Если X и Y периферически когомологически локально связны над R, то имеют место формулы Кюннета

$$(2)\quad 0\to \bigoplus_{p+q=n-1} \big(I_x^p(X;R)\otimes I_y^q(Y;R)\big)\to$$
$$\to I_{(x,y)}^n(X\times Y;R)\to \bigoplus_{p+q=n} \mathrm{Tor}\big(I_x^p(X;R), I_y^q(Y;R)\big)\to 0\,.$$

Следствие 4. *Пусть R — счётное кольцо. Если X и Y периферически когомологически локально связны над R, то и произведение $X\times Y$ обладает этим свойством.*

Доказательство. Свойство pclc_R пространства X эквивалентно счётности всех локальных модулей $I_x^n(R)$ (предложение 2). Если этим свойством обладают X и Y, то все модули $I_x^p(X;R)$ и $I_y^q(Y;R)$ счётны, и из рассмотрения точной последовательности 2 следствия 3 получаем, что все модули $I_{(x,y)}^n(X\times Y;R)$ также счётны, т.е. $X\times Y$ обладает свойством pclc_R.

В заключение выражаю благодарность Е. Г. Скляренко за постановку задачи и руководство работой.

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On Prohorov spaces *

by

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Abstract. A topological property is defined which implies the Prohorov property and in some eases is equivalent. Related properties are discussed, and our property is used to refute two reasonable conjectures concerning Prohorov spaces.

§ 1. Introduction. For a separable, metric space, X, we let $P(X)$ denote the space of probability measures on the Borel subsets of X, endowed with the weak topology. That is, $\mu_n\to\mu$ if and only if $\liminf\mu_n G\geqslant\mu G$ for every open set G. Let $P_t(X)$ denote the subspace of $P(X)$ consisting of those "tight" measures, μ, for which $\mu B=\sup\{\mu K\colon K\subseteq B,\ K \text{ is compact}\}$ for all Borel sets, B, in X.

We say that X is *Prohorov* if it is true that if M is a compact subset of $P_t(X)$, then if $\varepsilon>0$, there exists a compact subset H of X such that for each $\mu\in M$, $\mu H\geqslant 1-\varepsilon$. The first result concerning Prohorov spaces was that if X is Polish (complete, separable, metric), then X is Prohorov [Pv].

It is not at all obvious which separable metric spaces enjoy the Prohorov property. Indeed, the next series of results concerning Prohorov spaces consisted of examples of non-Prohorov spaces, culminating in the description of a non-Prohorov, σ-compact subspace of the unit square [D].

Shortly afterwards, Preiss [Ps] (and later, independently, Saint-Raymond [SR]) showed that among the co-analytic separable metric spaces, the only Prohorov spaces are the Polish spaces, so it is much easier to be non-Prohorov than was previously suspected. In the same paper, Preiss gives an example, assuming the continuum hypothesis (CH), of a non-Polish, Prohorov subspace of the unit interval.

The purpose of this paper is to examine the relationship between Polish spaces and Preiss's CH example in an attempt to topologically characterize separable, metric Prohorov spaces. We give a topological property, called an S-decomposition, which is sufficient, and many times necessary, for a space to be Prohorov. Related properties of S-decompositions are discussed, and the property is used to refute two reasonable conjectures about Prohorov spaces.

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* The research for this paper was conducted while the author was at University of London, Chelsea College.

§ 2. **Preliminaries.** Throughout this paper, Z denotes a compact metric space. We intend for Z to be some compactification of the space X, but no specific method for describing such a compactification is used.

DEFINITION. (X, B) is an *S-decomposition* of Z means that

(i) $X \cup B = Z$ and $X \cap B = \emptyset$; and

(ii) if for each compact subset K of B, $\langle O_{K1}, O_{K2}, \ldots\rangle$ is a decreasing sequence of open (relative to Z) sets containing K, and for each compact subset H of X, U_H is open (relative to Z) containing H, then there exists a sequence $\langle K_1, K_2, \ldots\rangle$ of compact subsets of B, and a compact subset H of X such that $\bigcup_{i=1}^{\infty} O_{K_i i} \cup U_H = Z$.

When working with S-decompositions, we always assume, sometimes without specifically saying so, that $K \subseteq B$ is compact and $H \subseteq X$ is compact.

Before proceeding, the reader's attention is drawn to certain properties which we denote by P, C'', and β. A space B is said to have *property P* if there exists a countable subset A of B such that if U is open and U contains A, then $B \setminus U$ is countable. B is said to have *property C''* if it is true that if for each $x \in B$, $\langle O_{x1}, O_{x2}, \ldots\rangle$ is a decreasing sequence of open sets containing x, then there exists a sequence $\langle x_1, x_2, \ldots\rangle$ of points of B such that $\bigcup_{i=1}^{\infty} O_{x_i i} = B$. (Notice the similarities to our definition of an S-decomposition.) B is said to have *property β* if every measure, μ, from $P(B)$ is supported by a countable set. It is relatively straightforward to see that $P \to C'' \to \beta$. For more details, see [K, pp. 527–528] and [MS].

In terms of the preceding definitions, Preiss uses CH to describe an uncountable P space, B, in the unit interval, and then shows that the complement, X, of B, is Prohorov but not Polish.

K. Kunen has informed the author that under Gödel's Axiom of Constructibility, V = L, there exists an uncountable, co-analytic, P subspace of the unit interval. Since this space has property C'', it follows by Theorems 3.2 and 3.4 below that the complement of this P space is Prohorov. Therefore, not only does Preiss's theorem require some condition in addition to that of X being Prohorov to imply that X is Polish—it is not sufficient that X be analytic. This answers Problem 20 of [SR].

The following lemmas will be useful.

LEMMA 2.1 (Lemma 1, [Ps]). *If $X \subseteq Y$ and M is compact in $P_t(X)$ and $\hat{M} = \{\hat{\mu} \in P_t(Y)\colon \hat{\mu}(B) = \mu(B \cap X)$ for some $\mu \in M\}$, then $\hat{M}$ is compact.*

LEMMA 2.2 (Lemma 5.3, [T]). *If M is a compact subset of $P_t(Y)$, and if F is a closed subset of Y with the property that if $\mu \in M$, $\mu F = 0$, then if $\varepsilon > 0$, there exists an open set G, $F \subseteq G$, such that if $\mu \in M$, $\mu G < \varepsilon$.*

We say that the decomposition (X, B) of Z is *universally measurable* if it is true that for each $\mu \in P(Z)$, $\mu^* X = \mu_* X$. (μ^* and μ_* denote outer and inner measure respectively. Also, if $\mu^* X = \mu_* X$, $\mu^* B = \mu_* B$.)

LEMMA 2.3 (First paragraph of proof of Theorem 4, [Ps]). *If (X, B) is a universally measurable decomposition of Z and M is a compact subset of $P_t(Z)$ and if $\mu \in M$ implies that $\mu^* B = 0$, then M is (homeomorphic to) a compact subset of $P_t(X)$.*

§ 3. **S-decompositions and Prohorov spaces.** To put the relationship between $P(X)$ and $P_t(X)$ into perspective, we prove

LEMMA 3.1. *If (X, B) is a universally measurable decomposition of Z, then $P_t(X) = P(X)$.*

Proof. Suppose $\mu \in P(X)$. Let $\hat{\mu} \in P(Z)$ be defined by $\hat{\mu} A = \mu(A \cap X)$. Since X is $\hat{\mu}$-measurable, we have that for Borel A in X, with $A = \hat{A} \cap X$, ($\hat{A}$ Borel in Z), $\mu A = \hat{\mu}(\hat{A} \cap X) = \sup\{\hat{\mu} K\colon K$ is a compact subset of $\hat{A} \cap X\} = \sup\{\mu K\colon K$ is a compact subset of $A\}$.

THEOREM 3.2. *If B is the union of a σ-compact set and a C'' set, then (X, B) is an S-decomposition of Z, where Z is a compact space containing B and X is the complement of B.*

Proof. Let, for compact $K \subseteq B$, $\langle O_{K_i}\rangle$ be a decreasing sequence of open sets containing K, and for compact $H \subseteq X$, U_H be open containing H. For convenience, write $B = (K_1 \cup K_3 \cup K_5 \cup \ldots) \cup C''$, where each K_{2i-1} is compact and C'' has that property. In C'', we can find a sequence $\langle x_2, x_4, x_6, \ldots\rangle$ such that C'' is covered by the sets $O_{\{x_{2i}\}2i}$. For even $2i$, let $K_{2i} = \{x_{2i}\}$, and now B is covered by the sets $O_{K_i i}$. But as $Z \setminus \bigcup_{i=1}^{\infty} O_{K_i i}$ is compact and lies in X, we call it H to get our conclusion.

THEOREM 3.3. *If (X, B) is an S-decomposition of Z, then (X, B) is universally measurable.*

Proof. Let $\mu \in P(Z)$. Let $\varepsilon > 0$. For compact $K \subseteq B$, let O_{Ki} be open containing K such that $\mu(O_{Ki} \setminus K) < \varepsilon/2^{i+1}$. For compact $H \subseteq X$, let U_H be open containing H such that $\mu(U_H \setminus H) < \varepsilon/2$. Denoting by H and $\langle K_1, K_2, \ldots\rangle$ the sets such that $\bigcup_{i=1}^{\infty} O_{K_i i} \cup U_H = Z$, we have that $\mu H \leqslant \mu_* X$ and $\mu(\bigcup_{i=1}^{\infty} K_i) \leqslant \mu_* B = 1 - \mu^* X$, so $\mu H + \mu(\bigcup_{i=1}^{\infty} K_i) \leqslant 1 - (\mu^* X - \mu_* X)$. On the other hand, $1 = \mu Z \leqslant \mu U_H + \mu(\bigcup_{i=1}^{\infty} O_{K_i i})$, and since $\mu U_H - \mu H < \varepsilon/2$ and $\mu(\bigcup_{i=1}^{\infty} O_{K_i i}) - \mu(\bigcup_{i=1}^{\infty} K_i) < \varepsilon/2$, $\mu^* X - \mu_* X < \varepsilon$. As ε was arbitrary, $\mu^* X = \mu_* X$.

THEOREM 3.4. *If (X, B) is an S-decomposition of the compact space Z, then X is Prohorov.*

Proof. Suppose that (X, B) is an S-decomposition of Z, but that X is not Prohorov. Let $M \subseteq P(X)$ be compact and $\varepsilon > 0$ such that if $H \subseteq X$ is compact, there exists $\mu \in M$ such that $\mu H < 1 - \varepsilon$. Denote by $\hat{M}$ the extension to $P(Z)$ of M. By Lemma 2.1, $\hat{M}$ is compact, and since it is true that if K is compact in B, $\mu K = 0$

for all $\mu \in \hat{M}$, then by Lemma 2.2, we can choose, for each i, O_{Ki} containing K such that $\mu O_{Ki} < \varepsilon/2^{i+1}$ for each $\mu \in \hat{M}$.

For each compact $H \subseteq X$, there exists $\mu_H \in \hat{M}$ such that $\mu_H H < 1-\varepsilon$, so let U_H be open containing H such that $\mu_H U_H < 1-\varepsilon$. Now given any $\bigcup_{i=1}^{\infty} O_{K_i i} \cup U_H$, μ_H must assign this measure less than 1, so it fails to contain Z. The contradiction concludes the proof.

Thus, an S-decomposition, (X, B) of Z is sufficient to ensure that X is Prohorov, and this property covers the Polish spaces as well as Preiss's example of a non-Polish, Prohorov space. It also follows from Theorem 3.2 and Preiss's result that for compact, metric Z, if B is analytic and X is Prohorov, then (X, B) is an S-decomposition, since B is σ-compact.

The next theorem shows that S-decompositions characterize Prohorov spaces under other circumstances. The significance of the theorem is that (in vague terminology), Preiss's example is essentially the best example of a non-Polish Prohorov space in the unit interval.

We remark that it is possible, with CH, to construct a Prohorov subspace X of $[0, 1]$ which is not universally measurable. Thus, S-decompositions can not completely characterize Prohorov spaces.

THEOREM 3.5. *If (X, B) is a universally measurable decomposition of the compact metric space, Z, and if compact sets in B are countable, and if B has dimension* 0, *then if X is Prohorov, (X, B) is an S-decomposition of Z.*

Proof. From the dimension requirement on B, we obtain a collection, $\mathscr{W}$, of open sets in Z, which forms a basis for B and is such that if W and W' are members of $\mathscr{W}$, then $W \subseteq W'$ or $W' \subseteq W$ or $W \cap W' = \emptyset$.

Suppose, now, that (X, B) is not an S-decomposition. We let, for compact $K \subseteq B$, $\langle O_{K1}, O_{K2}, \ldots \rangle$ be a sequence of open sets containing K, and for compact $H \subseteq X$, U_H be open containing H such that if $K_1^1, K_1^2, K_2^2, K_1^3, K_2^3, K_3^3, \ldots$ are compact sets from B, and H is compact in X, then Z is not contained in $[O_{K_1^1 1} \cup \cup (O_{K_1^2 2} \cup O_{K_2^2 2}) \cup (O_{K_1^3 3} \cup O_{K_2^3 3} \cup O_{K_3^3 3}) \cup \ldots] \cup U_H$. (This modification of S-decompositions is easily seen to be equivalent to the original formulation.) Furthermore, we may assume that for $x \in B$, $O_{\{x\}i} \in \mathscr{W}$, and we abbreviate this by O_{xi}.

Let $M = \{\mu \in P(Z)\colon$ if $x \in B$, $\mu O_{xi} \leqslant 1/i\}$. We shall show that M is nonempty (it is compact since $P(Z)$ is) and if H is compact in X, there exists $\mu \in M$ such that $\mu H = 0$ in the same argument. These conclusions will follow if we show that if $H \subseteq X$ is compact and $x_1, x_2, \ldots, x_p$ are points of B and n is a natural number, then $M' = \{\mu \in P(Z)\colon \mu O_{xi} \leqslant 1/i$ for $x \in \{x_1, \ldots, x_p\}$ and $i \leqslant n$, and $\mu U_H = 0\}$ is non-empty.

Consider all *finite* measures, η, for which $\eta O_{xi} \leqslant 1/i$ and $\eta U_H = 0$ for those x and i. Let $t = \sup\{\eta Z\colon \eta$ is such a measure$\}$. Now, if $t = \infty$, M' is obviously non-empty. So we assume that $t < \infty$, in which case, the value of t is actually attained by some ηZ.

Now, for this η and for the $x \in \{x_1, \ldots, x_p\}$ and $i \leqslant n$, $\{O_{xi}\colon \eta O_{xi} = 1/i\}$ must cover $Z \setminus U_H$, so there exists a minimal subcover. But from this minimal subcover, there exists $i \leqslant n$ and at least i different points x from $\{x_1, \ldots, x_p\}$ such that O_{xi} belongs. As the O_{xi} come from $\mathscr{W}$, and the subcover is minimal, these O_{xi} are mutually exclusive, and hence $\eta Z \geqslant 1$. Therefore, M' is non-empty, and consequently $M'' = \{\mu \in P(Z)\colon$ if $x \in B$, $\mu O_{xi} \leqslant 1/i$ and $\mu U_H = 0\}$ exists, as a subset of M.

Although M is a subset of $P(Z)$, by Lemma 2.3, M may be thought of as a subset of $P(X)$, since compact subsets of B are countable, and degenerate sets in B have measure 0 for all $\mu \in M$. Therefore, X is not Prohorov.

§ 4. Examples of non-Polish, Prohorov spaces. Our definition of an S-decomposition is related to the definition of property C'', and C'' implies property β, so one might conjecture that if $X \subseteq Z$ is Prohorov and universally measurable with respect to Z, then X is "essentially Polish"—that is, there exists a G_δ set in Z such that the symmetric difference, $(X \setminus G_\delta) \cup (G_\delta \setminus X)$ has property β. Preiss's example satisfies this condition. The following shows that not every example does.

EXAMPLE 4.1 (CH). There exists an S-decomposition, (X, B) of $Z = [0, 1]$ such that if K is σ-compact in Z, $(K \setminus B) \cup (B \setminus K)$ is not β.

Proof. Let N be a first category subset of Z such that if U is open in Z, $U \cap N$ contains a Cantor set. Order the dense open subsets of Z, $\{O_\alpha\}$, indexed by the countable ordinals.

Let K_α be a Cantor subset of $(Z \setminus N) \cap (\bigcap_{\sigma \leqslant \alpha} O_\sigma)$, and let $B = \bigcup_{\sigma < \omega_1} K_\alpha$. Furthermore, B can easily be arranged to be dense in Z. Let $X = Z \setminus B$.

(X, B) is an S-decomposition of Z, for given $\langle O_{Ki} \rangle$ for $K \subseteq B$ and U_H for $H \subseteq X$, we simply find a sequence $\langle K_1, K_3, K_5, \ldots \rangle$ whose union is dense in B. Now, as $\bigcup_{i=1}^{\infty} O_{K_{(2i-1)}(2i-1)}$ is dense and open in Z, it appears as some O_σ and so only countably many K_α fail to lie in it. Arrange these as $\langle K_2, K_4, \ldots \rangle$ and we have that $\bigcup_{i=1}^{\infty} O_{K_i i}$ contains B. Then take $H \subseteq X$ as $Z \setminus \bigcup_{i=1}^{\infty} O_{K_i i}$.

Now suppose that $K \subseteq Z$ is σ-compact. If K is second category in Z, then K contains an open set, U, and hence $K \setminus B$ contains a Cantor set and is consequently not β. On the other hand, if K is first category in Z, then $Z \setminus K$ contains $\bigcap_{\sigma < \alpha} O_\sigma$ for some α, and then K_α is a Cantor subset of $B \setminus K$. Now, as either $K \setminus B$ or $B \setminus K$ is not β, then $(K \setminus B) \cup (B \setminus K)$ is not β.

Finally, we make a note about the completeness of $P(X)$. X is known to be Polish if and only if $P(X)$ is [Pv]. Relationships between other completeness properties of X and $P(X)$ may be found in [B]. In particular, for Polish spaces as well as for complements (in Polish Z) of C'' sets, X, $P(X)$ is "PC" (contains a dense Polish subspace), the latter being true since $P(X)$ contains all continuous measures in $P(Z)$. This does not carry over to S-decompositions.

EXAMPLE 4.2. (CH). There exists an S-decomposition, (X, B) of $Z = [0, 1]$ such that $P(X)$ is not PC.

Proof. This comes from a stright-forward modification of Theorem 5 of Brown's paper [B], using the technique of Example 4.1. In addition to Brown's lists, $\{C_\alpha\}$, $\{Q_\alpha\}$, and $\{L_\alpha\}$, list the dense open sets in Z, $\{O_\alpha\}$ as well. When picking the set D to be B_α, in addition to Brown's requirements, force D to miss $Z \setminus \bigcap_{\sigma<\alpha} O_\sigma$. Take $B = \bigcup_{\alpha<\omega_1} B_\alpha$ and $X = Z \setminus B$.

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