such that for every* PL *involution $g\colon M \to M$ satisfying condition: $\sup\limits_{x \in M} \big(\varrho(f(x), g(x))\big) < \eta$, there exists a homeomorphism $h\colon \mathrm{Fix}(f) \to \mathrm{Fix}(g)$ taking a fixed point set of $f$ onto a fixed point set of $g$ and such that $\sup\limits_{x \in \mathrm{Fix}(f)} \big(\varrho(x, h(x))\big) < \varepsilon$.*

(1.1) is an extension of Theorem (5.1) of [6].

This easily follows from our Theorem (2.1) in Section 2, combined with Theorem (5.1) of [6].

Another consequence of (2.1) is the following Theorem (1.2), which justifies the title of the paper; $\mathscr{H}_{\mathrm{PL}}(M)$ denotes the space of all PL homeomorphisms of $M$ with compact-open topology.

THEOREM (1.2). *Let $f\colon M \to M$ be any* PL *involution of a closed* PL 3-*manifold $M$ such that the fixed point set* $\mathrm{Fix}(f)$ *of $f$ has no components of dimension* 1. *Then, for every neighbourhood $V$ of $\mathrm{id}_M$ in $\mathscr{H}_{\mathrm{PL}}(M)$, there is a neighbourhood $U$ of $f$ in $\mathscr{H}_{\mathrm{PL}}(M)$ such that for every* PL *involution $g\colon M \to M$ such that $g \in U$ there exists a* PL *isotopy $h_t\colon M \to M$, $t \in [0, 1]$ such that $h_t \in V$ for any $t \in [0, 1]$, $h_0 = \mathrm{id}_M$, and $h_1^{-1} \circ g \circ h_1 = f$.*

Theorem (1.2) implies that the space $I(M)$ of all PL involutions on $M$ is locally arcwise connected at every point $f \in I(M)$ such that $\mathrm{Fix}(f)$ has no components

[1] See „Added in proof" at the end of the paper.

of dimension 1 in the following sense: for every neighbourhood $U$ of $f$ in $I(M)$, there is a neighbourhood $V$ of $f$ in $U$ such that for every point $g\in V$ there is an arc $\xi\colon [0,1]\to U$ such that $\xi(0)=f$, $\xi(1)=g$. Theorem (1.2) can be contrasted with the situation in dimension 4; by Section 3, there are PL involutions of $S^4$, as close as we need to the standard PL involution $f$ on $S^4$, with $\mathrm{Fix}(f)=S^2\subset S^4$, which are not conjugate to $f$.

In the whole paper we use the following notation, which is also used in [6]: let $M$ be a compact PL 3-manifold. Then we denote by $\partial M$ the boundary of $M$ and by $\mathscr{H}_{\mathrm{PL}}(M,K)$ the space of PL homeomorphisms of $M$ onto itself which are identity on a subset $K\subset M$.

$\mathscr{H}_{\mathrm{PL}}(M)=\mathscr{H}_{\mathrm{PL}}(M,\emptyset)$. By $I(M)$ we denote the subspace of $\mathscr{H}_{\mathrm{PL}}(M)$ consisting of PL involutions. All the described spaces are considered with compact-open topology. In the whole paper we reserve the terms map, isotopy, homeomorphism, and action for the PL map, PL homeomorphism, PL isotopy, and PL action.

If any map is not PL, then we call it a *topological map*. If $h\in\mathscr{H}_{\mathrm{PL}}(M)$, we define $h*f\in I(M)$ by $h\circ f\circ h^{-1}=h*f$. For any $f\in I(M)$ we denote by $\mathrm{Fix}(f)$ the fixed point set of $f$ and by $\mathrm{Fix}^i(f)$ the sum of all $i$-dimensional components of $\mathrm{Fix}(f)$. We have fixed some metric $\varrho$ on $M$, coincident with the topology on $M$. Suppose that $K$ is a subset of $M$. Then the space of functions from $K$ to $M$ is a metric space with the metric $\varrho_K(f,g)=\sup_{x\in K}\big(\varrho(f(x),g(x))\big)$, and $I(M)$ is a metric subspace of $\mathscr{H}_{\mathrm{PL}}(M)$. For any $f\in I(M)$ the quotient space $M/f$ is a PL space (possibly not a manifold) obtained by the identification of $x$ and $f(x)$ for any $x\in M$ and the projection map, of $M$ onto $M/f$, which we shall denote by $p_f$, is PL. If $K$ is an $f$-invariant PL submanifold of $M$, then $I(K,f)$ denotes a space of all involutions $g\in I(M)$ such that $g|M\setminus K=f|M\setminus K$ and $I'(K,f)$ is a space of all involutions $g\in I(K)$ such that there is a $g'\in I(K,f)$ such that $g'|K=g$. For any space $W$ and any subspace $T$ of $W$, we denote by $\mathrm{Int}_W T$, and $\mathrm{Cl}_W T$ the topological interior of $T$ in $W$ and a closure of $T$ in $W$ respectively.

Finally we put $\mathrm{Fr}_W T=\mathrm{Cl}_W T\setminus\mathrm{Int}_W T$.

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**2. The main theorem.** Our goal is to prove the following

THEOREM (2.1). *Let $M$ be a closed 3-manifold, $f\in I(M)$ and let $L_1$ and $L_2$ be two closed, regular $f$-invariant neighbourhoods of $\mathrm{Fix}^1(f)$ in $M$ such that $L_1\subset\mathrm{Int}_M L_2$, and $L_2\cap(\mathrm{Fix}(f)\setminus\mathrm{Fix}^1(f))=\emptyset$. Then, for every neighbourhood $V$ of $\mathrm{id}_M$ in $\mathscr{H}_{\mathrm{PL}}(M)$ there exists a neighbourhood $U$ of $f$ in $I(M)$ such that for every $g\in U$ there is a homeomorphism $h_1\colon M\to M$, such that $h_1\in V$, and $h_1|L_1=\mathrm{id}_{L_1}$, and $h_1*g|M\setminus L_2=f|M\setminus L_2$. Moreover there exists an isotopy $h_t$, $t\in[0,1]$, joining $h_1$ with $h_0=\mathrm{id}_M$, such that $h_t\in V$ and $h_t|L_1=\mathrm{id}_{L_1}$.*



Proof of (2.1). Each 3-dimensional component of $\mathrm{Fix}(f)$ is a 3-dimensional PL-homology manifold, so it is a component of $M$, and so we can put $h_t=\mathrm{id}$ on it. Hence we can assume that $\dim(\mathrm{Fix}(f))\leqslant 2$.

If we find $h=h_1$ sufficiently close to $\mathrm{id}_M$ then the existence of $h_t$ automatically follows from Lemma (3.2) of [6].

Now Theorem (2.1) can be reduced to the following proposition:

PROPOSITION (2.1.a). *Let $M$ be a closed 3-manifold, $f\in I(M)$, $\dim(\mathrm{Fix}(f))\leqslant 2$, and let $L$ be a compact $f$-invariant, 3-dimensional PL-submanifold of $M$, such that $\mathrm{Fix}^1(f)\subset\mathrm{Int}_M L$ and $(\mathrm{Fix}(f)\setminus\mathrm{Fix}^1(f))\cap L=\emptyset$. Then, for every neighbourhood $V$ of $\mathrm{id}_M$ in $\mathscr{H}_{\mathrm{PL}}(M)$, there is a regular, $f$-invariant neighbourhood $K$ of $\mathrm{Fix}(f)\setminus\mathrm{Fix}^1(f)$ in $M$ such that $K\cap L=\emptyset$, and for every component $F$ of $\mathrm{Fix}(f)\setminus\mathrm{Fix}^1(f)$ there exists a neighbourhood $U_F$ of $f$ in $I(K_F,f)$, where $K_F$ is a component of $K$ containing $F$, such that for any $g\in U_F$ there is a homeomorphism $h\in V\cap\mathscr{H}_{\mathrm{PL}}(M,M\setminus K_F)$ such that $h*g=f$.*

We omit the reduction of (2.1) to (2.1.a), which can be easily done, using Theorem (3.1) of [6]. We have only to prove (2.1.a).

Proof of (2.1.a). Let $M,f$ and $L$ be as in (2.2), and let $V$ be any neighbourhood of $\mathrm{id}_M$ in $\mathscr{H}_{\mathrm{PL}}(M)$; then we can find a regular $f$-invariant neighbourhood $K$ of $\mathrm{Fix}(f)\setminus\mathrm{Fix}^1(f)$ in $M$, such that $K\cap L=\emptyset$ and that $p_f(K)$ is a regular neighbourhood of $p_f(\mathrm{Fix}(f)\setminus\mathrm{Fix}^1(f))$ in $M/f$. Moreover, we claim that $K$ is chosen so that for every 0-dimensional component $F$ of $\mathrm{Fix}(f)$ (such a component must be a point) $K_F$ is a 3-cell, so small that $\mathscr{H}_{\mathrm{PL}}(M,M\setminus K_F)\subset V$, and for every 2-dimensional component $F$ of $\mathrm{Fix}(f)$, which by [1], p. 76 and [12], p. 280 must be a PL surface in $M$, $K_F$ is a total space of a locally trivial PL fibre bundle $q_F\colon K_F\to F$, such that every fibre $q_F^{-1}(a)$, $a\in F$ is homeomorphic to the interval $[0,1]$, $\partial(q_F^{-1}(a))=\partial K_F\cap q_F^{-1}(a)$, and $q_F^{-1}(a)\cap F=\{a\}$. We fix $K_F$ and $q_F$ having the desired properties, for all the whole proof.

**The case of 0-dimensional components of** $\mathrm{Fix}(f)$. Now we suppose that $F$ is a 0-dimensional component of $\mathrm{Fix}(f)$, i.e. $F=\{a\}$ for some point $a$ of $M$. Then $K_F$ is a cone $c(\partial K_F)$ over $\partial K_F$ with vertex $a$. $f|\partial K_F$ is a free involution on $\partial K_F\cong S^2$, and so the quotient space $p_f(\partial K_F)=(\partial K_F)/(f|\partial K_F)$ is homeomorphic to the projective space $P^2$. $p_f(K_F)$ is a regular neighbourhood of $p_f(a)$ in $M/f$, and so it is a cone over $p_f(\partial K_F)=P^2$, i.e. $p_f(K_F)=c(P^2)$. Now, using Theorem (4.3) of [4], we find a neighbourhood $U_F$ of $f$ in $I(K_F,\varphi)$, such that for every $g\in U_F$, $g|K_F$ has a single fixed point.

Let $g\in U_F$ be any such involution, and let $a'\in K_F$ be a fixed point of $g|K_F$. Then we can choose a regular $g$-invariant neighbourhood $T$ of $a'$ in $K_F\setminus\partial K_F$, such that $p_g(T)$ is a regular neighbourhood of $p_g(a')$ homeomorphic to $c(P^2)$. $K_F\setminus(T\setminus\partial T)$ is homeomorphic to $S^2\times[0,1]$, and $\partial K_F$ and $\partial L$ correspond to $S^2\times\{0\}$ and $S^2\times\{1\}$, respectively. By the theorem of Livesay [8], p. 582, $p_g(K_F\setminus(T\setminus\partial T))$ is homeomorphic to $P^2\times[0,1]$. This implies that $K_F/(g|K_F)$ is homeomorphic to

$c(P^2)$, and so there is a homeomorphism $h'\colon K_F/(g|K_F) \to K_F/(f|K_F)$ which takes $p_g(a')$ onto $p_f(a)$. But the maps

$$p_g|K_F\setminus a'\colon K_F\setminus a' \to (K_F/(g|K_F))\setminus p_g(a')$$

and

$$p_f|K_F\setminus a\colon K_F\setminus a \to (K_F/(f|K_F))\setminus p_f(a)$$

are the universal covering maps, and so $h'$ lifts to a homeomorphism $h''\colon K_F \to K_F$ such that $p_f \circ h'' = h' \circ p_g$. Then, we have $h'' \circ g = f \circ h''$, and $h''|\partial K_F = \mathrm{id}_{K_F}$. So, we can take $h|K_F = h''$ and $h|M\setminus K_F = \mathrm{id}_{M\setminus K_F}$. Of course

$$h \in \mathscr{H}_{\mathrm{PL}}(M, M\setminus K_F) \subset V.$$

**The case of 2-dimensional components of** $\mathrm{Fix}(f)$. Then suppose, that $F$ is a 2-dimensional component of $\mathrm{Fix}(f)\setminus \mathrm{Fix}^1(f)$; then $F$ is a closed surface.

**The definition of the neighbourhoods** $V'$ **and** $U_1$. Let $V'$ be a neighbourhood of $\mathrm{id}_M$ in $V$. First, we find a neighbourhood $U_1$ of $f$ in $I(K_F, f)$, such that for every $g \in U_1$ satisfying condition: $F = \mathrm{Fix}(g|K_F)$, there exists an $H_g \in V \cap \mathscr{H}_{\mathrm{PL}}(M, M\setminus K_F)$ such that $H_g * g = f$. Let us denote by $\tilde{M}$ a 3-manifold with a boundary obtained from $M$ by splitting $M$ along the surface $F$. Then, there is a natural map $q\colon \tilde{M} \to M$, such that $q(\partial\tilde{M}) = F$, $q|\partial\tilde{M}\colon \partial\tilde{M} \to F$ is a double covering map, and

$$q|(\tilde{M}\setminus\partial\tilde{M})\colon (\tilde{M}\setminus\partial\tilde{M}) \to M\setminus F$$

is a homeomorphism.

Note that if $F$ is two-sided in $M$, then $\partial\tilde{M}$ is homeomorphic to two copies of $F$, and if $F$ is not two-sided, then $\partial\tilde{M}$ is a connected double covering of $F$. Let $\tilde{K}_F = q^{-1}(K_F)$. Then it is easy to see that for every $g \in I'(K_F, f)$, such that $F = \mathrm{Fix}(g)$ there is a unique free involution $\tilde{g}$ on $\tilde{K}_F$, such that $g \circ q|\tilde{K}_F = q \circ \tilde{g}$ and that for every homeomorphism $v \in \mathscr{H}_{\mathrm{PL}}((M, M\setminus K_F)) \cup F)$, there exists a unique homeomorphism $\tilde{v} \in \mathscr{H}_{\mathrm{PL}}((\tilde{M}, \tilde{M}\setminus\tilde{K}_F) \cup \partial\tilde{M})$ such that $v \circ q = q \circ \tilde{v}$. Let us define a space $\mathscr{A} \subset I(\tilde{K}_F)$ as follows: $g' \in \mathscr{A}$ iff there exists a $g \in I'(K_F, f)$ such that $g' = \tilde{g}$. Then, we define maps:

$\alpha\colon \mathscr{A} \to I(K_F, f)$ and $\beta\colon \mathscr{H}_{\mathrm{PL}}(\tilde{K}_F, \partial\tilde{K}_F) \to \mathscr{H}_{\mathrm{PL}}(M, (M\setminus K_F) \cup F)$ as follows: for any $g' \in \mathscr{A}$, $\alpha(g') = g$, where $g$ is a unique involution in $I(K_F, f)$ such that $\tilde{g}_{K_F} = g'$, where $g_{K_F} = g|K_F$, and for any $v' \in \mathscr{H}_{\mathrm{PL}}(K_F, \partial\tilde{K}_F)$, $\beta(v') = v$, where $v'$ is a unique homeomorphism in $\mathscr{H}_{\mathrm{PL}}(M, (M\setminus K_F) \cup F)$ such that $\tilde{v}|\tilde{K}_F = v'$. It is easy to see that $\alpha$ is a homeomorphism of $\mathscr{A}$ onto $\alpha(\mathscr{A}) \subset I(K_F, f)$ and $\beta$ is a homeomorphism of $\mathscr{H}_{\mathrm{PL}}(\tilde{K}_F, \partial\tilde{K}_F)$ onto $\mathscr{H}_{\mathrm{PL}}(M, (M\setminus K_F) \cup F)$. Then let

$$\tilde{V} = \beta^{-1}\big(V' \cap \mathscr{H}_{\mathrm{PL}}(M, (M\setminus K_F) \cup F)\big).$$

$\tilde{V}$ is an open neighbourhood of $\mathrm{id}_{\tilde{K}_F}$ in $\mathscr{H}_{\mathrm{PL}}(\tilde{K}_F, \partial\tilde{K}_F)$. Then we can use Theorem (3.1) of [6] to find an open neighbourhood $\tilde{U}$ of $\tilde{f}_{K_F}$ in $I(\tilde{K}_F)$ where $f_{K_F} = f|K_F$, such that for any $g \in \tilde{U} \cap \mathscr{A} = \{g \in \tilde{U}\colon g|\partial K_F = \tilde{f}_{K_F}|\partial\tilde{K}_f\}$ there exists an $h_g \in \tilde{V}$, such that $h_g * g = \tilde{f}_{K_F}$. Then $\alpha(\tilde{U} \cap \mathscr{A})$ is an open neighbourhood of $f$ in $\alpha(\mathscr{A})$, and so there is an open neighbourhood $U_1$ of $f$ in $I(K_F, f)$ such that $U_1 \cap \alpha(\mathscr{A}) = \alpha(\tilde{U} \cap \mathscr{A})$. This $U_1$ is the required neighbourhood, because, for every $g \in U_1$ such that $\mathrm{Fix}(g) = F$, we have $g \in \alpha(\mathscr{A})$ and $\alpha^{-1}(g) \in \tilde{U} \cap \mathscr{A}$, and so there is an $h_{\alpha^{-1}(g)} \in \tilde{V}$ such that $h_{\alpha^{-1}(g)} * \alpha^{-1}(g) = \tilde{f}_{K_F}$; but this implies that if $H_g = \beta(h_{\alpha^{-1}(g)}) \in V$ then $H_g * g = f_{K_F}$.



**The definition of the neighbourhoods** $U_2$ **and** $U_2'$. Then we find a neighbourhood $U_2$ of $f$ in $U_1$ and $\delta > 0$ such that the following condition is satisfied: if $g \in U_2$, and there is a homeomorphism $h_g'\colon F_g \to F$, where $F_g = \mathrm{Fix}(g|K_F)$ satisfying $\varrho_{F_g}(h_g', \mathrm{id}_{F_g}) < \delta$, then there exists a homeomorphism $H_g' \in V' \cap \mathscr{H}_{\mathrm{PL}}(M, M\setminus K_F)$ such that $H_g'|F_g = h_g$, and $H_g' * g \in U_1$. Then, clearly, $\mathrm{Fix}(H_g' * (g|K_F)) = F$.

The existence of $U_2$ and $\delta$ easily follows from Theorem (1) of [11].

Let $U_2'$ be a neighbourhood of $f|K_F$ in $I'(K_F, f)$ defined as follows: $g \in U_2'$ iff there exists a $g' \in U_2$ such that $g'|K_F = g$.

**The definition of the neighbourhood** $U_3'$, **and the proof of** (2.1.a). Finally, we shall find a neighbourhood $U_3'$ of $f|K_F$ in $U_2'$ such that for any $g \in U_3'$ there exists a homeomorphism $h_g'\colon F_g \to F$ with $\varrho_{F_g}(h_g', \mathrm{id}_{F_g}) < \delta$, where $\delta$ is a number we have chosen together with $U_2$, and $F_g = \mathrm{Fix}(g)$. This will complete the proof of (2.1.a) in the case of $\dim(F) = 2$, because then we can take

$$U = \{g \in I(K_F, f) = g|K_F \in U_3'\}.$$

Then, by the choice of $U_3'$, for any $g \in U$ there is a homeomorphism $h_g'\colon F_g \to F$, with $\varrho_{F_g}(h_g', \mathrm{id}_{F_g}) < \delta$; by the choice of $\delta$ and $U_2$, we can find a homeomorphism $H_g' \in V' \cap \mathscr{H}_{\mathrm{PL}}(M, M\setminus K_F)$ such that $g' = H_g' * g \in U_1$, and $\mathrm{Fix}(g') = F$. Then, by the choice of $U_1$, there is a homeomorphism $H_{g'} \in V'$ such that $H_{g'} * g' = f$ and $H_{g'} \in V'$. Then $f = H_{g'} * (H_g' * g) = (H_{g'} \circ H_g') * g$ and $H_{g'} \circ H_g' \in V$ if only $V'$ is sufficiently small.

We have only to find $U_3'$ satisfying our requirements. We shall use the following notation: $f \in I'(K_F, f)$, $F_f = \mathrm{Fix}(f)$ and if $D \subset K_F$, then $f[D] = D \cup f(D)$. If $T$ is a subpolyhedron of a surface $L$, then $\hat{T}$ is a closure in $L$ of the set of all points of $T$, having in $T$ the neighbourhood homeomorphic to $R^2$.

To describe $U_3'$ we shall need some lemmas:

Lemma (2.2). *There is a neighbourhood $U'$ of $f|K_F$ in $I'(K_F, f)$, such that for every $g \in U'$ the fixed point set $F_g = \mathrm{Fix}(g)$ is a surface and the map $q_F|F_g\colon F_g \to F$ induces an isomorphism*

$$(q_F|F_g)_*\colon H_i(F_g, Z_2) \to H_i(F, Z_2)$$

*on $Z_2$-homology for any $i$. In particular $q_F|F_g\colon F_g \to F$ is onto.*

Proof of (2.2). The proof is analogous to the proof of (5.5) in [6]. The only difference is that $F$ is a surface, not a circle. This implies, that if $U'$ is chosen sufficiently small, then, for any $g \in U'$, $F_g$ is a surface such that $H_i(F_g, Z_2) \cong H_i(F, Z_2)$ (see [4], Theorem (4.3)). So, for any $g \in U'$, $F_g$ is homeomorphic to one of two surfaces, $F'$ and $F''$ which have the same $Z_2$-homology as $F$. Then, instead of the map $s\colon S^1 \to F_1$ used in the proof of (5.5) of [6], we consider a homeomorphism $s\colon \bar{F} \to F_g$, where $\bar{F}$ is this of the surfaces $F'$ and $F''$ which is homeomorphic to $F_g$.

Then the proof goes as in [6]. To show that $q_F|F_g$ is onto $F$, we suppose that on the contrary it is not. Then there is a point $d\in F$ such that $(q_F|F_g)(F_g)\subset F\setminus\{d\}$. This implies that $(q_F|F_g)_*(H_2(F_g,Z_2))=0$, because $H_2(F\setminus\{d\},Z_2)=0$. But

$$(q_F|F_g)_*(H_2(F_g,Z_2))=H_2(F,Z_2)=Z_2.$$

This is a contradiction.

LEMMA (2.3). *If $D$ is a compact, connected PL-surface in $F$ and $g\in I'(K_F,f)$ then $g[q_F^{-1}(D)]\setminus F_g$ has no more than two components.*

Proof of (2.3). Let us denote $\bar D=g[q_F^{-1}(D)]$, and $D^0=\mathrm{Int}_{K_F}\bar D$. $\bar D\setminus F_g$ is $g$-invariant, so $g'=g|\bar D\setminus F_g$ is a free involution, and $p_{g'}\colon \bar D\setminus F_g\to(\bar D\setminus F_g)/g'$ is connected. Actually, $D$ is connected, so $q_F^{-1}(\mathrm{Int}_F D)$ is connected, and so $D^0$ is connected. This implies that $D^0/g''$ is connected, where $g''=g|D^0$. Then from Lemma (2.1) on page 198 in [1], applied to $g$ it follows that $p_g(F_g\cap\bar D)\subset\partial(p_g(D^0))$ (note that $p_g(D^0)$ is a manifold). This fact and the fact that $p_g(\mathrm{Cl}_{K_F}D^0)\supset(\bar D\setminus F_g)/g'\supset D^0/g''$ imply that $(\bar D\setminus F_g)/g'$ is connected. But $p_{g'}$ is a double covering map, whence there are no more than 2 components of $\bar D\setminus F_g$.

LEMMA (2.4). *Let $W$ be a compact, 3-dimensional PL submanifold of $K_F$ and $T$ be a closed 2-dimensional PL submanifold of $K_F$, such that $T$ is transversal to $\partial D$, $T\subset K_F\setminus\partial K_F$, and let $T_1=T\cap D$. Then $\hat T_1$ is a compact surface in $D$ such that $\partial\hat T_1\subset\partial D$ and if $T\cap\mathrm{Int}_{K_F}D=\emptyset$ then $\hat T_1\neq\emptyset$.*

The proof of (2.4) is easy, and so we omit it.

LEMMA (2.5). *Let $A$ be a closed PL-disc in $F$ and $W$ be a 3-dimensional PL-submanifold of $q_F^{-1}(\mathrm{Int}_F A)$. Suppose that $g\in I'(K_F,f)$ is such that $F_g$ is transversal to $\partial(q_F^{-1}(A))$ and that $q_F|F_g\colon F_g\to F$ induces isomorphism on $Z_2$-homology. Then, for every component $T$ of $W\cap F_g$ we have $\hat T=\emptyset$ or $\hat T$ is a sphere with holes.*

Proof of (2.5). First we prove that $\hat T$ is orientable. Let $T_1$ be the component of $q_F^{-1}(A)\cap F_g$ containing $\hat T$. If $\hat T$ were non-orientable, then $\hat T_1$ would be a non-orientable surface in a 3-cell $q_F^{-1}(A)$, with $\partial\hat T_1\subset\partial q_F^{-1}(A)$. Such surfaces do not exist. By the classification of surfaces every orientable compact surface is a sphere with some holes and some handles attached. Let us suppose that, for some component $T$ of $W\cap F_g$, $\hat T$ is a surface containing at least one handle. Then there is a $z\in H_1(\hat T,Z_2)$ such that $i_*(z)\neq 0$, where $i_*$ is a homomorphism induced on $Z_2$-homology by the inclusion $i\colon \hat T\to F_g$. On the other hand $(q_F|F_g)_*\colon H_1(F_g,Z_2)\to H_1(F,Z_2)$ is an isomorphism, and the homomorphism $i\colon H_1(A,Z_2)\to H_1(F,Z_2)$ induced by the inclusion $i'\colon A\to F$ is trivial, because $H_1(A,Z_2)=0$. $q_F|\hat T$ maps $\hat T$ into $A$, and so $q_F\circ i=i'\circ(q_F|\hat T)$, thus

$$0=(i'\circ(q_F|\hat T))_*(z)=(q_F\circ i)_*(z)=((q_F)_*\circ i_*)(z)\neq 0.$$

This is a contradiction, and so $\hat T$ is a sphere with holes or $\hat T=\emptyset$.

LEMMA (2.6). *Let $A$ and $B$ be the closed PL-disc in $F$, such that $g[q_F^{-1}(B)]\subset\mathrm{Int}_{K_F}(q_F^{-1}(A))$, and $g\in I'(K_F,f)$. Suppose that $F_g$ is transversal to $q_F^{-1}(\partial A)$. Then $q_F^{-1}(B)\cap F_g$ is contained in certain component of $q_F^{-1}(A)\cap F_g$.*



Proof of (2.6). Suppose that on the contrary, there are points $a,b\in q_F^{-1}(B)\cap F_g$ which are contained in distinct components $T_1$ and $T_2$ of $q_F^{-1}(A)\cap F_g$. $q_F^{-1}(A)$ is a 3-cell and $a,b\in\mathrm{Int}_{K_F}(q_F^{-1}(A))$, because $B\subset\mathrm{Int}_F A$. This implies, that $\hat T_1$ and $\hat T_2$ are compact surfaces in $q_F^{-1}(A)$, such that $\partial\hat T_i\subset\partial(q_F^{-1}(A))$ for $i=1,2$. From this and from duality it follows that each $\hat T_i$ disconnects $q_F^{-1}(A)$, and so $q_F^{-1}(A)\setminus\hat T_1\setminus\hat T_2$ has at least 3 components. The fact that $\hat T_i\cap\mathrm{Int}_{K_F}(g[q_F^{-1}(B)])\neq\emptyset$ for $i=1,2$ implies that each of these components has a non-empty intersection with $g[q_F^{-1}(B)]$. So $g[q_F^{-1}(B)]\setminus\hat T_1\setminus\hat T_2$ has at least 3 components, and so $g[q_F^{-1}(B)]\setminus F_g$ has at least 3 components. This contradicts Lemma (2.3).

LEMMA (2.7). *Let $R$ be a PL surface in $I\times(3I)^2$ where $nI=[-n,n]\subset R^1$. Suppose that $\partial R\cap\big(I\times\partial((3I)^2)\big)\neq\emptyset$, $\partial R(I\times\partial I^2)\neq\emptyset$, $\partial R\subset I\times\big(\partial(3I^2)\cup\partial I^2\big)$ and that $R$ intersects $I\times\partial I^2$ transversaly. Then $I\times(3I)^2\setminus R$ contains at least two components, having a non empty intersection with $I\times\partial((2I)^2)$. Consequently, if $R_1$ and $R_2$ satisfy the described conditions then $I\times(3I)^2\setminus(R_1\cup R_2)$ contains at least three components having a non empty intersection with $I\times\partial((2I)^2)$.*

Proof of (2.7) easily follows from the duality.

LEMMA (2.8). *Let $A$ and $B$ be the closed PL discs in $F$, such that $B\subset\mathrm{Int}_F A$. Suppose that $g\in I'(K_F,f)$ is sufficiently close to $f$. Then for $n=0,1$ each map $S^n\to q_F^{-1}(\mathrm{Int}_F B)\cap F_g$ is a homotopic to a constant in $q_F^{-1}(\mathrm{Int}_F A)\cap F_g$.*

Proof of (2.8). Let $A_0,A_1,A_2,A_3,A_4$ be the closed PL discs in $F$ such that $A_0=A$, $A_4=B$, and $A_{i+1}\subset\mathrm{Int}_F A_i$. We assume that $g$ is so close to $f$ that

(1) $g[q_F^{-1}(A_{i+1})]\subset q_F^{-1}(A_i)$ for $i\leqslant 3$ and $q_F|F_g\colon F_g\to F$ induces isomorphism on $Z_2$-homology.

For the technical reasons, we assume first, that

(2) $F_g$ is transversal to $q_F^{-1}(\bigcup_{i\leqslant 4}\partial A_i)$.

If $n=0$ then (2.8) follows from (2.6). Suppose, that $n=1$ and let $\xi\colon S^1\to q_F^{-1}(B)\cap F_g$ be a PL-map, which is homotopic to a constant in $q_F^{-1}(A_1)\cap F_g$. Let $T$ be a component of $q_F^{-1}(A_1)\cap F_g$, containing $\xi(S^1)$. By (1) and (2.5) $\hat T$ is a sphere with holes, containing $\xi(S^1)$. $\xi$ is not homotopic to a constant in $\hat T$, so $\hat T\setminus\xi(S^1)$ has two components, $T_1,T_2$ such that $T_i\cap\partial\hat T\neq\emptyset$ for $i=1,2$. Let $R_i=T_i\cap q_F^{-1}(A_1\setminus\mathrm{Int}_F B)$. Then $R_i$ is a sphere with holes, whose boundary is contained in $q_F^{-1}(\partial A_1\cup\partial B)$ and intersects both $q_F^{-1}(\partial A_1)$ and $q_F^{-1}(\partial B)$. By (2.7) $q_F^{-1}(A\setminus\mathrm{Int}_F B)\setminus R_1\setminus R_2$ has at least 2 components intersecting $q_F^{-1}(A_2\setminus\mathrm{Int}_F A_3)$, hence $g[q_F^{-1}(A_2\setminus\mathrm{Int}_F A_3)\setminus F_g$ has more than 2 components (note that by (1) $g[q_F^{-1}(A_2\setminus\mathrm{Int}_F A_3)]\subset A_1\setminus A_4$).

This contradicts (2.3) and so proves that if (2) is satisfied then each map $\xi\colon S^n\to q_F^{-1}(B)\cap F_g$ is homotopic to a constant in $q_F^{-1}(A_1)\cap F_g$.

Now we consider the general case, i.e. we do not assume that (2) is satisfied. Let $\xi\colon S^n\to q_F^{-1}(\mathrm{Int}_F B)\cap F_g$ be any map. By the general position theorem there exists a PL-homeomorphism $h\colon K_F\to K_F$ such that $h|\partial K_F=\mathrm{id}$ and $h(F_g)$ is transversal to $q_F^{-1}(\bigcup_{i\leqslant 4} \partial A_i)$. We can choose $h$ so close to $\mathrm{id}_M$ that $g' = h * g$ satisfies the conditions:

$$g'[q_F^{-1}(A_{i+1})] \subset q_F^{-1}(A_i) \quad \text{for} \quad i \leqslant 3,$$

$$h\xi(S^1) \subset q_F^{-1}(\mathrm{Int}_F B) \quad \text{and} \quad h^{-1} q_F^{-1}(A_1) \subset q_F^{-1}(A).$$

By our previous consideration applied to $g'$ and to the map $\xi' = h\xi$ we infer that $h\xi\colon S^n \to q_F^{-1}(A_1) \cap h(F_g)$ is homotopic to a constant. Then $\xi\colon S^n \to q_F^{-1}(A) \cap F_g$ is homotopic to a constant.

LEMMA (2.9). *Let $N$ be a closed surface. Then for every open cover $\alpha$ of $N$, there exists an open cover $\beta$ of $N$ such that for every closed surface $M$ and maps $\xi_1\colon M \to N$ and $\xi_2\colon N \to M$ such that $\xi_1 \circ \xi_2\colon N \to N$ is $\beta$-close to $\mathrm{id}_N$, there exists a* PL*-homeomorphism $h\colon M \to N$ which is $\alpha$-close to $\xi_1$ (if $\beta$ is an open cover of $N$, and $h_1, h_2\colon M \to N$ are the maps, then we say that $h_1$ and $h_2$ are $\beta$-close, if for every $x \in M$ there exists $b \in \beta$, with $\{h_1(x), h_2(x)\} \subset b$).*

Proof of (2.9). We shall use the $\alpha$-approximation theorem of Chapman and Ferry [3]. This theorem was proved in [3] for dimensions $n \geqslant 5$, but the proof can be extended to the case $n = 2$ (see [7], Theorem (1.2)).

Now to get (2.3), we need only to combine this theorem with Theorem (3) of [5] and with Theorem (5.1) p. 88 of [2], which in view of compactness of $M$ and $N$ and of the local contractability of ANR-spaces implies that if the mapping $\xi_1 \circ \xi_2\colon N \to N$ is sufficiently close to $\mathrm{id}_N$ then it is homotopic to $\mathrm{id}_N$, and moreover homotopy can be chosen to be small. Then Theorem (3) of [5] and (1.2) of [7] imply the existence of a topological homeomorphism $f'\colon M \to N$ approximating $\xi_1$, which by [9] p. 63 can be approximated by the PL-homeomorphism $h\colon M \to N$.

**The description of** $U_3'$. Now, to find the required $U_3'$ we use Lemma (2.9), with $N = F$ and $\alpha$ — any cover of $F$, such that for $a \in \alpha$, $\operatorname{diam} a \leqslant \delta$ where $\delta$ is a number found together with $U_2$. Let $\beta$ be a cover of $F$, guaranteed by (2.9). We intend to find $U_3'$ so small, that for any $g \in U_3'$ we are able to find $\xi_1\colon F \to F_g$ and $\xi_2\colon F_g \to F$, such that $\xi_2 \circ \xi_1$ is $\beta$-close to the identity. Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be covers of $F$ by the closed PL discs such that $\alpha_4 = \beta$ and there exist maps $\omega_i\colon \alpha_i \to \alpha_{i+1}$ for $i \leqslant 3$, satisfying conditions: for $a \in \alpha_i$, $i = 1, 3$ or $4$, $a \subset \mathrm{Int}_F(\omega_i(a))$ and for $a \in \alpha_2$, $\mathrm{St}^2 a \subset \mathrm{Int}_F(\omega_i(a))$. Then we claim, that $U_3'$ is so small, that for any $g \in U_3'$, $g$ satisfies the requirements of (2.2) and of (2.8) with $A = a$, $B = \omega(a)$, for any $a \in \alpha_i$, and $i = 1$ or $3$. Then we take $M = F_g$, $\xi_2 = q_F | F_g$, and we have only to construct $\xi_1\colon F \to F_g$. Let $\tau$ be a triangulation of $F$, such that for any vertex $v \in \tau^0$, $\mathrm{St}(v) \subset a$ for certain $a \in \alpha_1$, and we establish $\xi_1 | v$ to be any point of $q_F^{-1}(V) \cap F_g$ (this set is $\neq \emptyset$, by (2.2) and our choice of $U_3'$). Then, we use (2.8) with $n = 0$, to extend $\xi_1$ to $\tau^1$ in such a way, that for any 1-simplex, $\sigma^1$, $\xi_1(\sigma^1) \subset q_F^{-1}(b)$ for a certain $b \in \alpha_2$ ($b$ may be chosen to be $\omega_1(a)$ where $a \in \alpha_1$ is a disc containing the star of one of the vertices of $\sigma^1$). Then, the image by $\xi^1$ of the boundary of each 2-simplex $\sigma^2$ of $\tau^2$ is contained in $q_F^{-1}(\mathrm{St}^2(b))$, for a certain $b \in \alpha_2$, and so in $q_F^{-1}(c)$, where



$c = \omega_2(b) \in \alpha_3$. Then we use (2.8) again, with $n = 1$, to extend $\xi_1$ to the map of $F$ onto $F_g$ such that for each $\sigma^2 \in \tau^2$ $(\sigma^2) \subset q_F^{-1}(d)$ for a certain $d \in \alpha_4$.

Let us note that $\xi_1, \xi_2$ satisfy our requirements. Actually, for $x \in F$, $\xi_2 \circ \xi_1(x)$ is $\beta$-close to $x$. This fact and (2.9) complete the description of $U_3'$, and so the proof of (2.1).

**3. The failure of the 4-dimensional analogue.** Theorem (1.2) may be contrasted with the situation in dimension 4. Let $f$ be a standard orthogonal involution on $S^4$, such that $\mathrm{Fix}(f)$ is a standard unknotted $S^2 \subset S^4$. Then there exists a sequence $f_i$ of PL involutions on $S^4$, such that $\varrho_{S_4}(f, f_i) \to 0$, and no $f_i$ is conjugate to $f$. $f_i$ are constructed as follows: let $K_i$ be a sequence of $f$-invariant 4-cells in $S^4$, such that $K_i \cap \mathrm{Fix}(f)$ is a 2-cell in $K_i$ for every $i$, and that $\operatorname{diam}(K_i) \to 0$. Then we take $f_i | S^4 \setminus K_i = f | S^4 \setminus K_i$, and on $K_i$ we define $f_i$ to be a non-standard involution, as described in [8], p. 347.

**Added in proof.** Since the time this paper and [6] have been written, there appeared the "equivariant Dehn Lemma" (W. H. Meeks III and S.-T. Yau, Comment. Math. Helv. 56 (1981), pp. 225–239) and a proof of the "Smith conjecture". These two theorems make it possible to strengthen the result of [6] and to obtain it with much less effort. This, together with the result of the present paper, gives the following theorem, which is an extension of both (1.1) and (1.2):

THEOREM. *Let $f\colon M \to M$ be any* PL *map of a closed* PL *3-manifold $M$ such that $f^p = \mathrm{id}_M$, for a prime number $p$. Then, for every neighbourhood $V$ of $\mathrm{id}_M$ in $\mathscr{H}_{\mathrm{PL}}(M)$ there is a neighbourhood $U$ of $f$ in $\mathscr{H}_{\mathrm{PL}}(M)$ such that, for any $g \in U$, with $g^p = \mathrm{id}_M$ there exists a* PL*-isotopy $h_t\colon M \to M$, $h_t \in V$ such that $h_0 = \mathrm{id}_M$ and $h_1^{-1} \circ g \circ h_1 = f$.*

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